1. $P_0(y) = \frac{1}{2} e^{-|y|}$ Solution to Problem Set 3

$P_1(y) = e^{-2|y|}$

$$L(y) = \frac{P_1(y)}{P_0(y)} = \frac{e^{-2|y|}}{\frac{1}{2} e^{-|y|}} > \gamma$$

$$e^{|y| - 2|y|} > \frac{\gamma}{2}$$

$$e^{-|y|} > \frac{\gamma}{2}$$

$$-|y| > \ln\left(\frac{\gamma}{2}\right)$$

$$|y| < -\ln\left(\frac{\gamma}{2}\right)$$

$$\Rightarrow y \in \left[\ln\left(\frac{\gamma}{2}\right), -\ln\left(\frac{\gamma}{2}\right)\right]$$

This holds when $\gamma < 2$.
When $\gamma \geq 2$, it is always H_0.
$\therefore$ $PFA = 0$, $P_D = 0$.

$$P_{FA} = \int_{\ln\left(\frac{\gamma}{2}\right)}^{-\ln\left(\frac{\gamma}{2}\right)} \frac{1}{2} e^{-|y|} \, dy$$

$$= \int_{\ln\left(\frac{\gamma}{2}\right)}^{0} \frac{1}{2} e^{y} \, dy + \int_{0}^{-\ln\left(\frac{\gamma}{2}\right)} \frac{1}{2} e^{-y} \, dy$$

$$= \frac{1}{2}\left[e^{y}\right]_{\ln\left(\frac{\gamma}{2}\right)}^{0} - \frac{1}{2}\left[e^{-y}\right]_{0}^{-\ln\left(\frac{\gamma}{2}\right)}$$

$$= \frac{1}{2}\left[1 - \frac{\gamma}{2}\right] - \frac{1}{2}\left[\frac{\gamma}{2} - 1\right]$$

$$\cancel{= \frac{1}{2} - \frac{\gamma}{4} - \frac{1}{\gamma} + \frac{1}{2}} \qquad = \frac{1}{2} - \frac{\gamma}{4} - \frac{\gamma}{4} + \frac{1}{2}$$

$$\cancel{= 1 - \frac{\gamma}{4} - \frac{1}{\gamma}} \qquad = 1 - \frac{\gamma}{2}$$

$$PFA = \alpha$$

$$\Rightarrow 1 - \frac{\gamma}{4} - \frac{1}{\gamma} = \alpha$$

$$4\gamma - \gamma^2 - 4 = \alpha\gamma$$

$$\gamma^2 + (\alpha - 4)\gamma + 4 = 0$$

$$\gamma = \frac{(4-\alpha) \pm \sqrt{(\alpha-4)^2 - 16}}{2}$$

$$PFA = \alpha.$$

$$\Rightarrow 1 - \frac{\gamma}{2} = \alpha.$$

$$\Rightarrow 2 - \gamma = 2\alpha.$$

$$\boxed{\gamma = 2 - 2\alpha}$$

$$P_D = \int_{\ln(\frac{\gamma}{2})}^{-\ln(\frac{\gamma}{2})} e^{-2|y|}\,dy = \int_{\ln(\frac{\gamma}{2})}^{0} e^{2y}\,dy + \int_{0}^{-\ln(\frac{\gamma}{2})} e^{-2y}\,dy.$$

$$= \frac{1}{2}\left[e^{2y}\right]_{\ln(\frac{\gamma}{2})}^{0} - \frac{1}{2}\left[e^{-2y}\right]_{0}^{-\ln(\frac{\gamma}{2})}.$$

$$= \frac{1}{2}\left[1 - \left(\frac{\gamma}{2}\right)^2\right] - \frac{1}{2}\left[\left(\frac{\gamma}{2}\right)^2 - 1\right].$$

$$= 1 - \frac{\gamma^2}{4}.$$

$$\Rightarrow P_D = 1 - \frac{(2-2\alpha)^2}{4}$$

$$\Rightarrow P_D = 1 - (1-\alpha)^2 = (2-\alpha)\alpha.$$

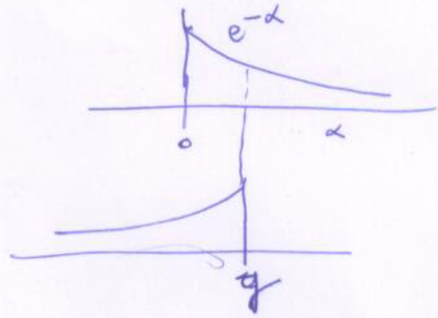
$$\boxed{P_D = \alpha(2-\alpha)}.$$

(2) (a) $\mathcal{H}_0 : Y = N$

$\mathcal{H}_1 : Y = N + S$

N, S independent each with pdf $p(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$

$$L(y) = \frac{p_1(y)}{p_0(y)}$$



$$p_0(y) = p(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_1(y) = \int_0^y e^{-\alpha} e^{-y+\alpha} \, d\alpha$$

$$= y e^{-y} \text{ for } y \geq 0$$

$$L(y) = \frac{y e^{-y}}{e^{-y}} = y \text{ for } y \geq 0$$

(b) $P_F = \int_\tau^\infty e^{-y} dy = \alpha$ $\qquad \left.\frac{e^{-y}}{-1}\right|_\tau^\infty = e^{-\tau} = \alpha \Rightarrow \boxed{\tau = \ln\left(\frac{1}{\alpha}\right)}$

$$P_D = \int_\tau^\infty \cancel{y e^{-y} dy} = \cancel{\int_\tau^\infty y \, d(-e^{-y})} = -\cancel{y e^{-y}}\Big|_\tau^\infty - \int_\tau^\infty \cancel{(e^{-y})} dy$$

$$= \cancel{\tau e^{-\tau}} + \cancel{\frac{e^{-y}}{-1}}\Big|_\tau^\infty \quad ✓$$

$$= \cancel{\tau e^{-\tau} - e^{-\tau}} = \cancel{(\tau-1)e^{-\tau}} = \underline{\alpha\left(\ln\frac{1}{\alpha} - 1\right)}$$

$\cancel{(\tau-1)e^{-\tau} = \alpha}$

$$P_D = \int_\tau^\infty y e^{-y} dy = (\tau - 1) e^{-\tau} = \alpha\left(\ln\frac{1}{\alpha} - 1\right).$$

(c) $\mathcal{H}_0 : Y_k = N_k \qquad k = 1, \dots, n$

$\mathcal{H}_1 : Y_k = N_k + S \qquad k = 1, \dots, n.$

$n > 1$, $N_1, \dots, N_n, S$ independent $\sim p(x) = \begin{cases} e^{-x} & x \geqslant 0 \\ 0 & x < 0. \end{cases}$

$$L(y) = \frac{p_1(y)}{p_0(y)}$$

$$p_0(y) = \prod_{k=1}^{n} e^{-y_k}, \quad y_k \geqslant 0 \ \forall k.$$

$$p_1(y|S=s) = \prod_{k=1}^{n} p_1(y_k|S=s) = \prod_{k=1}^{n} e^{-(y_k - s)}, \quad \underline{y_k \geqslant s} \ \forall k.$$

$$= \left(\prod_{k=1}^{n} e^{-y_k}\right) e^{+ns}, \quad \underline{y_k \geqslant s \ \forall k.}$$

$$p_1(y) = \left(\prod_{k=1}^{n} e^{-y_k}\right) \cdot \int_0^{\min(y_1, \dots, y_n)} e^{+ns} e^{-s} ds$$

Need $s \leq y_k \ \forall k$

$$= \left(\prod_{k=1}^{n} e^{-y_k}\right) \cdot \left. \frac{e^{+(n-1)s}}{(n-1)} \right|_0^{\min(y_1, \dots, y_n)}$$

$$= \left(\prod_{k=1}^{n} e^{-y_k}\right) \frac{1}{n-1} \left[e^{(n-1)\min(y_1, \dots, y_n)} - 1 \right]$$

$$L(y) = \frac{1}{n-1} \left[\underbrace{e^{(n-1)\min(y_1, \dots, y_n)}} - 1 \right]$$

(d) Need $L(y)$ to be compared with τ.

$$\left(\tau' = \frac{\ln(\tau(n-1) + 1)}{(n-1)} \right)$$

$(\equiv)$ $\underbrace{\min(y_1, \dots, y_n)}$ compared with τ'.

$$P_F = \int_{\Gamma_1} p_0(y)\, dy = \int_{\tau'}^{\infty} (p_{\min}(y))\, dy$$

$p_{\min}(y)$: density of $\min(Y_1, \dots, Y_n)$ under $\mathcal{H}_0$.

$$P\left[\min(Y_1, \cdots, Y_n) \le \alpha\right]$$

$$p(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0. \end{cases}$$

$$= 1 - P\left[\min(Y_1, \cdots, Y_n) > \alpha\right]$$

$$\int_{\alpha}^{\infty} e^{-y}\, dy = \underline{e^{-\alpha}}.$$

$$= 1 - \prod_{k=1}^{n} P[Y_k > \alpha]$$

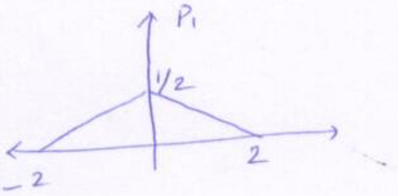
$$= 1 - \prod_{k=1}^{n} e^{-\alpha} = 1 - e^{-n\alpha}.$$

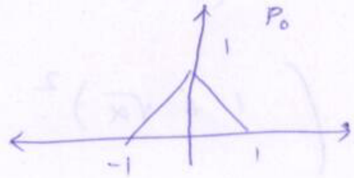
$$p_{min}(y) = \begin{cases} n e^{-ny}. & y \geq 0. \\ 0 & \text{else.} \end{cases}$$

$$P_F = \int_{\tau'}^{\infty} n e^{-ny}\, dy = n \left.\frac{e^{-ny}}{-n}\right|_{\tau'}^{\infty} = \cancel{n}\frac{e^{-n\tau'}}{\cancel{n}} = e^{-n\tau'}.$$

$$\underline{\tau' = \frac{1}{n}\ln\left(\frac{1}{\alpha}\right)}$$

3. $p_1(y) = \frac{2-|y|}{4}, |y|<2 \quad p_0(y) = 1-|y| \quad , |y|<1$





for $2>|y|>1$, alway H_1

for $|y|<1$, we ~~perform~~ find the NP test

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{2-|y|}{4(1-|y|)} > \gamma$$

$$\Rightarrow \quad 2-|y| > 4\gamma(1-|y|)$$

$$\Rightarrow \quad (4\gamma-1)(|y|) > 4\gamma-2$$

$$\Rightarrow \quad |y| > \frac{4\gamma-2}{4\gamma-1}$$

if $\frac{4\gamma-2}{4\gamma-1} < 0$ or $\gamma \in \left(\frac{1}{4},\frac{1}{2}\right)$ then $|y|>0$

in which case $P_{FA}=1$, $P_D=1$

for case of $\gamma \notin \left(\frac{1}{4},\frac{1}{2}\right)$ or $\alpha \neq 1$

$$P_{FA} = \int_{\frac{4\gamma-2}{4\gamma-1}}^{1} 1-|y| \, dy = 2 \times \frac{1}{2} \times \left[1-\left(\frac{4\gamma-2}{4\gamma-1}\right)\right]^2$$

$$= \left[\frac{(4\gamma-1)-(4\gamma-2)}{(4\gamma-1)}\right]^2 = \left(\frac{1}{4\gamma-1}\right)^2 = \alpha$$

$$\Rightarrow \quad 4\gamma-1 = \frac{1}{\sqrt{\alpha}} \quad \text{or} \quad \gamma = \frac{1}{4}\left(\frac{1}{\sqrt{\alpha}}+1\right)$$

$$P_D = \int_{\frac{4\gamma-2}{4\gamma-1}}^{2} p_1(y)dy = \frac{1}{4}\left(2-\frac{4\gamma-2}{4\gamma-1}\right)^2 = \frac{1}{4}\left(\frac{(8\gamma\cancel{-2})-(4\gamma\cancel{-2})}{(4\gamma-1)}\right]^2$$

$$= \left(\frac{2\gamma}{4\gamma-1}\right)^2$$

$$= \frac{1}{4}\left(\frac{(\sqrt{\alpha})^{-1}+1}{(\sqrt{\alpha})^{-1}\cancel{+1}\cancel{-1}}\right)^2$$

$\cancel{\lambda(y) = \frac{P_1(y)}{P_0(y)}}$

$$= \frac{1}{4}\left(\frac{(\sqrt{\alpha})^{-1}+1}{(\sqrt{\alpha})^{-1}}\right)^2$$

$$= \frac{1}{4}\left(1+\sqrt{\alpha}\right)^2$$

$\therefore$ $P_D = \frac{1}{4}\left(1+\sqrt{\alpha}\right)^2$ when $\alpha \neq 1$

$\cancel{P_D = 1, \alpha = 1}$

$P_D = 1$, when $\alpha = 1$