contributions

Scattering by a Dielectric Cylinder of Arbitrary Cross Section Shape

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Abstract—The theory and equations are developed for the scattering pattern of a dielectric cylinder of arbitrary cross section shape. The harmonic incident wave is assumed to have its electric vector parallel with the axis of the cylinder, and the field intensities are assumed to be independent of distance along the axis. Solutions are readily obtained for inhomogeneous cylinders when the permittivity is independent of distance along the cylinder axis.

Although other investigators have approximated the field within the dielectric body by the incident field, we treat the total field as an unknown function which is determined by solving a system of linear equations.

In the case of the dielectric cylindrical shell of circular cross section, this technique yields results which agree accurately with the exact classical solution. Scattering patterns are also presented in graphical form for a dielectric shell of semicircular cross section, a thin homogeneous plane dielectric sheet of finite width, and an inhomogeneous plane sheet. The effects of surface-wave excitation and mutual interaction among the various portions of the dielectric shell are included automatically in this solution.

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I. Introduction

Although rigorous solutions are available for scattering by homogeneous dielectric cylinders of circular and elliptical cross section, only approximate solutions exist for homogeneous or inhomogeneous cylinders of other shapes with finite cross section dimensions. A ray-optics technique is commonly employed by radome designers. Each ray passing through the dielectric body is assumed to undergo the same reflection and phase delay that is experienced by a plane wave passing through an infinitely wide plane sheet of the same thickness and with the same angle of incidence. This ray-optics method often provides reasonably accurate results for slightly curved dielectric shells, but it is inadequate for rapidly curving shells and the edge region of a truncated shell.

The ray-optics solution has been refined and extended considerably by Kouyoumjian, Peters and Thomas. [1] Their technique has proven quite successful with circular dielectric cylinders, spheres, and a few other shapes. However, the method becomes somewhat complicated in the general case and it does not always provide accurate results for small or irregular dielectric bodies.

Rhodes [2] has developed an iteration technique for dielectric scattering problems. The first-order solution is obtained by approximating the total field in the dielectric body by the incident field, and then calculating the scattered field by considering the equivalent volume currents in the dielectric region to radiate in unbounded free space. Useful results are obtained for thin dielectric shells having a dielectric constant near unity. By a similar technique, Andreasen [3] has obtained data for thin spherical shells; Stickler [4] has calculated the scattering patterns of dielectric cylinders of rectangular cross section and Philipson [5] has made an analysis of the scattering properties of thin dielectric rings.

Using a variational formulation, Cohen [6] has obtained accurate solutions for the circular dielectric cylinder. This approach becomes rather complicated, however, and the calculations become lengthy when the dielectric body has an arbitrary shape and dielectric constant.

The technique developed in this paper is based on the integral equation for the field of a harmonic source in the presence of a dielectric cylinder of arbitrary cross section shape. The dielectric cylinder is divided into square cells which are small enough so that the electric field intensity is nearly uniform in each cell. The total electric field intensity within each cell is initially considered to be an unknown quantity. A system of linear equations is obtained by enforcing at the center of each cell the condition that the total field must equal the sum of the incident and scattered fields. This system of equations is solved with the aid of a digital computer to evaluate the electric field intensity in each cell. It is then a rather simple procedure to calculate the scattered field at any other point in space. The technique has the following advantages.

1) The solution approaches the exact solution if a sufficiently large number of cells is employed.

2) Solutions are obtained for a dielectric shell of arbitrary shape as quickly and systematically as for a circular shell.

3) Simply by inserting the appropriate equations for the incident field, one obtains the solution for any two-dimensional source (such as a line source, any array of line sources, or a plane-wave source) in the presence of a dielectric cylinder.

4) Solutions are readily obtained for dielectric shells of tapered thickness and inhomogeneous dielectric shells.

5) The effects of surface-wave excitation and interaction among the various parts of the dielectric cylinder are included automatically in the solution.

6) Accurate solutions are obtained for dielectric cylinders having cross section dimensions up to several wavelengths. The ray-tracing or geometrical-optics methods often fail when the cross section dimensions are on the order of one wavelength or less.

7) Once a solution has been obtained for any particular source location, a relatively simple calculation will provide the new solution corresponding to a rotated or translated source.

The following sections develop the theory and the equations involved in this technique and present numerical results for homogeneous and inhomogeneous dielectric cylinders of various cross section shapes.

II. The Basic Theory

Consider a harmonic wave incident in free space on a dielectric cylinder of arbitrary cross section as suggested in Fig. 1. The time-factor $e^{j\omega t}$ is understood. It is assumed that the incident electric field intensity $\boldsymbol{E}^i$ has only a z component and it is not a function of z, where the z axis is taken to be parallel with the axis of the cylinder. That is,

$$\boldsymbol{E}^i = \hat{z}E^i(x, y) \tag{1}$$

The dielectric cylinder is assumed to have the same permeability as free-space ($\mu=\mu_0$). The dielectric material is assumed to be linear and isotropic, but it may be inhomogeneous with respect to the transverse coordinates as follows:

$$\epsilon = \epsilon(x, y) \tag{2}$$

where ϵ represents the complex permittivity.

Let $\boldsymbol{E}$ represent the total field; that is, the field set up by the source in the presence of the dielectric cylinder. The "scattered field" is defined to be the difference between the total and the incident fields. Thus,

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s \tag{3}$$

Under the assumed conditions, the total and the scattered electric field intensities will have only z components.

The scattered field $\boldsymbol{E}^s$ may be generated by an equivalent electric current J radiating in unbounded free space, where

$$J = j\omega(\epsilon - \epsilon_0)E \tag{4}$$

with ω representing the angular frequency $2\pi f$. This equivalent current density is often called the "polarizarion current."

The field of an electric current filament dI parallel with the z axis in free space is given by

$$d\boldsymbol{E}^s = -\hat{z}(\omega\mu/4)H_0^{(2)}(k\rho)dI \tag{5}$$

where $H_0^{(2)}(k\rho)$ is the Hankel function of order zero, ρ is the distance from the current filament to the observation point and $k=\omega\sqrt{\mu_0\epsilon_0}=2\pi/\lambda$. The free-space wavelength is denoted by λ. The increment of electric current which generates the scattered field is given by

$$dI = J\,dS = j\omega(\epsilon - \epsilon_0)E\,dS \tag{6}$$

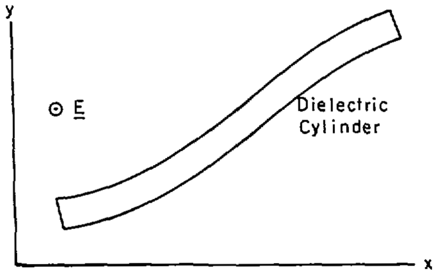

Fig. 1. Cross section of a dielectric cylinder showing the coordinate system.

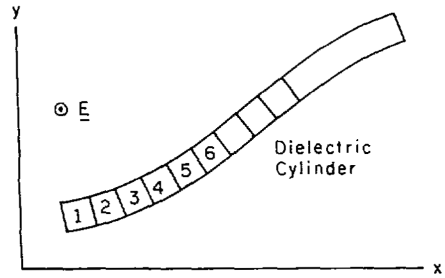

Fig. 2. The cross section of the dielectric cylinder is divided into small cells.

where dS is the increment of surface area on the cross section of the dielectric cylinder. From (5) and (6), the scattered field is given by

$$E^s(x, y) = -(jk^2/4) \iint (\epsilon_r - 1)E(x', y')H_0^{(2)}(k\rho)dx'dy' \quad (7)$$

where (x, y) and (x', y') are the coordinates of the observation point and the source point, respectively, ϵ_r is the complex relative dielectric constant ($\epsilon_r = \epsilon/\epsilon_0$) and

$$\rho = \sqrt{(x - x')^2 + (y - y')^2} \quad (8)$$

The integration in (7) is to be performed over the cross section of the dielectric cylinder. In the inhomogeneous case the relative dielectric constant is considered to be a function of the source coordinates

$$\epsilon_r = \epsilon_r(x', y')$$

Equation (7) is valid for the scattered field at any point inside or outside the dielectric region. The integral equation for the total field E is obtained from (3) and (7),

$$E(x, y) + (jk^2/4) \iint (\epsilon_r - 1)E(x', y')H_0^{(2)}(k\rho)dx'dy' = E^i(x, y) \quad (9)$$

Let us divide the cross section of the dielectric cylinder into cells sufficiently small so that the dielectric constant and the electric field intensity are essentially constant over each cell. The division into cells is indicated in Fig. 2. If (9) is enforced at the center of cell m, the following expression is obtained:

$$E_m + (jk^2/4) \sum_{n=1}^{N} (\epsilon_n - 1)E_n \cdot \iint_{\text{cell } n} H_0^{(2)}(k\rho)dx'dy' = E_m^i \quad (10)$$

where ϵ_n and E_n represent the complex relative dielectric constant and the electric field intensity at the center of cell n and

$$\rho = \sqrt{(x' - x_m)^2 + (y' - y_m)^2} \quad (11)$$

By taking $m = 1, 2, 3, \cdots, N$, (10) yields a system of N linear equations, where N represents the total number of cells. These can be solved to determine the total electric field intensity at the center of each cell ($E_1, E_2, E_3, \cdots, E_N$). Having thus determined the total field $E(x, y)$ in the dielectric region, it is then possible to calculate the scattered field of the dielectric cylinder at any point in space by means of (7). The details of the solution are described in the following section.

III. Surface Integrals of the Hankel Function

The surface integrals in (10) can be evaluated by numerical integration formulas such as the trapezoidal rule and Simpson's rule. These methods have been employed successfully, but it was found that the calculations are quite lengthy, and care must be exercised in integrating through the singularity that exists when the observation point is at the center of cell n. The region of integration (over cell n) is square or rectangular in the simplest case, and a closed-form solution for this integral is not known. A simple solution is, however, available for the integral of the zero-order Hankel function over a circular region. It is given by

$$(jk^2/4) \underset{\text{cell } n}{\int_0^{2\pi} \int_0^a} H_0^{(2)}(k\rho)\rho' d\rho' d\phi'$$
$$= (j/2)[\pi ka H_1^{(2)}(ka) - 2j] \qquad \text{if } m = n$$
$$= (j\pi ka/2)J_1(ka)H_0^{(2)}(k\rho_{mn}) \qquad \text{if } m \neq n \quad (12)$$

where ρ is given by (11) and ρ' and ϕ' are polar coordinates based on a coordinate origin at the center of cell n. The first solution given in (12) applies if the observation point is at the center of the circular region (i.e., if $m = n$). The second solution applies if the observation point is at a distance ρ_{mn} from the center of the circular region, where ρ_{mn} is greater than the radius a of the circular region. Numerical calculations have shown that little error is incurred in approximating square cells with circular cells of the same cross section area to take advantage of the simple expressions given in (12). The distance between the centers of cells m and n is given by

$$\rho_{mn} = \sqrt{(x_m = x_n)^2 + (y_m - y_n)^2} \tag{13}$$

Now the system of linear equations represented by (10) can be written in the following form:

$$\sum_{n=1}^{N} C_{mn}E_n = E_m{}^i \qquad \text{with } m = 1, 2, \cdots, N \tag{14}$$

If a_n represents the radius of the equivalent circular cell which has the same cross section area as cell n, the coefficients C_{mn} are given by

$$C_{mn} = 1 + (\epsilon_m - 1)(j/2)[\pi k a_m H_1{}^{(2)}(ka_m) - 2j] \qquad \text{if } n = m \tag{15}$$

$$C_{mn} = (j\pi k a_n/2)(\epsilon_n - 1)J_1(ka_n)H_0{}^{(2)}(k\rho_{mn}) \qquad \text{if } n \neq m \tag{16}$$

IV. Formulating the Scattered Field

Once the system of linear equations (14) has been solved, the scattered field of the dielectric cylinder can be calculated at any point in space by means of (7). To simplify the surface integral in (7), it is convenient again to divide the cylinder into small cells whose cross section shape is approximately square. From (7) and (12), the scattered field at any point outside the dielectric body is given by

$$E^s(x, y) = -j(\pi k/2)\sum_{n=1}^{N}(\epsilon_n - 1)E_n a_n J_1(ka_n)H_0{}^{(2)}(k\rho_n) \tag{17}$$

where ϵ_n represents the average dielectric constant over cell n, a_n is the radius of the circle having the same area as cell n and ρ_n is the distance from the observation point to the center of cell n,

$$\rho_n = \sqrt{(x - x_n)^2 + (y - y_n)^2} \tag{18}$$

The distant scattering pattern of the dielectric cylinder is obtained by employing the asymptotic form for the Hankel function of large argument and taking

$$\rho_n = \rho_0 - x_n \cos\phi - y_n \sin\phi \tag{19}$$

where ρ_0 and ϕ are the polar coordinates of the distant observation point. The distant scattered field is given by

$$E^s(\rho_0, \phi) = -j(\pi k/2)\sqrt{2j/\pi k\rho_0} \cdot e^{-jk\rho_0}\sum_{n=1}^{N}(\epsilon_n - 1)E_n a_n J_1(ka_n) \cdot e^{jk(x_n \cos\phi + y_n \sin\phi)} \tag{20}$$

The plane-wave scattering properties of any cylindrical body of infinite length are conveniently described in terms of the echo width [7] which is denoted by $W(\phi)$ and is defined as follows:

$$W(\phi) = \lim_{\rho_0 \to \infty} 2\pi\rho_0 \left|\frac{E^s(\rho_0, \phi)}{E^i}\right|^2 \tag{21}$$

From (20) and (21), the echo width of a dielectric cylinder of arbitrary cross section shape is given as follows:

$$W(\phi) = \frac{\pi^2 k}{|E^i|^2}\left|\sum_{n=1}^{N}(\epsilon_n - 1)E_n a_n J_1(ka_n)e^{jk(x_n \cos\phi + y_n \sin\phi)}\right|^2 \tag{22}$$

V. Numerical Results for the Circular Cylindrical Shell

Using (14), (15), (16), (20) and (22), numerical solutions were obtained for a circular dielectric cylindrical shell. Figure 3 shows the electric field distribution in the dielectric shell set up by an incident plane wave, and Fig. 4 shows the distant scattering pattern. Figure 4 also shows the exact classical solution which involves an infinite series of cylindrical mode functions. It may be noted that the integral-equation technique yields results which show excellent agreement with the exact solution. In view of the nature of the integral-equation formulation, it is believed that highly accurate results can also be obtained for dielectric cylinders of other cross section shapes where an exact solution is not available.

In Fig. 3 it may be noted that the electric field intensity in the dielectric region varies from 0.8 to 1.58 in magnitude. The incident plane wave has an electric field intensity of 1.0. Thus, poor accuracy is to be expected if the electric field intensity in the dielectric region is approximated by the incident electric field intensity.

In addition to the plane-wave solution previously described, the integral-equation technique readily yields solutions when the incident wave is the field of a line source or any array of line sources of infinite length. The line sources are assumed to be parallel with the axis of the dielectric cylinder. Figure 5 shows the scattering pattern of a circular dielectric cylindrical shell when the incident field is generated by a nearby line source having an omnidirectional pattern. In Fig. 5 the echo width is referred to the incident power density at the center of the dielectric cylindrical shell. Similar calculations show that the scattering pattern approaches the plane-wave solution shown in Fig. 4 when the line source is at a great distance from the dielectric shell. This provides one check on the accuracy and validity of the integral-equation technique as applied to the dielectric cylinder in the presence of a line source.

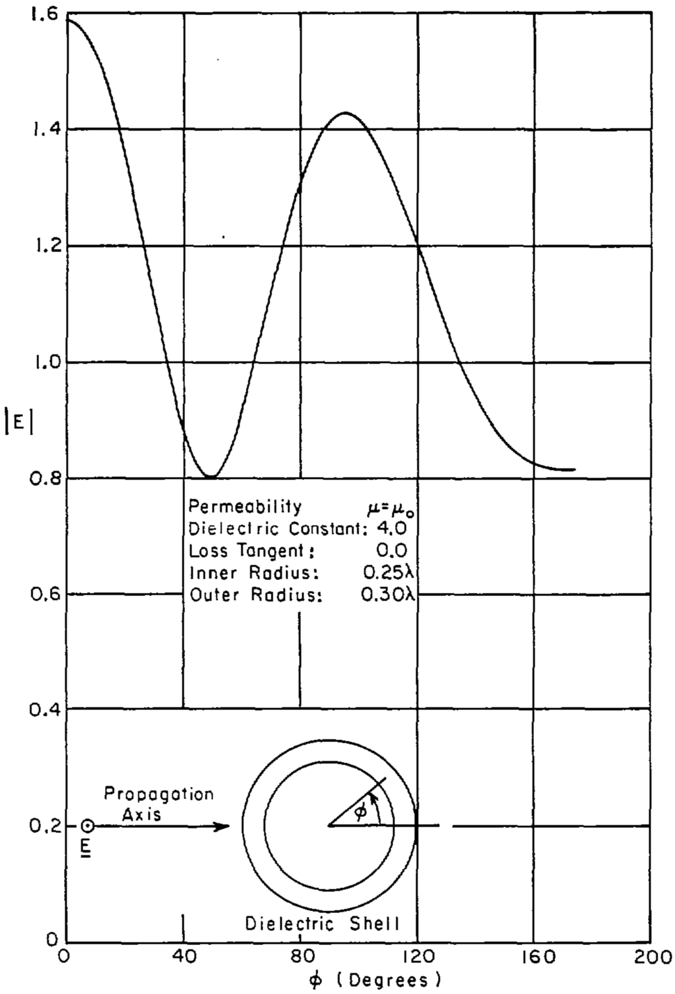

Fig. 3. Electric field distribution in circular dielectric cylindrical shell with plane-wave incident, calculated with the integral-equation technique.

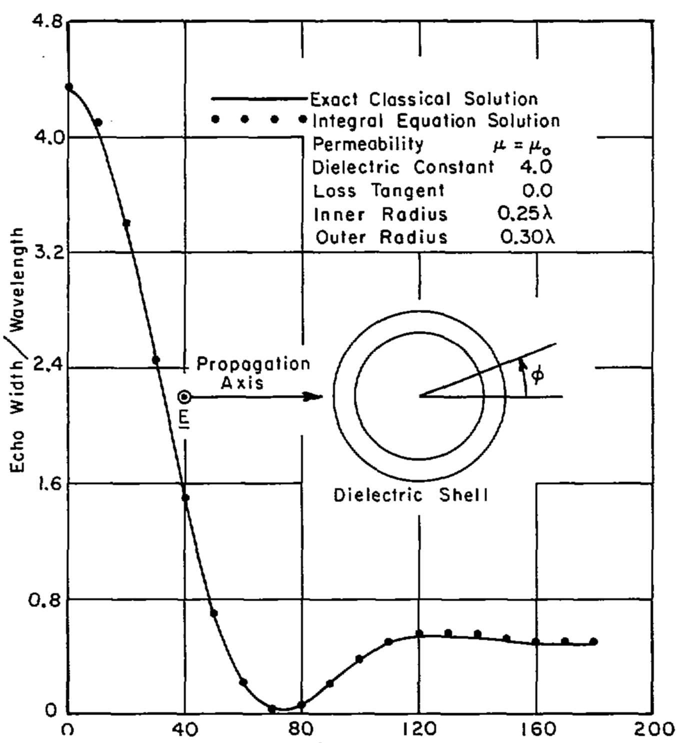

Fig. 4. Distant scattering pattern of circular dielectric cylindrical shell with plane-wave incident.

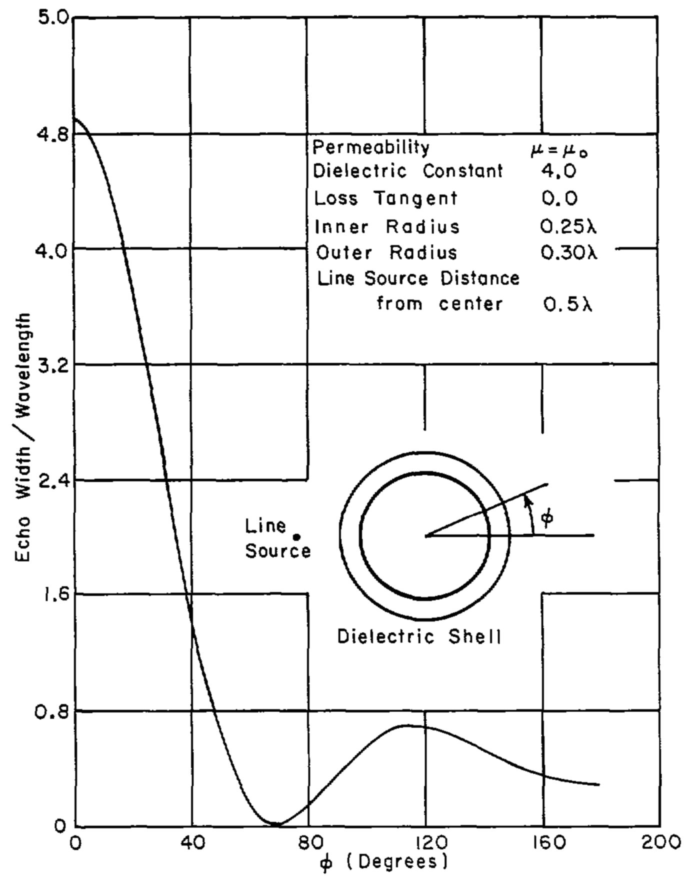

Fig. 5. Scattering pattern of a circular dielectric cylindrical shell in the presence of a nearby parallel line source, calculated with the integral-equation technique.

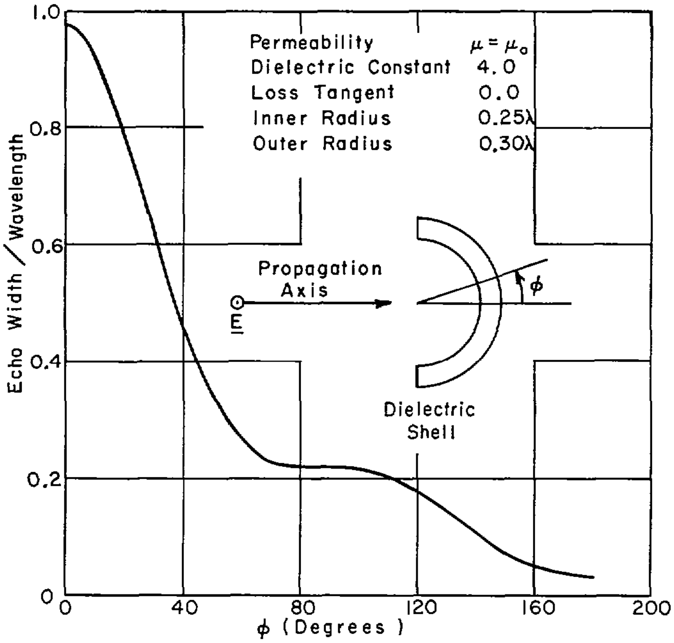

Fig. 6. Scattering pattern of a semicircular dielectric cylindrical shell with plane-wave incident, calculated with the integral-equation technique.

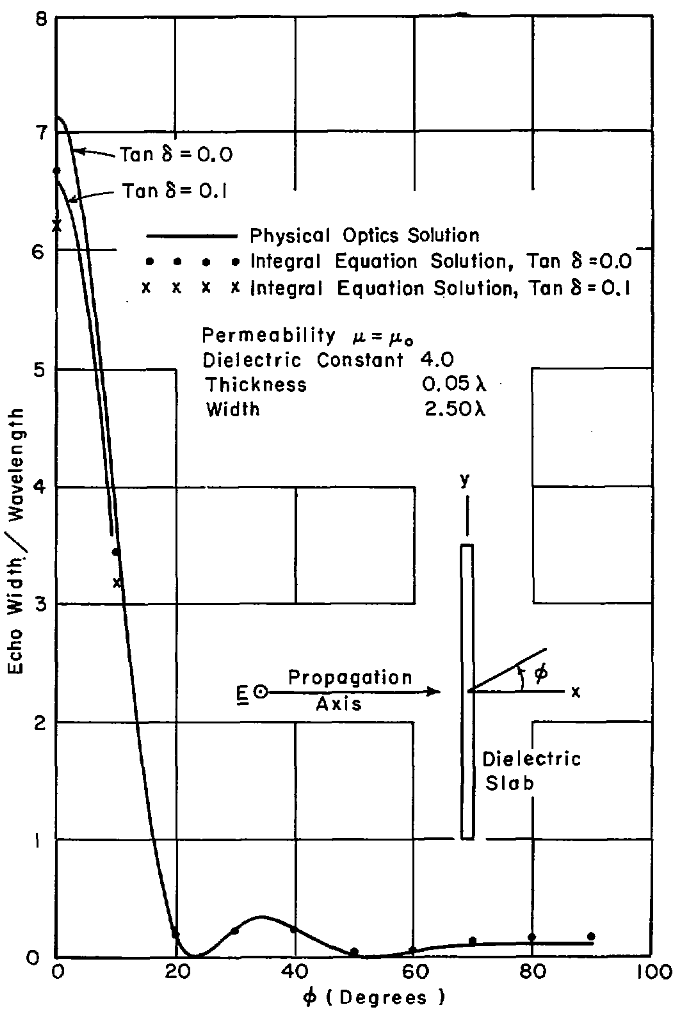

Fig. 7. Calculated scattering patterns of a homogeneous plane dielectric slab of finite thickness and width with a plane wave having normal incidence.

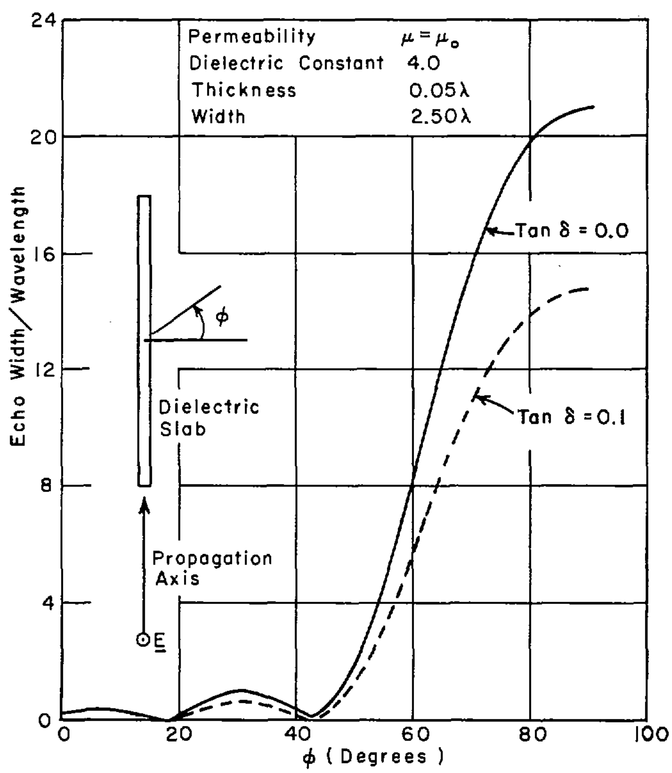

Fig. 8. Calculated scattering patterns of a homogeneous plane dielectric slab with a plane wave at grazing incidence.

VI. Numerical Results for Semicircular Cylindrical Shell

Figure 6 shows the plane-wave scattering pattern of a semicircular cylindrical shell, computed with the integral-equation technique developed in Sections II and III. This particular semicircular dielectric shell has much smaller echo width than the corresponding circular shell illustrated in Fig. 4. However, it is possible to obtain a considerable increase in the backscatter echo width of a semicircular shell (compared with the circular shell of the same radii) by proper choice of the inner and outer radii and the wavelength. No exact solution is available for comparison with the results shown in Fig. 6, and the geometric-optics solution is not likely to be accurate for a shell of such small cross section dimensions.

It should be pointed out that the solution for the semicircular shell (or any other cross section shape) is as simple, straightforward and systematic as that for the circular cylindrical shell when the integral-equation technique is employed. This represents a distinct advantage over the boundary-value solution, the geometric-optics solutions and the variational solutions.

VII. Numerical Results for Plane Homogeneous Dielectric Slabs

Figure 7 shows the calculated scattering patterns of a plane dielectric slab of finite thickness and width for the case where a plane wave is incident normally on the slab. Results are shown both for lossless and dissipative homogeneous slabs.

For comparison, the physical optics solutions are also shown. The physical optics solution is based on the approximation that the electric field intensity within the dielectric body is the same as that in a dielectric slab of infinite width. Thus, the edge effects and surface-wave phenomena are usually neglected in the physical-optics approximation in contrast with the integral-equation solution. Although accurate agreement is not to be expected between the two methods, reasonably good agreement is observed in Fig. 7.

It should be mentioned that slightly dissipative dielectric cylinders are handled as easily as lossless dielectric bodies with the integral-equation technique. In problems involving perfectly conducting or highly conducting scatterers, a somewhat different formulation has been employed with good success. Some modifications must be made in the techniques developed here to study the scattering properties of highly dissipative dielectric bodies, since in this case it is not reasonable to assume the electric field intensity is nearly uniform over each cell of the type employed here.

Figure 8 shows the scattering patterns of the same homogeneous dielectric slab for grazing incidence. Results are shown both for the lossless and dissipative cases. The physical-optics solution is not shown for this case simply because the optical solutions do not yield useful results for grazing incidence unless considerable effort is made to include the effects of surface-wave excitation in the dielectric slab. It may be noted in Fig. 8 that the forward scattering intensity is much greater for grazing incidence than for normal incidence, and that the results are much more sensitive to the loss tangent (tan δ) for grazing incidence.

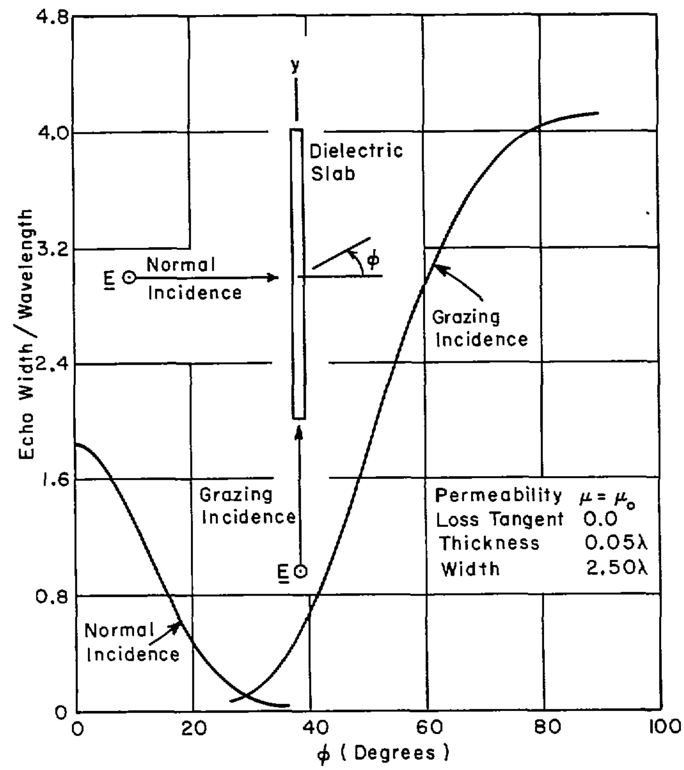

Fig. 9. Calculated scattering patterns of an inhomogeneous plane dielectric slab with a plane wave at normal and grazing incidence. Dielectric constant varies linearly from 4.0 at center to 1.0 at edges of slab.

VIII. Numerical Results for Plane Inhomogeneous Dielectric Slabs

Figure 9 shows the scattering patterns for a lossless inhomogeneous dielectric slab for normal incidence and grazing incidence. The dielectric constant is assumed to vary linearly from 4.0 at the center of the slab to 1.0 at the edges, being a function of the y coordinate only. A physical-optics solution for this problem would be somewhat involved. With the integral-equation technique the solution is straightforward and systematic.

IX. Conclusions

A technique is developed for calculating the scattered fields of a dielectric cylinder or cylindrical shell of arbitrary cross section shape. The solution is based on the integral equation for the problem, and it involves the solution of a system of linear equations. The first step in the solution is to determine the electric field distribution within the dielectric body. This field distribution is often of considerable interest in itself, since it promotes an understanding of the phenomena which are important in the scattering process. For example, any surface waves which may be excited in the dielectric body are revealed in the standing waves in the field distribution, and the edge effects are evident in the field perturbations near the edges of a dielectric shell.

The solution is best accomplished with the aid of a digital computer. This integral-equation solution is more systematic, more general and more accurate than the optical methods or the variational solutions. For an inhomogeneous dissipative dielectric shell of any cross section shape, one merely inserts in the equations the coordinates which define the shape of the shell, and the appropriate dielectric constant and loss tangent associated with each point in the dielectric body. The incident wave may be a plane wave or any combination of plane waves, or it may be the field generated by a parallel line source or any array of parallel line sources. One merely inserts the value of the incident field associated with each point in the dielectric body. The effects of surface waves and edge diffraction are included automatically in the solution.

Examples are included to show the scattering patterns of circular and semicircular dielectric cylindrical shells and plane dielectric slabs of finite thickness and width. Results are illustrated for inhomogeneous and dissipative dielectric slabs as well as their simpler counterparts.

The technique can be extended to apply to incident waves having arbitrary polarization, to dielectric bodies having a permeability differing from that of free space, and even to three-dimensional scattering problems. There will, however, be a corresponding increase in the complexity and the computational costs.

This integral-equation technique is applicable primarily to problems in which the cross section area of the dielectric cylinder is not too large. The dielectric cross section is divided into cells which are square or nearly so. For accurate results, the edge dimensions of each cell should not exceed $0.2/\sqrt{\epsilon_r}$ wavelengths. If, for example, the dielectric constant is 4.0, a solution can be obtained at reasonable cost for a dielectric shell of thickness up to 0.1 wavelength and width up to 10 wavelengths. This will involve the solution of 100 linear equations, which is not an unreasonable number. By employing interpolation formulas, it is possible to handle a shell of thickness 0.1 wavelength and width up to 40 wavelengths by solving a set of 100 linear equations. The computation time is determined by the cross section area of the dielectric cylinder. It is believed that a cross section area up to four square wavelengths can be handled at reasonable cost and without difficulty by this technique with the aid of a computer such as the IBM 7094.

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