Capacity of Network-Coded Wireless Multicast using Node-Based Scheduling

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***Abstract*— Multicast throughput from one source to multiple sinks in a network can be improved using network coding. In wireless networks, the maximum possible multicast throughput has been recently evaluated using a linear programming optimization model that incorporates constraints due to interference between nodes. The model uses conflict graphs involving all possible hyperarcs in a graph representing the wireless network to determine an interference-free transmission schedule. In this paper, we propose a simple pseudo-random transmission scheduling algorithm that selects non-interfering hyperarcs node-by-node in an iterative fashion. This results in a significant reduction in the number of variables in the optimization model, while still achieving an identical multicast throughput. For instance, in a 10×3 rectangular grid network, our approach requires only 74 hyperarcs out of a total of 336 hyperarcs for achieving the maximum multicast throughput of 2/3.**

I. Introduction

Data communications between nodes of a network is a problem central to all modern communication systems. The flow of information between such networks of nodes has undergone a paradigm change with the recent advent of network coding as introduced in [1]. By allowing simple combination of information at participating nodes in addition to traditional routing, the overall capacity of the network can be increased significantly using ideas of network coding.

The capacity of wireless networks under suitable models, has been determined by Gupta and kumar in [2]. They show that in a network comprising of n identical nodes, with atmost $n/2$ transmit-receive pairs, the throughput capacity per node is $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$ bits-per-second assuming random node placement and $\Theta(\frac{1}{\sqrt{n}})$ bits-per-second assuming optimal node placement and communication pattern. Each node does interference aware routing to forward their data. This analysis does not consider network coding and multicast transmission.

Jain *et al* in [3] model interference using a conflict graph and present methods for computing bounds on the optimal throughput for a network with specified source-destination pairs and traffic patterns using a linear programming model. They achieve throughput gains over shortest path routing since they use an interference aware routing protocol.

With the advent of network coding, multicast capacity of network coding for random networks has been studied in [4] for random wireline networks and random wireless networks. However, [4] does not account for interference between wireless links. Sagduyu *et al* in [5] proposed a link scheduling based MAC with network coding, which improves the performance of lossless wireless networks in terms of throughput and energy efficiency over routing algorithms. It has been shown that the throughput optimization and network code design are separable. In [5], new wireless network codes have also been designed. Network codes for lossy networks is considered in [6] which shows that random network codes work well for lossy networks also.

Smith *et al* in [7] have found the transport capacity of a wireless broadcast network with interference constraints. It has been shown in [7] that the transport capacity can be achieved through routing and network coding provides only a constant gain when employed.

In [8] Park *et al* have formulated another optimization model to estimate wireless multicast throughput for an adhoc network. The model considers losses in the network, interference-aware scheduling and broadcast nature of wireless nodes. Our approach follows the model in [8]. The novelty in our work is in the determination of the scheduling of transmissions for minimizing interference and contention. Our approach for the determination of non-interfering subgraphs is node-based, as opposed to the hyperarc-based approach in [8]. This results in a significant reduction in the complexity of optimization enabling determination of maximum multicast throughput in larger networks. We assume the existence of random network codes for our model, following [8].

By running the optimization on a $n_r \times 3$ rectangular grid of nodes for $3 \leq n_r \leq 10$, we find that we obtain the maximum throughput of $2/3$ with a fraction of the total number of hyperarcs possible in the network. This results in a significant reduction in complexity. In addition, by running optimizations with suitable sets of active hyperarcs, we find that *receiver selection* ability, i.e., the ability of a Node i to determine a subset of receivers from its neighbors for each transmission, is central to the gains obtained by network coding over routing in wireless networks.

Our results suggest that the optimal multicast throughput of a large wireless network might be approachable by considering a fraction of the possible hyperarcs by first selecting nodes that can transmit with limited interference. Such a node-based approach could result in further generalization of the optimization process incorporating advanced physical-layer transmission and reception methods at nodes of a wireless network.

The rest of this paper is organized as follows. Section II describes the model of the wireless network. The proposed node-based scheduling algorithm along with the optimization model for network coding is explained in Section III. The performance of the proposed algorithm on an $n_r \times 3$ grid network is detailed in Section IV. Complexity analysis for random networks is done in Section V. Section VI concludes the paper.

II. Model of wireless network

Consider a wireless packet network consisting of n nodes. We assume that all the nodes are identical i.e. they have uniform transmission power, transmission range, interference range and omnidirectional antennas. We model the wireless network as a graph $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ is the set of nodes and $E \subseteq \{(i, j) : i, j \in V\}$ is the set of edges. An edge (i, j) a Node i to Node j exists if $d_{ij} \le r(n)$, where d_{ij} be the Euclidean distance between Node i and Node j, and $r(n)$ be the transmission range of a node.

Each wireless node has inherent *wireless multicast advantage*. If we assume nodes have omnidirectional transmission, it can reach more than one receiver in its transmission range in a single hop. This is modeled as a hyperarc as in [8]. Notation (i, J_i) indicates that Node i reaches receivers in J_i ($J_i \subset V$) in a single hop.

Scheduling is an interference avoidance scheme which defines a maximal independent set (or a non-interfering subgraph) of non-interfering hyperarcs which can be active simultaneously. We follow the protocol model of [2] to define non-interfering links. Slotted transmission is assumed where one of the non-interfering subgraphs is activated at every slot boundary to transmit packets.

III. Node-based scheduling algorithm and network coding

A. Selection of non-interfering subgraphs

The node-based scheduling(NBS) algorithm outputs a hyperarc set A and the non-interfering subgraphs A_k $(1 \le k \le M)$ for simultaneous transmission in a network represented by a directed graph $G = (V, E)$. We assume there are no incoming links to the source node, $s \in V$. Let $J_i = \{j \in V : (i, j) \in E\}$ denote the neighbors of Node i in G. Initialize the first n non-interfering subgraphs as follows:

$$A_k = \{(k, J_k) : k \in V\}, \qquad 1 \le k \le n.$$

Consider each hyperarc of the source in $\{(s, K) : K \subseteq J_s, K \neq \emptyset\}$ as a virtual node in a new directed graph $G' = (V', E')$. Denoting the virtual node corresponding to a hyperarc (s, K) as v_K, the nodes and edges of G' are defined as

$$V' = \{V \setminus s\} \bigcup (\bigcup_K v_K),$$
$$E' = (E \setminus \{(s, j) \in E\}) \cup E'_2,$$

where

$$E'_2 = \bigcup_{\substack{K \subseteq J_s \\ K \neq \emptyset}} \{(v_K, k) : k \in K\}.$$

In other words, the graph G' is constructed by replacing s in G with the virtual nodes representing the hyperarcs from s as depicted in Fig. 1 Hence, $d_{v_i j}$ (in G') $= d_{sj}$ (in G) and

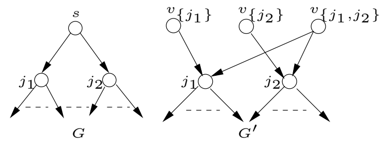

Fig. 1. Construction of G'

$d_{v_i v_j} = 0$.

We now present an algorithm that provides one pseudo-random non-interfering subgraph of the network. Let $J'_i = \{j \in V' : (i, j) \in E'\}$ denote the neighbors of Node i in G'. Let $R_i = \{j \in V' : d_{ij} \le r(n)\}$ denote the set of nodes present in the transmission range of Node i. Note that $J_i \subset R_i$, for some arbitrary networks, the direction of the link between Node i and Node j is specified for maximizing the wireless network throughput.

1) Arrange the nodes in V' in a random order. Assign $A_k = \{(v, J'_v)\}$, where $v \in V'$ is the first node in the random arrangement. Assign $T = v$, $N = J'_v$, and $i = 2$.
   The set T denotes the set of transmitting nodes selected in A_k. The set N denotes the set of active neighbors of T in A_k.
2) Select the i-th node m in the random arrangement for possible inclusion in A_k. The set A_k is altered under the following conditions:
   a) If $m \in N$, then the selected node is not included in A_k. Go to Step 3.
      If $m \notin N$, proceed to (b) to determine a suitable non-interfering hyperarc emanating from m for possible inclusion in A_k.
   b) Determine the set of neighbors of m that are within the transmission range of nodes in T i.e. compute
   $$D_1 = \{j \in J'_m : d_{tj} \le r(n), \text{ for some } t \in T\}.$$
   c) Determine the set of nodes in N that are within the transmission range of Node m. i.e Determine the neighbors of m that are present in N. Compute $D_2 = R_m \cap N$
   d) If $|D_1| \le 1$ and $|D_2| \le 1$, then
      - Let t be the transmitter connected with D_2.
      - Update A_k and N as,
      $$\begin{aligned} A_k &= A_k \setminus (t, J_t), \\ A_k &= A_k \bigcup \{(t, J_t \setminus D_2), (m, J'_m \setminus D_1)\}, \\ N &= \{N \setminus D_2\} \cup \{J'_m \setminus D_1\}. \end{aligned}$$

      In determining a suitable non-interfering hyperarc above, only a subset of nodes are selected as receivers for each selected transmitting node. Nodes in D_1 are not selected as receivers for node m, and nodes in D_2 are not selected in N. For a Node m, if $J'_m = R_m$, then $D_1 = D_2$

e) Update $T = T \cup \{m\}$.

3) If $i < |V'|$, increment i by 1 and go to Step 2.

The above algorithm is repeated several times to obtain distinct non-interfering subgraphs A_1, A_2, $\cdots$, A_M that are maximal in the sense that $A_i \nsubseteq A_j$. All distinct hyperarcs present in $\bigcup_{k=1}^{M} A_k$ are collected to form the hyperarc set A.

B. Linear Programming Model

We now follow the linear programming formulation presented in [8] for determining the maximum multicast throughput from a source to multiple sinks. As before, we assume that the wireless network is represented as a directed graph $G = (V, E)$. We construct non-interfering subgraphs A_k $(1 \leq k \leq M)$ with hyperarcs from the set A as discussed in Section III-A. An important difference in our optimization when compared to [8] is that we consider only hyperarcs in the set A.

Let z_{iJ} be the average broadcast rate at which packets are injected into the hyperarc $(i, J) \in A$. Let L packets per unit time be the link capacity. The unreliability of wireless links results in a packet loss in the link with probability p. If a packet is injected on to the hyperarc (i, J) at an average rate z_{iJ}, it is shown in [6] that the packets reach a subset $K \subseteq J$ at an average rate z_{iJK} given by

$$z_{iJK} = z_{iJ}(1-p)^{|K|} p^{(|J|-|K|)},$$

assuming losses in each link of a hyperarc are independent.

As in [8], we assume that the subgraph A_k is used in a particular slot of a slotted transmission schedule. Let λ_k be the fraction of time for which the subgraph A_k is active. Scheduling constraints that limit the average rate of packet injection at hyperarcs are given by the following set of equations:

$$\sum_k \lambda_k \leq 1.$$

$$\sum_k \lambda_k c_k(i, J) - z_{iJ} \geq 0 \quad \forall (i, J) \in A,$$

where

$$c_k(i, J) = \begin{cases} L \text{ if } (i, J) \in A_k, \\ 0 \text{ otherwise.} \end{cases}$$

We denote the average multicast throughput as f, the set of destinations as T $(T \subset V)$, and the information flow rate in link (i, j) of hyperarc (i, J) towards sink $t \in T$ as $x_{iJj}^{(t)}$. We maximize f over a rate vector $\underline{z} \in [0, L]^{|A|}$ with linear constraints. The optimization problem is stated precisely as follows:

$$\text{maximize} \quad f$$
$$\text{subject to}$$
$$\sum_k \lambda_k c_k(i, J) - z_{iJ} \geq 0, \quad \forall (i, J) \in A.$$
$$\sum_k \lambda_k \leq 1.$$
$$\sum_{\{L \subseteq J : L \cap K \neq \emptyset\}} z_{iJL} - \sum_{j \in K} x_{iJj}^{(t)} \geq 0, \qquad \forall (i, J) \in A, K \subseteq J, t \in T.$$
$$\sum_{\{J:(i,J)\in A\}} \sum_{j \in J} x_{iJj}^{(t)} - \sum_{\{j:(j,I)\in A, i \in I\}} x_{jIi}^{(t)} = \begin{cases} f \text{ if } i = s \\ -f \text{ if } i = t \\ 0 \text{ otherwise} \end{cases}, \qquad \forall i \in V, t \in T.$$
$$z_{iJ} \geq 0, \quad \forall (i, J) \in A.$$
$$x_{iJj}^{(t)} \geq 0, \quad \forall (i, J) \in A, j \in J, t \in T.$$
$$\lambda_k \geq 0 \quad \forall k.$$

It is shown in [8] and [6] that a feasible solution of the above optimization problem always results in a valid network code. Once a feasible rate vector $\underline{z}^*$ is determined, a network code can be designed, theoretically, to achieve the maximum multicast throughput f.

IV. Results and Discussion

We present results on the maximization of multicast throughput on a $n_r \times 3$ rectangular grid network as depicted in Fig. 2 for $n_r = 4$. The nodes are identical with a transmission range, $r(n) = 1$. The operational area of a $n_r \times 3$ grid is $2(n_r - 1)$ units. The number of rows n_r is varied from 3 to 10. The center node in the first row is considered as the source node. The three nodes in the n_r-th row are taken to be the sinks. The connections in the graph have been depicted for $n_r = 4$ in Fig. 2. For other values of n_r, similar connections have been assumed for nodes in each row.

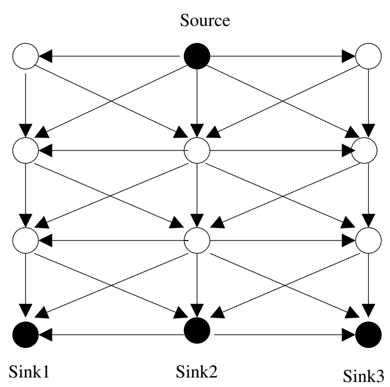

Fig. 2. An example - 4 x 3 grid network

The maximum multicast throughput for pure routing schemes is 1/3 for all values of n_r. With network coding, a multicast throughput of 2/3 can be readily achieved for all values of n_r by using flows down the first and the third column of the grid.

Now, we present results regarding the performance of our proposed node-based interference-free subgraph selection algorithm assuming that the link error probability p equals 0. The subgraph selection algorithm is run for several times and value of M is tabulated. Table I presents a comparison between the proposed method and the method in [8] in terms of the number of hyperarcs generated by the scheduling algorithms, the number of variables in the optimization process, and the maximum multicast throughput. From the table, we observe that there is a significant decrease in the number of hyperarcs selected by our algorithm compared to [8], which results in a significant decrease in the number of variables in the optimization process. However, the maximum multicast throughput remains the same.

From Table I, we see that the performance of the algorithm depends on the number of interference-free subgraphs M. To study the effect of M on the maximum multicast throughput, we considered a 7×3 grid and varied M from 10 to 300 in steps of 25. Fig. 3 shows the resulting maximum multicast throughput of our algorithm as a function of M. We notice that

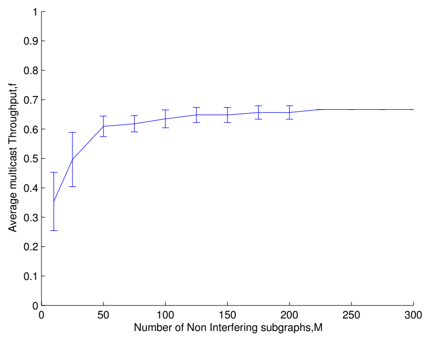

Fig. 3. Maximum f versus M

the maximum throughput increases monotonically and reaches a maximum of 2/3 at large values of M. As the value of M increases, our algorithm constructs all hyperarcs of interest from the source to destination, which results in maximum achievable throughput. The error bars depict the variation due to a choice of different random seeds in the subgraph selection algorithm.

We now present results for the case when the link error probability is non-zero. We consider a 4×3 grid network depicted in Fig. 2. A plot of the maximum throughput versus p is presented in Fig. 4. When the losses in links are independent, it is proved in [6] that the average rate of each link is proportional to the link success probability. Hence, as shown in Fig. 4, the multicast throughput scales down with the link success probability. Note that our results agree with those presented in [8] for the 4×3 grid both with and without link error.

In all the above results, the receiver selection ability of each

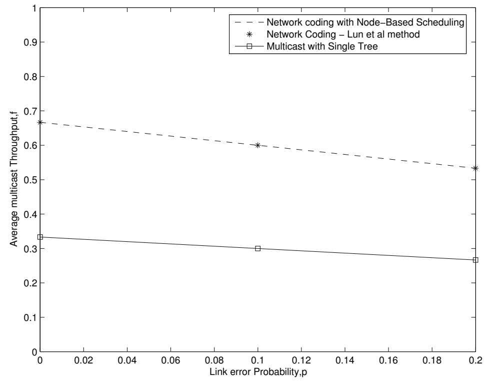

Fig. 4. Maximum f versus p

node is crucial in achieving the gains from network coding, i.e., the multicast throughput of $2/3$. It was observed that if every node is assumed to have a fixed broadcast set, the throughput was $1/3$ irrespective of the row size, n_r. Even if the source alone is allowed to unicast and other nodes broadcast to the fixed set of receivers as above, the throughput remains at $1/3$, for all n_r. As reported in [8], this is equal to the throughput in routing schemes.

V. Complexity Analysis for Random Networks

In this section, we analyse the complexity reduction obtained using the NBS algorithm for a n node random network. A random network is modeled as a random geometric graph, $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the set of vertices and $E = \{(i, j) : d_{ij} \leq r(n)\}$ is the set of edges. For n nodes located according to a uniform distribution in a disk of area P, it is shown in [10], that the network is asymptotically connected as $n \to \infty$, if the minimum transmission range of a node is $r(n) = \sqrt{\frac{P(\log n + \gamma_n)}{\pi n}}$, ($\gamma_n \to \infty$ as $n \to \infty$). We compare the expected number of hyperarcs for the n node random network and the number of hyperarcs generated in the NBS algorithm.

A. Expected number of hyperarcs of a n node random network

It is known that for a random geometric graph, if the nodes are uniformly placed, the node-degree distribution is binomial with probability q, where, q is the probability that Node i and Node j are connected. The range of values of q is given by , $\frac{\pi r^2(n)}{2P} \leq q \leq \frac{\pi r^2(n)}{P}$. The total number of hyperarcs present in a given n node random network,

$$N = \sum_{i=1}^{n} (2^{d_i} - 1),$$

where d_i is the degree of Node i. Therefore, the expected number of hyperarcs for a n node random network is

$$E[N] = \sum_{i=1}^{n} E(2^{d_i} - 1).$$

n_r	M	$\|A\|$	$\|A\|$ in [8]	No. of Variables	No. of Variables in [8]	Maximum f
3	24	26	77	245	628	0.6667
4	61	44	114	420	929	0.6667
5	106	49	151	503	1230	0.6667
6	179	54	188	614	1531	0.6667
7	219	59	225	692	1832	0.6667
8	306	64	262	817	2133	0.6667
9	377	69	299	926	2434	0.6667
10	410	74	336	997	2735	0.6667

TABLE I
COMPARISON OF PARAMETERS IN THE OPTIMIZATION PROBLEM.

Using the fact that all the node degree distributions are identical, we can rewrite $E[N]$ as

$$\begin{aligned} E[N] &= nE[2^{d_i}-1], \\ &= n\sum_{k=0}^{n-1}\binom{n-1}{k}q^k(1-q)^{n-1-k}(2^k-1), \\ &= n\left((1+q)^{n-1}-1\right). \end{aligned}$$

B. Expected number of hyperarcs generated in NBS algorithm

Let N' be the number of hyperarcs generated in the NBS algorithm. An upper bound on the value of $N^{'}$ is found using the knowledge on the size of the non-interfering subgraph, A_k, which inturn depends on the spatial reuse. As mentioned earlier, we choose M non-interfering subgraphs to find the multicast throughput. We find the maximum number of non-interfering nodes that can be active simultaneously. This is the number of circles that can be packed in a disk of area P. Each node has atleast a coverage area of $\frac{\pi r^2(n)}{2}$. Therefore, the maximum number of nodes with non-overlapping transmission area is $\frac{2P}{\pi r^2(n)}$ and the maximum number of hyperarcs present in a non interfering subgraph is upper bounded by $\frac{2Pn}{\log n}$. If we assume hyperarcs present in all non interfering subgraphs are distinct, the upper bound on $N^{'}$ is

$$N^{'} \leq \frac{2PMn}{\log n}.$$

Note that N' is linear with n and M. We can choose the value of M to reduce the complexity. As in Fig.3, higher values of M acheive optimal multicast throughput. Therefore, a trade-off exists between throughput and complexity reduction.

VI. CONCLUSION

In this paper, we have considered the problem of reducing the complexity of optimization in determining the maximum multicast throughput in a wireless network. We proposed a node-based pseudo-random transmission scheduling algorithm. This algorithm results in a significant reduction in variables, which would be useful in practice. We have analyzed the performance of our algorithm for various sizes of a rectangular grid network, link error probability, and number of iterations. We have shown that our simpler algorithm results in the same multicast throughput as more complicated versions in some example networks. We have analysed the complexity reduction in the optimization model using our algorithm.

The simpler optimization method appears to suggest that a crucial factor in the gains in throughput because of network coding is the capability of *receiver selection*. Simulation of receiver selection in a practical network is one avenue of future research. Other avenues of future work are the following: (1) the extension of our optimization method to random networks with large number of nodes, (2) deriving a tighter bound for N' and (3) the extension of the optimization to more advanced physical layer implementations in the transmitting and receiving nodes.

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