# ELL212 - Tutorial 4 Solutions, Sem II 2015-16

**Problem 1**

a) The charge density $\rho(r)$ is given by:

$$\rho(r) = e(p(r) - n(r)) = en_0\left(\exp\left(-\frac{e\phi(r)}{k_BT}\right) - \exp\left(\frac{e\phi(r)}{k_BT}\right)\right) \tag{1}$$

For small $Q$, the potential $\phi(r)$ set-up in the plasma is also small, and thus the charge density can approximated by a linear expression in $\phi(r)$:

$$\rho(r) = -\frac{2e^2n_0}{k_BT}\phi(r) \tag{2}$$

Substituting into the poisson's equation:

$$\nabla^2\phi(r) = \frac{2e^2n_0}{\epsilon_0 k_BT}\phi(r) \tag{3}$$

Using the sperhical coordinate system (wherein for a $r$ dependant potential, $\nabla^2\phi(r) \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}(r^2\phi(r))) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi(r)))$, we obtain: In spherical coordinates, use Laplacian expression,

$$\boxed{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi(r)) = \frac{2e^2n_0}{\epsilon_0 k_BT}\phi(r)} \tag{4}$$

b) The solution to Eq. 4 can be written as

$$\phi(r) = A_1\frac{\exp(r/\lambda_D)}{r} + A_2\frac{\exp(-r/\lambda_D)}{r} \tag{5}$$

where $\lambda_D = \sqrt{\epsilon_0 k_BT/2n_0e^2}$. Clearly $A_1 = 0$ since the potential (and hence the electric field) cannot go to $\infty$ as $r \to \infty$. To compute $A_2$, we make use of the fact that there is a point charge $Q$ sitting at the origin. The electric field at a distance $r$ from the origin:

$$\mathbf{E}(r) = -\hat{r}\frac{\partial\phi(r)}{\partial r} = \hat{r}\frac{A_2}{r^2}\left(1 + \frac{r}{\lambda_D}\right)\exp(-r/\lambda_D) \tag{6}$$

Consider a guassian surface of a very small radius $R$ centered at the origin, then the flux of $\mathbf{E}$ through this surface is given by:

$$\Phi = \int \mathbf{E}\cdot d\mathbf{S} = 4\pi A_2\left(1 + \frac{R}{\lambda_D}\right)\exp(-R/\lambda_D) \tag{7}$$

The charge enclosed within this surface is given by

$$Q_{\text{enc}} = Q + 4\pi\int_0^R \rho(r)r^2dr = Q - \frac{8\pi e^2n_0}{k_BT}\int_0^R \phi(r)r^2dr \tag{8}$$

It is easy to show that $\int_0^R \phi(r)r^2dr = A_2\lambda_D(\lambda_D(1-\exp(-R/\lambda_D)) - R\exp(-R/\lambda_D)) \to 0$ as $R \to 0$ and $\Phi \to 4\pi A_2$ as $R \to 0$. Thus, the gauss' law ($\Phi = Q_{\text{enc}}/\epsilon_0$) results in $A_2 = Q/4\pi\epsilon_0$ and hence

$$\boxed{\phi(r) = \frac{Q}{4\pi\epsilon_0 r}\exp(-r/\lambda_D)}$$

c) The potential $\phi_0(r)$ due to $Q$ in the absence of plasma is trivially given by

$$\phi_0(r) = \frac{Q}{4\pi\epsilon_0 r} \tag{9}$$

and thus $\phi(r) = \phi_0(r)\exp(-r/\lambda_D)$. Thus, the effect of the charge $Q$ is screened by the surrounding mobile plasma and becomes negligible over length scales of the order of $\lambda_D$ which can be taken to be an estimate of the screening length.

**Problem 2**

a) $|\psi(r)|^2$ is the probability of finding the electron per unit volume at a distance $r$ from the origin. Since the electron must be found somewhere in all space,

$$\int_0^\infty |\psi(r)|^2 4\pi r^2 dr = 1 \tag{10}$$

Using $\psi(r) = A\exp(-r/a)$, we obtain:

$$\boxed{A = \frac{1}{\sqrt{\pi a^3}}} \tag{11}$$

b) Consider a small volume $\Delta V$ in space. The probability of finding the electron in that volume is $\mid \psi \mid^2 dV$. Thus the average charge $\Delta Q$ in this volume is:

$$\Delta Q = -e|\psi(r)|^2 \Delta V \tag{12}$$

and hence the average charge density $\rho(r)$

$$\rho(r) = \frac{\Delta Q}{\Delta V} = -e|\psi(r)|^2 = -\frac{e}{\pi a^3}\exp(-2r/a) \tag{13}$$

To calculate the field at $r$ due to this charge distribution, consider a spherical gaussian surface of radius $r$ centered at the origin and apply the gauss' law:

$$E(r)4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r \frac{e}{\pi a^3}\exp(-2R/a)4\pi R^2 dR \tag{14}$$

which, after some simplification, gives:

$$\boxed{E(r) = -\frac{e}{4\pi\epsilon_0 r^2}\left[1 - \exp(-2r/a)\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]} \tag{15}$$

For small $r$, we can expand $E(r)$ into a taylor series in $r$ (or use $\exp(-2r/a) \approx 1 - 2r/a + 2r^2/a^2 - 4r^3/3a^3$) to obtain:

$$\boxed{E(r) \approx -\frac{er}{3\pi\epsilon_0 a^3}} \tag{16}$$

c) Consider an external field $E_{\text{ex}}$. This displaces the electron cloud from positive nucleus. We ignore the distortion in the shape of electron cloud. If the electron cloud displaced by distance $d$ from nucleus due to the field. In equilibrium, the net force on nucleus is 0 and hence ($E_{\text{ex}}$ and $E$ refer only to the magnitudes of the external electric field and the electric field due to the electron cloud):

$$E(d) = E_{\text{ex}} \tag{17}$$

For small $d$, using Eq. 16

$$\boxed{p = ed = 3\pi\epsilon_0 a^3 E_{\text{ex}}} \tag{18}$$

where $p$ is the dipole formed by the atom. Therefore the polarisability is given by $\alpha = 3\pi\epsilon_0 a^3$.