EE6340 - Information Theory

Problem Set 3 Solution

February 23, 2013

1. a) No. of sequences containing 3 or fewer ones $= \binom{100}{3} + \binom{100}{2} + \binom{100}{1} + \binom{100}{0} = 166751$
   Given that all the codewords need to be of the same length, Minimum length required is

$$L_{min} = \lceil \log 166751 \rceil = \lceil 17.347 \rceil = 18 \tag{1}$$

   b) Probability of a sequence not being assigned any codeword

   =Probability of observing a sequence having greater than 3 ones

   $=1 - (0.995)^{100} -^{100} C_1 (0.995)^{99}(0.005) -^{100} C_2 (0.995)^{98}(0.005)^2 -^{100} C_3 (0.995)^{97}(0.005)^3$

   $=0.00167$

2. a) $H(X) = \frac{7}{8}\log\frac{8}{7} + \frac{1}{8}\log 8$
   $x^8 \in A_\epsilon{}^{(8)} \implies 2^{-8(H(X)+\epsilon)} \le p(x_1, x_2, ...x_8) \le 2^{-8(H(X)-\epsilon)}$
   Lowest value of $\epsilon = 0$(known)
   Since $n = 8$ and $\mathbb{P}(1) = \frac{7}{8}$, $\mathbb{P}(0) = \frac{1}{8}$, consider the case of 1 zero and 7 ones(8 such sequences).
   Pr(each sequence) $= \frac{1}{8}(\frac{7}{8})^7$
   Now $H(X) = \frac{1}{8}\log_2[(\frac{8}{7})^8 8] \implies 2^{8H(X)} = (\frac{8}{7})^8 \times 8$
   $n = 8 \implies 2^{-nH(X)} = (\frac{7}{8})^7(\frac{1}{8})$
   Thus $2^{-8\epsilon} \le \frac{p(x_1,x_2,x_3,.....x_8)}{(\frac{7}{8})^7(\frac{1}{8})} \le 2^{8\epsilon}$
   Each sequence with 7 ones has probability $(\frac{7}{8})^7(\frac{1}{8})$. Thus, the typical set with $\epsilon = 0$ has 8 sequences.
   When $\epsilon = \frac{\log_2 7}{8}$, $2^{-8\epsilon} = \frac{1}{7}$
   $(\frac{1}{7})(\frac{1}{8})(\frac{7}{8})^7 \le p(x_1, x_2, ...x_8) \le (\frac{7}{8})^8$
   $\implies$ for this ϵ, $A_\epsilon^{(8)}$ contains sequences of all 1's, 7 1's and 6 1's, $= 1 + 8 + 28 =$37 sequences.
   Thus for ϵ values such that $0 \le \epsilon \le \frac{\log_2 7}{8}$, the typical set contains exactly 8 sequences.

   b) Elements of $A_\epsilon^{(8)} = \{01111111, 10111111, ..., 11111110\}$
   $\mathbb{P}(A_\epsilon^{(8)}) = 8(\frac{1}{8})(\frac{7}{8})^7 = (\frac{7}{8})^7$

   c) Let $N(\epsilon) =$ No. of elements in $A_{\epsilon'}^{(n)}$ for $\epsilon' = \epsilon$. Change in no.of sequences occurs at $\epsilon = k\frac{\log_2 7}{8}, k = 1, 2, ...7$
   $N(\epsilon) = \{8, 37, 93, ..., 256(\text{all sequences})\}$

3. a) This follows from the property of the typical set that $\mathbb{P}(x^n \in A_\epsilon^{(n)}) \ge 1 - \epsilon_1$. Hence, $\mathbb{P}(x^n \in A_\epsilon^{(n)}) \to 1$.

   b) From Law of Large Numbers, we can write $\mathbb{P}(x^n \in B^n) \ge 1 - \epsilon_2$, i.e, there exists an n_0 such that for every $n \ge n_0$, $\mathbb{P}(x^n \in B^n) \ge 1 - \epsilon_2$.
   Let n_0 be such that both $\mathbb{P}(x^n \in A^n) \ge 1 - \epsilon_1$ and $\mathbb{P}(x^n \in B^n) \ge 1 - \epsilon_2$ are true $\forall n \ge n_0$

$$\begin{aligned}\mathbb{P}(A^n \cap B^n) =& \mathbb{P}(A^n) + \mathbb{P}(B^n) - \mathbb{P}(A^n \cup B^n) \\ \ge& 1 - \epsilon_1 + 1 - \epsilon_2 - 1(\text{ Since } \mathbb{P}(A^n \cup B^n) \le 1) \\ =& 1 - \epsilon_1 - \epsilon_2 \\ \to& 1 \text{ as } n \to \infty\end{aligned}$$

c) $|A^n \cap B^n| \le |A^n| \le 2^{n(H+\epsilon)}$ (Intersection property)

d) Choose ϵ_1,ϵ_2 such that $\epsilon_1 + \epsilon_2 < \frac{1}{2}$ for some large n
$\implies \frac{1}{2} \le \mathbb{P}(A^n \cap B^n)$ (from part(b))

$$\begin{aligned}\frac{1}{2} &\le \mathbb{P}(A^n \cap B^n) \\ &= \sum_{x^n \in A^n \cup B^n} \mathbb{P}(x^n) \\ &\le \sum_{x^n \in A^n \cup B^n} \mathbb{P}(x^n \in A^n)(\text{Since } A^n \cup B^n \subset A^n) \\ &\le 2^{-n(H-\epsilon)}|A^n \cap B^n|\end{aligned}$$

Thus, $|A^n \cap B^n| \ge \frac{1}{2}2^{n(H-\epsilon)}$

4.

$$\begin{aligned}&\lim_{n\to\infty}(p(X_1, X_2, ...X_n)^{\frac{1}{n}}) \\ &= \lim_{n\to\infty} 2^{\log_2 (p(X_1,X_2,....X_n))^{\frac{1}{n}}} \\ &= \lim_{n\to\infty} 2^{\frac{1}{n}[\sum_{i=1}^{n} \log_2 (p(X_i)]} \text{ (X_is are independent)} \\ &= 2^{\frac{n}{n}\mathbb{E}(\log_2 p(X))} \text{ (LLN)} \\ &= 2^{-H(X)} \text{ (Assuming H(X) exists)}\end{aligned}$$

5. a) Since $X_1, X_2, ...X_n$ are i.i.d $\sim p(x)$,

$$\begin{aligned}&\lim_{n\to\infty} -\frac{1}{n}\log q(X_1, X_2, ...X_n) \\ &= -\frac{1}{n}\sum_{i=1}^{n}\log q(X_i) \text{ (X_is are i.i.d)} \\ &= -\mathbb{E}(\log q(X)) \text{ (From Law of large numbers)} \\ &= -\sum p(x)\log q(x) \text{ (Since each X_i is drawn from p(X))} \\ &= D(p||q) + H(p)\end{aligned}$$

b) Limit of the log likelihood ratio

$$\begin{aligned}&= \lim_{n\to\infty} -\frac{1}{n}\log\frac{q(X_1, X_2, ...X_n)}{p(X_1, X_2, ...X_n)} \\ &= -\frac{1}{n}\sum_{i=1}^{n}\log\frac{q(X_i)}{p(X_i)} \text{ (X_is are i.i.d)} \\ &= -\mathbb{E}(\log\frac{q(X)}{p(X)}) \\ &= -\sum p(x)\log\frac{q(x)}{p(x)} \\ &= D(p||q)\end{aligned}$$