EE613 Problem Set 5

①

$$p(\underline{x};\sigma^2) = \begin{cases} \prod_{n=0}^{N-1} \frac{x[n]}{\sigma^2} \exp\left(-\frac{1}{2}\frac{x^2[n]}{\sigma^2}\right) & x[n] > 0 \text{ for } n=0,\dots,N-1 \\ 0 & \text{else} \end{cases}$$

$$= \frac{\prod_{n=0}^{N-1} x[n]}{(\sigma^2)^N} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right) u\left(\min_n x[n]\right).$$

$$= \underbrace{\frac{1}{(\sigma^2)^N} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right)} \underbrace{u\left(\min x[n]\right) \prod_{n=0}^{N-1} x[n]}.$$

$\sum_{n=0}^{N-1} x^2[n]$ is a sufficient statistic.

②

$$p(\underline{x};\theta) = \frac{1}{(2\pi)^{N/2}\theta^{N/2}} \exp\left\{-\frac{1}{2\theta}\sum_{n=0}^{N-1}(x[n]-\theta)^2\right\}.$$

$$= \frac{1}{(2\pi)^{N/2}\theta^{N/2}} \exp\left\{-\frac{1}{2\theta}\left(\sum_{n=0}^{N-1} x^2[n] + N\theta^2 - 2\theta\sum_{n=0}^{N-1} x[n]\right)\right\}$$

$$= \frac{1}{(2\pi)^{N/2}\theta^{N/2}} \exp\left\{-\frac{1}{2\theta}\sum_{n=0}^{N-1} x^2[n] - \frac{N\theta}{2} + \sum_{n=0}^{N-1} x[n]\right\}.$$

$\sum_{n=0}^{N-1} x^2[n]$ is a sufficient statistic.

③

$$p(\underline{x};\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\underbrace{\sum_{n=0}^{N-1}(x[n]-A)^2}_{\text{sufficient statistic.}}\right\}.$$

$$T(\underline{x}) = \sum_{n=0}^{N-1}(x[n]-A)^2. \qquad E\left[T(\underline{x})\right] = \sum_{n=0}^{N-1}\sigma^2 = N\sigma^2.$$

$$\widehat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-A)^2 \quad \rightarrow \text{MVUE}.$$

④ $$P(\underline{x}) = \prod_{n=0}^{N-1} p(x[n]) = \theta^{k}(1-\theta)^{N-k} \qquad k = \sum_{n=0}^{N-1} x[n].$$

$$= \left(\frac{\theta}{1-\theta}\right)^{\sum_{n=0}^{N-1} x[n]}(1-\theta)^N$$

$\sum_{n=0}^{N-1} x[n]$ is a sufficient statistic. $E\left[\sum_{n=0}^{N-1} x[n]\right] = N\theta$

$$\hat{\theta} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]. \quad \rightarrow \text{MVUE}.$$

⑤ $$p(x[n];\theta) = \begin{cases} \exp\{-(x[n]-\theta)\} & x[n] > \theta \\ 0 & x[n] < \theta. \end{cases}$$

$$p(\underline{x};\theta) = \prod_{n=0}^{N-1} \exp\{-(x[n]-\theta)\}\, u(x[n]-\theta)$$

$$= \underbrace{\exp\left\{-\sum_{n=0}^{N-1} x[n]\right\}}\ \underbrace{\exp\{N\theta\}\, u(\min(x[n])-\theta)}_{g(T(\underline{x}),\,\theta)}$$

$T(\underline{x}) = \min x[n]$ is a sufficient statistic.

Let us find $E[\min x[n]]$.

$$Pr[\min x[n] \le \alpha] = 1 - Pr[\min x[n] > \alpha]$$
$$= 1 - (1 - F_{x[n]}(\alpha))^N$$

$U = \min x[n]$.

$$F_U(\alpha) = 1 - (1 - F_x(\alpha))^N$$

$$p_U(\alpha) = N(1 - F_x(\alpha))^{N-1} p_x(\alpha)$$

$$F_x(\alpha) = \int_\theta^\alpha e^{-(x-\theta)}\,dx \quad \text{for } \alpha > \theta$$

$$= e^\theta \int_\theta^\alpha e^{-x}\,dx = e^\theta \left.\frac{e^{-x}}{-1}\right|_\theta^\alpha = e^\theta\left(-e^{-\alpha} + e^{-\theta}\right)$$

$$= 1 - e^{-(\alpha-\theta)}.$$

$$1 - F_x(\alpha) = e^{-(\alpha-\theta)} \quad \text{for } \alpha > \theta.$$

$$p_U(\alpha) = N\, e^{-(\alpha-\theta)(N-1)}\, e^{-(\alpha-\theta)} \quad \text{for } \alpha > \theta.$$

$$= N\, e^{-N(\alpha-\theta)} \quad \text{for } \alpha > \theta.$$

$$E[U] = \int_\theta^\infty \alpha N e^{-N(\alpha-\theta)}\,d\alpha = N \int_\theta^\infty \alpha\, d\left(\frac{e^{-N(\alpha-\theta)}}{(-N)}\right).$$

$$= -\int_\theta^\infty \alpha\, d\left(e^{-N(\alpha-\theta)}\right)$$

$$= -\alpha e^{-N(\alpha-\theta)}\Big|_\theta^\infty + \int_\theta^\infty e^{-N(\alpha-\theta)}\,d\alpha$$

$$= (0 + \theta) + \left.\frac{e^{-N(\alpha-\theta)}}{(-N)}\right|_\theta^\infty$$

$$= \theta + \frac{1}{N}.$$

Therefore, $\hat{\theta} = \min x[n] - \frac{1}{N}$ is an unbiased estimate that is a fn. of the sufficient statistic. If we assume that the sufficient statistic is complete, then $\hat{\theta} = \min x[n] - \frac{1}{N}$ is the MVUE.