EE613: Estimation Theory
Problem Set 7

1. For the signal model
$$s[n] = \begin{cases} A & 0 \le n \le M-1 \\ -A & M \le n \le N-1 \end{cases},$$
find the LSE of A and the minimum LS error. Assume that $x[n] = s[n] + w[n]$ for $n = 0, 1, \cdots, N-1$ are observed. If now $w[n]$ is WGN with variance σ^2, find the PDF of the LSE.

2. Show that
$$(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^T(\mathbf{x} - \mathbf{H}\boldsymbol{\theta}) = (\mathbf{x} - \mathbf{H}\hat{\boldsymbol{\theta}})^T(\mathbf{x} - \mathbf{H}\hat{\boldsymbol{\theta}}) + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\mathbf{H}^T\mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$
where
$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}.$$
Use this to argue that $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ is the LSE.

3. In this problem we prove that a projection matrix $\mathbf{P}$ must be symmetric. Let $\mathbf{x} = \boldsymbol{\xi} + \boldsymbol{\xi}^{\perp}$, where $\boldsymbol{\xi}$ lies in a subspace which is the range of the projection matrix or $\mathbf{P}\mathbf{x} = \boldsymbol{\xi}$, and $\boldsymbol{\xi}^{\perp}$ lies in the orthogonal subspace or $\mathbf{P}\boldsymbol{\xi}^{\perp} = \mathbf{0}$. For arbitrary vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in R^N show that
$$\mathbf{x}_1^T\mathbf{P}\mathbf{x}_2 - \mathbf{x}_2^T\mathbf{P}\mathbf{x}_1 = 0$$
by decomposing $\mathbf{x}_1$ and $\mathbf{x}_2$ as discussed above. Finally, prove the desired result.

4. Prove the following properties of the projection matrix
$$\mathbf{P} = \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T.$$
   (a) $\mathbf{P}$ is idempotent.
   (b) $\mathbf{P}$ is positive semidefinite.
   (c) The eigenvalues of $\mathbf{P}$ are either 1 or 0.

5. If the signal model is
$$s[n] = A + B(-1)^n \qquad n = 0, 1, \ldots, N-1$$
and N is even, find the LSE of $\boldsymbol{\theta} = [A \;\; B]^T$. Now assume that $A = B$ and repeat the problem using the constrained LS approach. Compare your results.