EE2025

Lecture 5

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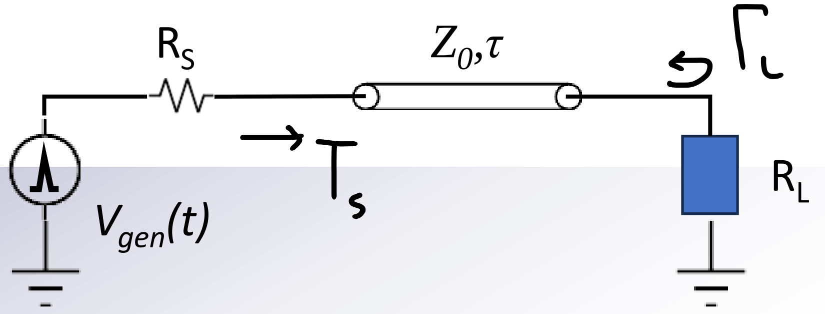
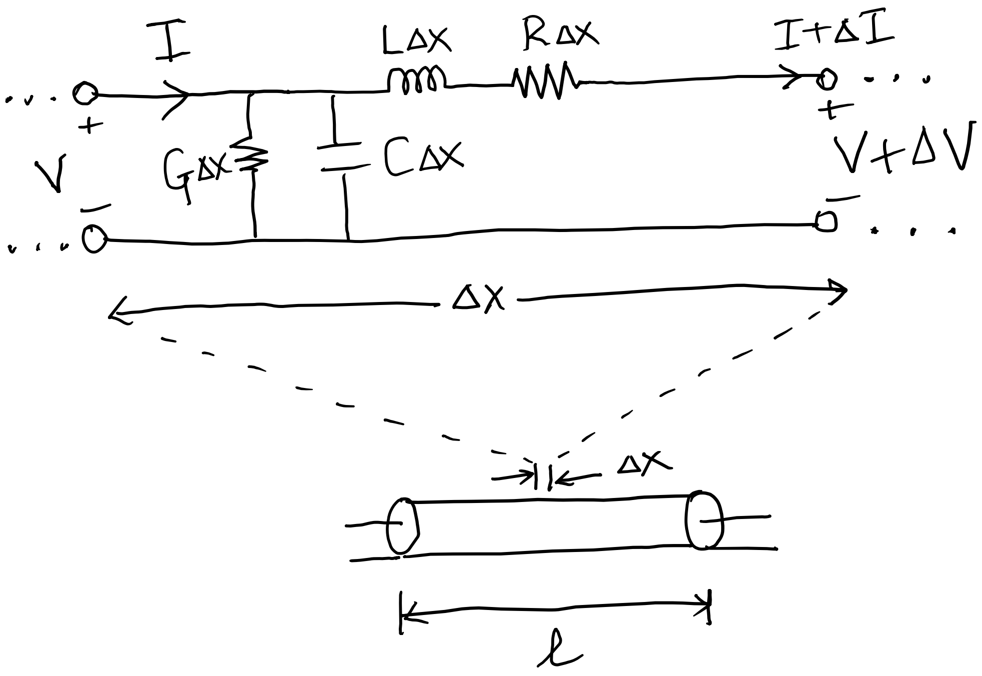
Summary

- Since the electromagnetic field underlying voltage and current signals travel at a finite speed, a delay is induced in carrying information from the source to the load over a finite piece of a transmission line.
- A lossless transmission line can be modeled using a distributed LC ladder circuit.
- The model leads to wave-like solutions (t±x/v) traveling in opposite directions
- The transmission line is characterized by an impedance $Z_0=\sqrt{L/C}$, and the delay $\tau=\ell\sqrt{LC}$
- When there is a discontinuity in the transmission line, reflections occur causing echoes on the line.
- The reflection and transmission coefficient characterizes the voltages and currents at the junction.
- Since reflection and transmission are local events they are not influenced by non-local impedances.

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \qquad T_s = \frac{Z_0}{Z_0 + R_s}$$



Generalized distributed model



$$\left.\begin{aligned}\Delta V &= -(R+j\omega L)\,\Delta x\; I\\ \Delta I &= -(G+j\omega C)\,\Delta x\; V\end{aligned}\right\} \Rightarrow \begin{aligned}\frac{dV}{dx} &= -(R+j\omega L)\,I\\ \frac{dI}{dx} &= -(G+j\omega C)\,V\end{aligned}$$

$$\frac{d^2V}{dx^2} = \gamma^2 V$$

$$\frac{d^2I}{dx^2} = \gamma^2 I$$

where

$$\gamma^2 = (R+j\omega L)(G+j\omega C)$$

$$\gamma = \alpha + j\beta$$

$$V(t) = \mathrm{Re}\,[Ve^{j\omega t}]$$

$$I(t) = \mathrm{Re}\,[Ie^{j\omega t}]$$

$$V = V^+ e^{-\gamma x} + V^- e^{\gamma x}$$

$$I = I^+ e^{-\gamma x} + I^- e^{\gamma x}$$

where is the $\left(t - \frac{x}{v}\right)$?

Phase and attenuation constant

$V = V^+ e^{-\gamma x} + V^- e^{\gamma x}$ where $\gamma^2 = (R+j\omega L)(G+j\omega C) = (\alpha + j\beta)^2$

$I = I^+ e^{-\gamma x} + I^- e^{\gamma x}$

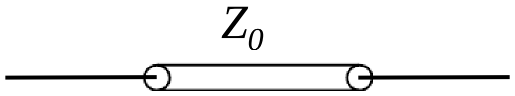
$V(t) = \mathrm{Re}\left[V^+ e^{-\alpha x} e^{j(\omega t - \beta x)} + V^- e^{-\alpha x} e^{j(\omega t + \beta x)}\right]$ $\left(\text{Comparing to } \left(t - \frac{x}{v}\right) \quad e^{j\omega\left(t - \frac{\beta x}{\omega}\right)} \Rightarrow v = \frac{\omega}{\beta} \right)$

Phase $\phi_{V^+} = \angle V^+ + \omega t - \beta x$

$\beta \Delta x = 2\pi \Rightarrow \Delta x = \frac{2\pi}{\beta} = \lambda$ or $\beta = \frac{2\pi}{\lambda}$ $\left(\frac{\text{rads}}{\text{m}}\right)$

Amplitude $= |V^+| e^{-\alpha x}$ $\quad \alpha \left(\frac{\text{neper}}{\text{m}} \text{ or } \frac{\text{dB}}{\text{m}}\right)$

Wave amplitudes and Impedance



$$\frac{dV}{dx} = -(R+j\omega L)I$$

$$\frac{dI}{dx} = -(G+j\omega C)V$$

$$V = V^+ e^{-\gamma x} + V^- e^{\gamma x}$$

$$I = I^+ e^{-\gamma x} + I^- e^{\gamma x}$$

$$\frac{V^+}{I^+} = -\frac{V^-}{I^-} = Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$V = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$$

$$I = \frac{V^+}{Z_0} e^{-\gamma x} - \frac{V^-}{Z_0} e^{\gamma x}$$

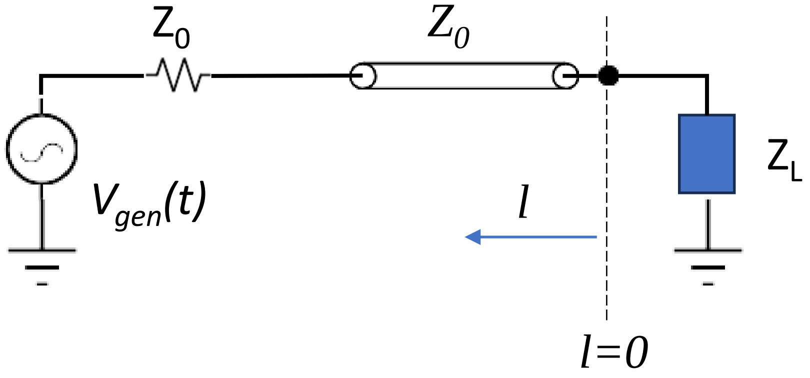
Check $R = G = 0$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Reflection coefficient

$$\Gamma(l) = \frac{V^- e^{\gamma(-l)}}{V^+ e^{-\gamma(-l)}}$$

$$= \Gamma(0)\, e^{-2\gamma l}$$



$$\Gamma(0) = \Gamma_L$$

$$V(l) = V^+ e^{\gamma l}\left[1 + \Gamma(l)\right]$$

$$I(l) = \frac{V^+}{Z_0} e^{\gamma l}\left[1 - \Gamma(l)\right]$$

$$Z(l) = Z_0\left[\frac{1+\Gamma(l)}{1-\Gamma(l)}\right] \quad \text{or} \quad \Gamma(l) = \frac{Z(l) - Z_0}{Z(l) + Z_0}$$

$$\Rightarrow \Gamma_L = \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Impedance transformation

$$Z(l) = Z_0\left[\frac{1+\Gamma(l)}{1-\Gamma(l)}\right]$$

$$= Z_0\left[\frac{1+\Gamma(0)e^{-2\gamma l}}{1-\Gamma(0)e^{-2\gamma l}}\right]$$

$$\Gamma(0) = \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

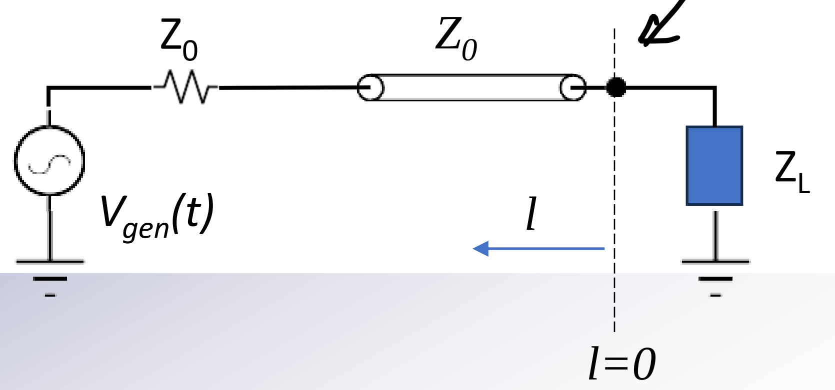
$$Z(l) = Z_0\left[\frac{Z_L\cosh(\gamma l) + Z_0\sinh(\gamma l)}{Z_L\sinh(\gamma l) + Z_0\cosh(\gamma l)}\right]$$

"normalized impedance"

$$\bar{z}(l) = \frac{Z(l)}{Z_0} = \frac{\bar{z}_L\cosh(\gamma l) + \sinh(\gamma l)}{\bar{z}_L\sinh(\gamma l) + \cosh(\gamma l)}$$

where $\bar{z}_L = \dfrac{Z_L}{Z_0}$

nothing special about this pt.



Loss-less and low loss transmission lines

Loss-less : $R = G = 0 \Rightarrow \gamma = j\omega\sqrt{LC} \Rightarrow \alpha = 0,\ \beta = \omega\sqrt{LC}$

$$v = \frac{1}{\sqrt{LC}} = \frac{\omega}{\beta}$$

Low-loss : $R \ll \omega L,\ G \ll \omega C$

$$\gamma = \sqrt{j\omega L\left(1+\frac{R}{j\omega L}\right) j\omega C\left(1+\frac{G}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\left\{1 - \frac{jR}{2\omega L} - \frac{jG}{2\omega C}\right\}$$

$$\beta = \omega\sqrt{LC} \qquad \alpha = \left[\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right]$$

$$\alpha \ll \beta$$

VSWR

Loss-less $\Rightarrow \alpha = 0$ $\quad \gamma = j\beta = j\omega\sqrt{LC}$

$$V(l) = V^+ e^{j\beta l} + V^- e^{-j\beta l}$$

$$I(l) = \frac{V^+}{Z_0} e^{j\beta l} - \frac{V^-}{Z_0} e^{-j\beta l}$$

$$\Gamma(l) = \frac{V^-}{V^+} e^{-2j\beta l} = \Gamma_L e^{-2j\beta l}$$

(No other source for V^- other than refl. @ load.)

Γ_L is complex in general $\quad \Gamma_L = |\Gamma_L| e^{j\phi}$

$$V(l) = V^+ e^{j\beta l}\left[1 + |\Gamma_L| e^{j(\phi - 2\beta l)}\right]$$

$$I(l) = \frac{V^+}{Z_0} e^{j\beta l}\left[1 - |\Gamma_L| e^{j(\phi - 2\beta l)}\right]$$

when $(\phi - 2\beta l) = 2m\pi$, $m \in \mathbb{Z}$: V is maximum, I is minimum

when $\phi - 2\beta l = (2n+1)\pi$, $n \in \mathbb{Z}$: V is min, I is max

VSWR (contd...)

$$V_{max} = |V^+|(1+|\Gamma_L|)$$

$$V_{min} = |V^+|(1-|\Gamma_L|)$$

$$I_{min} = \frac{|V^+|}{Z_0}(1-|\Gamma_L|)$$

$$I_{max} = \frac{|V^+|}{Z_0}(1+|\Gamma_L|)$$

$$\boxed{\text{VSWR}\ (\rho) = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \qquad \text{or} \qquad |\Gamma_L| = \frac{\rho-1}{\rho+1}}$$

Impedance transformation on loss-less line

$$Z(l) = \frac{V(l)}{I(l)} = Z_0\left[\frac{1+\Gamma_L e^{-2j\beta l}}{1-\Gamma_L e^{-2j\beta l}}\right]$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\boxed{Z(l) = Z_0\left[\frac{Z_L\cos\beta l + jZ_0\sin\beta l}{jZ_L\sin\beta l + Z_0\cos\beta l}\right]}$$

$$\bar{Z}(l) = \frac{Z(l)}{Z_0} = \left[\frac{\bar{Z}_L\cos\beta l + j\sin\beta l}{j\bar{Z}_L\sin\beta l + \cos\beta l}\right]$$

normalized

$$Z(l)_{max} = \frac{V_{max}}{I_{min}} = Z_0\left[\frac{1+|\Gamma_L|}{1-|\Gamma_L|}\right] = R_{max} = Z_0\rho$$

$$Z(l)_{min} = \frac{V_{min}}{I_{max}} = \frac{Z_0}{\rho} = R_{min}$$

Periodicity

$$Z(\ell) = Z_0 \left[\frac{Z_L \cos\beta\ell + j Z_0 \sin\beta\ell}{j Z_L \sin\beta\ell + Z_0 \cos\beta\ell} \right]$$

Case I

put $\underline{\ell \to \ell + \frac{\lambda}{2}}$

$\left(\text{remember } \beta = \frac{2\pi}{\lambda} \right)$

$$\cos\beta\left(\ell + \frac{\lambda}{2}\right) = -\cos\beta\ell$$

$$\sin\beta\left(\ell + \frac{\lambda}{2}\right) = -\sin\beta\ell$$

$$\Rightarrow \boxed{Z\left(\ell + \frac{\lambda}{2}\right) = Z(\ell)}$$

Case II

put $\underline{\ell \to \ell + \lambda/4}$

$$\cos\beta\left(\ell + \frac{\lambda}{4}\right) = -\sin\beta\ell$$

$$\sin\beta\left(\ell + \frac{\lambda}{4}\right) = \cos\beta\ell$$

$$\frac{Z(\ell)}{Z_0} = \frac{-Z_L \sin\beta\ell + j Z_0 \cos\beta\ell}{j Z_L \cos\beta\ell - Z_0 \sin\beta\ell} = \frac{j Z_L \sin\beta\ell + Z_0 \cos\beta\ell}{Z_L \cos\beta\ell + j Z_0 \sin\beta\ell} = \frac{Z_0}{Z(\ell)}$$

Impedance inverts itself

Z_0

$$z(\ell) = Z_0 \left[\frac{Z_L \cos\beta\ell + j Z_0 \sin\beta\ell}{j Z_L \sin\beta\ell + Z_0 \cos\beta\ell} \right]$$

$$\left.\begin{aligned} &\text{when} \quad Z_L = Z_0 \\ &z(\ell) = Z_0 \quad \forall \ \ell \\ &\Gamma(\ell) = 0 \quad \forall \ \ell \end{aligned}\right\} \text{"Matched load"}$$