EE613 Problem Set 6

① $\underline{x} \sim N(\underline{1}A, AI)$.

Sample mean estimator $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$.

$$E[\hat{A}] = A$$
$$\mathrm{var}(\hat{A}) = \frac{A}{N}.$$

The CRLB for estimation of $A = \frac{A}{N}\left(\frac{A}{A+\frac{1}{2}}\right)$. (derived in class)

$$\mathrm{var}(\hat{A}) > CRLB(A).$$

$$\frac{\mathrm{var}(\hat{A}) - CRLB(A)}{CRLB(A)} = \frac{\frac{A}{N}\left(\frac{\frac{1}{2}}{A+\frac{1}{2}}\right)}{\frac{A}{N}\left(\frac{A}{A+\frac{1}{2}}\right)} = \frac{1}{2A}$$ → this does not decrease with N.

$$\mathrm{var}(\hat{A}) - CRLB(A) = \frac{A}{N}\left(\frac{\frac{1}{2}}{A+\frac{1}{2}}\right).$$

$$\hat{A}_{ML} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}$$ (derived in class).

As $N \to \infty$, $\hat{A}_{ML}$ is unbiased and efficient.

$\hat{A}$ is not asymptotically efficient. ⇒ For large N, $\hat{A}_{ML}$ is better.

② $$p(\underline{x};\lambda) = \begin{cases} \prod_{n=0}^{N-1} \lambda \exp(-\lambda x[n]) & x[n] > 0 \text{ for all } n. \\ 0 & \text{else}. \end{cases}$$

$$\ln p(\mathbf{x};\lambda) = \sum_{n=0}^{N-1} \ln\left(\lambda \exp(-\lambda x[n])\right)$$

$$= \sum_{n=0}^{N-1} \ln\lambda - \lambda x[n] .$$

$$\frac{\partial \ln p(\mathbf{x};\lambda)}{\partial\lambda} = \frac{N}{\lambda} - \sum_{n=0}^{N-1} x[n] .$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{1}{\frac{1}{N}\sum_{n=0}^{N-1} x[n]}} .$$

$$\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial\lambda^2} = \frac{-N}{\lambda^2} < 0 .$$

③ $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \sim N\left(A, \frac{\sigma^2}{N}\right)$.

$$P\left[|\hat{A} - A| > \epsilon\right] \leq \frac{\sigma^2/N}{\epsilon^2}.$$

As $N \to \infty$, $P[|\hat{A} - A| > \epsilon] \to 0$.

$\Rightarrow \hat{A}$ is consistent.

④
$$Pr(\underline{x}; p) = p^{(\text{number of 1's in } \underline{x})} (1-p)^{(\text{number of 0's in } \underline{x})}$$

$$= p^{n_1}(1-p)^{N-n_1} = \left(\frac{p}{1-p}\right)^{n_1}(1-p)^N$$

$$\ln Pr(\underline{x}; p) = n_1 \ln p + (N - n_1)\ln(1-p).$$

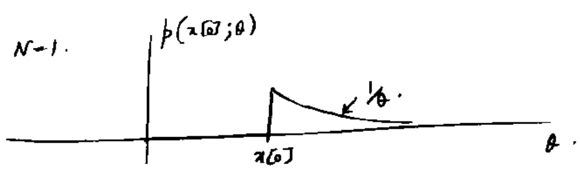
$$\frac{\partial \ln Pr(\underline{x}; p)}{\partial p} = \frac{n_1}{p} + \frac{N-n_1}{(1-p)}(-1)$$

$$\frac{n_1}{\hat{p}} = \frac{N-n_1}{1-\hat{p}} \Rightarrow \frac{1-\hat{p}}{\hat{p}} = \frac{N-n_1}{n_1} \Rightarrow \frac{1}{\hat{p}} = \frac{N}{n_1}$$

$$\Rightarrow \underline{\hat{p} = \frac{n_1}{N}}.$$

⑤
$$p(\underline{x}; \theta) = \begin{cases} \frac{1}{\theta^N} & \max x[n] < \theta \text{ or } \min x[n] < 0. \\ 0 & \max x[n] > \theta \text{ or } \min x[n] > 0. \end{cases}$$

For $N=1$.



$\hat{\theta}_{ML} = x[0].$

Similarly, for any N: $\hat{\theta}_{ML} = \max_n x[n]$.

(6)

$$p(\mathbf{x};\theta) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\frac{1}{\theta}}} \exp\left\{-\frac{x^2[n]}{2(\frac{1}{\theta})}\right\}.$$

$$\ln p(\mathbf{x};\theta) = \sum_{n=0}^{N-1}\left\{\ln\left(\frac{1}{\sqrt{2\pi/\theta}}\right) - \frac{x^2[n]}{2/\theta}\right\}$$

$$= +\frac{N}{2}\ln\frac{\theta}{2\pi} - \frac{\theta}{2}\sum_{n=0}^{N-1}x^2[n]$$

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta} = \frac{N}{2}\cdot\frac{2\pi}{\theta}\cdot\frac{1}{2\pi} - \frac{1}{2}\sum_{n=0}^{N-1}x^2[n].$$

$$= \frac{N}{2\theta} - \frac{1}{2}\sum_{n=0}^{N-1}x^2[n].$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{N}{\sum_{n=0}^{N-1}x^2[n]} = \frac{1}{\frac{1}{N}\sum_{n=0}^{N-1}x^2[n]}.$$

Define $T = \frac{1}{N}\sum_{n=0}^{N-1}x^2[n]$. $\quad E[T] = \frac{1}{\theta} \quad \mathrm{var}(x^2[n]) = \frac{3}{\theta^2} - \frac{1}{\theta^2} = \frac{2}{\theta^2}$

$$\hat{\theta}_{ML} = \frac{1}{T}$$

For large N:

$$\frac{1}{T} \approx \frac{1}{(1/\theta)} + \left(\frac{-1}{T^2}\right)_{T=\frac{1}{\theta}}\left(T - \frac{1}{\theta}\right)$$

$$= \theta - \theta^2\left(T - \frac{1}{\theta}\right)$$

$$E[\hat{\theta}_{ML}] \approx \theta,\quad \mathrm{var}(\hat{\theta}_{ML}) \approx \theta^4\,\mathrm{var}(T) = \theta^4\frac{2}{N\theta^2} = \frac{2\theta^2}{N}.$$

$$\hat{\theta}_{ML} \overset{a}{\approx} N\left(\theta, \frac{2\theta^2}{N}\right).$$

(7) $$p(\underline{x};\underline{\theta}) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x[n]-A)^2}{2\sigma^2}\right\}.$$

$$\underline{\theta} = \begin{bmatrix} A \\ \sigma^2 \end{bmatrix}.$$

$$\hat{\underline{\theta}}_{ML} = \begin{bmatrix} \frac{1}{N}\sum_{n=0}^{N-1} x[n] \\ \frac{1}{N}\sum_{n=0}^{N-1} (x[n]-\hat{A})^2 \end{bmatrix}$$

$$SNR = \alpha = \frac{A^2}{\sigma^2}.$$

$$\hat{\alpha}_{ML} = \frac{\hat{A}^2}{\widehat{\sigma^2}} = \frac{\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right)^2}{\frac{1}{N}\sum_{n=0}^{N-1}(x[n]-\hat{A})^2}.$$