ELL212 - Tutorial 1 Solutions, Sem II 2015-16

Q.1: Consider a monochromatic wave function given by

$$\psi(\vec{r}) = A(\vec{r})\cos(\omega t + \theta(\vec{r}))$$

Show that if $\psi(\vec{r})$ satisfies the wave-equation then so does

$$\xi(\vec{r}) = A(\vec{r})\exp[j\{\theta(\vec{r}) + \omega t\}]$$

and vice-versa.

A.1: The wave equation is given by

$$\nabla^2\psi(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$
$$\nabla^2(A(\vec{r})\cos(\omega t + \theta(\vec{r}))) = -\frac{\omega^2}{c^2}(A(\vec{r})\cos(\omega t + \theta(\vec{r})))$$

if we break the trignometric terms of

$$\cos(\omega t + \theta(\vec{r}))$$

and equate with the corresponding terms on both sides of above quation, we get

$$\nabla^2(A(\vec{r})\cos(\theta(\vec{r})) = -\frac{\omega^2}{c^2}A(\vec{r})\cos(\theta(\vec{r})$$
$$\nabla^2(A(\vec{r})\sin(\theta(\vec{r})) = -\frac{\omega^2}{c^2}A(\vec{r})\sin(\theta(\vec{r})$$
$$\nabla^2(A(\vec{r})\exp(j\theta(\vec{r}))) = \nabla^2(A(\vec{r})\cos(\theta(\vec{r})) + j\nabla^2(A(\vec{r})\sin(\theta(\vec{r})))$$
$$\nabla^2(A(\vec{r})\exp(j\theta(\vec{r}))) = -\frac{\omega^2}{c^2}A(\vec{r})\exp(j\theta(\vec{r})))$$

Hence $\xi(\vec{r}) = A(\vec{r})\exp[j\{\theta(\vec{r}) + \omega t\}]$ satisfies the wave equation too.

Q.2: Consider the two dimensional wave equation in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Obtain an expression for an azimuthally symmetric (i.e. ϕ independant) cylindrical wave at frequency ω for large ρ. (Hint: Use the substitution $\psi(\rho,t) = f(\rho,t)/\sqrt{\rho}$).

A.2: The wave equation at large ρ is given by

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = -\frac{\omega^2}{c^2}\psi$$
$$\frac{1}{\rho}\left(\rho\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial\psi}{\partial\rho}\right) = -\frac{\omega^2}{c^2}\psi$$

To first order, $\psi \sim \exp(-j\omega\rho/c), \exp(j\omega\rho/c)$

$$\psi(\rho,t) = f(\rho)\exp(-j\frac{\omega\rho}{c})$$

After computing $\left(\frac{\partial\psi}{\partial\rho}\right)$ and $\left(\rho\frac{\partial^2\psi}{\partial\rho^2}\right)$ and putting in the main equation, we get

$$\frac{\partial^2 f(\rho)}{\partial\rho^2} - 2j\frac{\omega}{c}\frac{\partial f(\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial f(\rho)}{\partial\rho} - j\frac{\omega f(\rho)}{c\rho} = 0$$

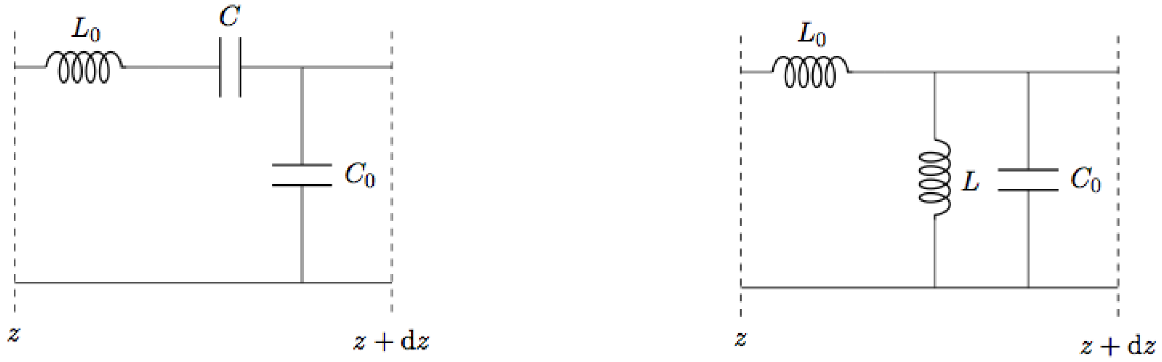
using approximation considering ρ to be very large

$$\frac{\partial f(\rho)}{\partial\rho} + \frac{f(\rho)}{2\rho} = 0$$

Solving the above differential equation we get $f(\rho) = \frac{K}{\sqrt{\rho}}$, where K is some constant

$$\tilde{\psi} = \frac{K}{\sqrt{\rho}} \exp(j\omega t) \exp(-j\frac{\omega\rho}{c})$$
$$\psi = Re(\tilde{\psi}) = \frac{K}{\sqrt{\rho}} \cos(\omega t - \frac{\omega\rho}{c}) + \phi(c)$$

Q.3: Approximate distributed circuit models of (lossless) a lossless transmission operating in high frequency modes is shown in fig.1. Note that L has units H·m, C has units F·m, L_0 has units H/m and C_0 has units F/m. Obtain expressions for the propagation constant β and the characteristic impedance Z_0 of the line for both circuits at frequency ω.



(a) (b)

Fig. 1. Distributed circuit models for **Q.4**

A.3: By considering the voltage and current on an infinresimal section of transmission line, the inductance and capacitance of the infinitesimal section of the line are LΔz and CΔz respectively.

Since the transmission medium is lossless, the differential equation govering voltage and current can be written as $\frac{\partial^2 V}{\partial z^2} = \beta^2 V$ OR $\frac{\partial^2 I}{\partial z^2} = \beta^2 I$

1) Image Q.3a

Transmission line equations:

$$V(z) - j(\omega L_0 - \frac{1}{\omega c})\partial z I(z) = V(z + \partial z)$$
$$-j(\omega L_0 - \frac{1}{\omega c})I(z) = \frac{\partial V}{\partial z}$$
$$V(z) = \frac{1}{j\omega C_0 \partial z}\Big(I(z) - I(z + \partial z)\Big)$$
$$V(z) = -\frac{1}{j\omega C_0}\frac{\partial I}{\partial z}$$

Using the two above differential equation of I and V and comparing with the differential equations of voltage and current we get the value of Propagation constant β

$$\beta = \sqrt{(}\omega^2 L_0 C_0 (1 - \frac{1}{\omega^2 L_0 C}))$$

Characteristic Impedance calculation:

$$\frac{\partial I}{\partial z} = -j\beta I(z) = -j\omega C_0 V(z)$$
$$Z_0 = \frac{V}{I} = \frac{\beta}{\omega C_0} = \sqrt{(}\frac{L_0}{C_0}(1 - \frac{1}{\omega^2 L_0 C}))$$

2) Image Q.3b

Transmission line equations:

$$V(z) - j\omega L_0 \partial z - I(z) = V(z + \partial z)$$
$$-j\omega L_0 I(z) = \frac{\partial V}{\partial z}$$
$$V(z) = -\frac{j\omega L}{1 - \omega^2 L C_0} \frac{\partial I}{\partial z}$$

Using the two above differential equation of I and V and comparing with the differential equations of voltage and current we get the value of Propagation constant β

$$\beta = \sqrt{(}\omega^2 L_0 C_0 (1 - \frac{1}{\omega^2 L C_0}))$$

Characteristic Impedance calculation:

$$\frac{\partial V}{\partial z} = -j\beta V = -j\omega L_0 I(z)$$
$$Z_0 = \frac{V}{I} = \frac{\omega L_0}{\beta} = \sqrt{(}\frac{L_0}{C_0(1 - \frac{1}{\omega^2 L C_0})})$$

Q.4: Consider a one dimensional wave satisfying the wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

and the following boundary conditions:

$$\psi(x = 0, t) = \psi_0 \exp\left(-\frac{t^2}{\tau^2}\right) \text{ and } \left.\frac{\partial \psi(x, t)}{\partial x}\right|_{x=0} = \frac{\psi_0 t}{2v\tau^2} \exp\left(-\frac{t^2}{4\tau^2}\right)$$

Also assume that the function $\psi(x, t)$ is fourier transformable with respect to time t at every x. Obtain an expression for $\psi(x, t)$ for all x and t.

A.4: The wave equation is given by $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi(x, \omega)} \exp(j\omega t) d\omega$$

Calculating $\frac{\partial^2 \psi}{\partial x^2}$ and $\frac{\partial^2 \psi}{\partial t^2}$ from the above equation and using the given one dimensional wave equation, we get

$$\frac{\partial^2 \tilde{\psi}}{\partial x^2} = \frac{\omega^2}{v^2} \tilde{\psi}$$

After solving the above differential equation, we get

$$\tilde{\psi(x, \omega)} = A(\omega) \exp(j\frac{\omega x}{v}) + B(\omega) \exp(-j\frac{\omega x}{v})$$

Given the boundary condition, we can easily find out the values of $A(\omega)$ and $B(\omega)$ which is as follows

$$B(\omega) = \frac{\psi_0 \tau \sqrt{\pi}}{2} \left(2 \exp(-\omega^2 \tau^2) + \exp(-\frac{\omega^2 \tau^2}{4})\right)$$
$$A(\omega) = \frac{\psi_0 \tau \sqrt{\pi}}{2} \left(\exp(-\frac{\omega^2 \tau^2}{4}) - 2 \exp(-\omega^2 \tau^2)\right)$$

Thereafter calculating $\tilde{\psi(x, \omega)}$ we can put it in the main equation of $\psi(x, t)$ and can easily obtain the desired expression

$$\psi(x, t) = \frac{\psi_0}{2} \exp(-\frac{(t + \frac{x}{v})^2}{\tau^2}) + \frac{\psi_0}{2} \exp(-\frac{(t - \frac{x}{v})^2}{\tau^2}) - \frac{\psi_0}{2} \exp(-\frac{(t + \frac{x}{v})^2}{4\tau^2}) + \frac{\psi_0}{2} \exp(-\frac{(t - \frac{x}{v})^2}{4\tau^2})$$