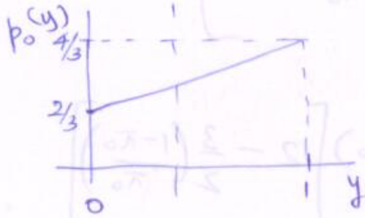
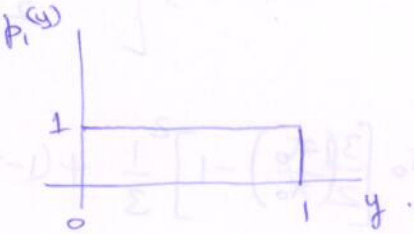
Problem set 2

(1)

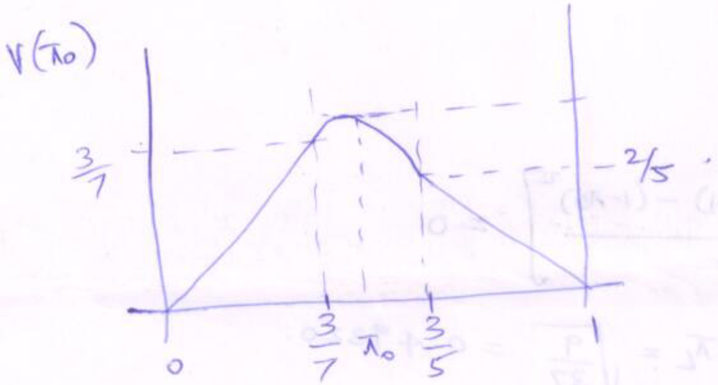




From problem 2 in PS1, Bayes rule is

$$\delta_B(y) = \begin{cases} 1 & \frac{2}{3}(y+1) < \frac{\pi_1}{\pi_0} \\ 0 & \frac{2}{3}(y+1) \geq \frac{\pi_1}{\pi_0} \end{cases}$$

$$\Gamma_1 : \left\{ \frac{2}{3}(y+1) < \frac{\pi_1}{\pi_0} = \frac{1-\pi_0}{\pi_0} \right\} \text{ (or) } \left\{ y < \frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right) - 1 \right\}$$



[when $\underbrace{\frac{1-\pi_0}{\pi_0} < \frac{2}{3}}$, $\Gamma_1 = \phi$, Decide H_0 always $\Rightarrow$ Bayes risk $V(\pi_0) = \pi_1 = 1 - \pi_0$.

$\Rightarrow 3 - 3\pi_0 < 2\pi_0 \Rightarrow \pi_0 > \frac{3}{5}$.]

[when $\frac{2}{3} < \frac{1-\pi_0}{\pi_0}$ and $\underbrace{\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right) - 1 > 1}$, $\Gamma_0 = \phi$, Decide H_1 always $\Rightarrow$ Bayes risk $V(\pi_0) = \pi_0$

$\frac{1-\pi_0}{\pi_0} > \frac{4}{3} \Rightarrow 4\pi_0 < 3 - 3\pi_0 \Rightarrow \pi_0 < \frac{3}{7}$.

when $\frac{3}{7} < \pi_0 < \frac{3}{5}$, $\Gamma_1 = \left\{ y : 0 \leq y < \frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right) - 1 \right\}$

$$\Gamma_0 = \left\{ y : \frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right) - 1 \leq y < 1 \right\}$$

$$V(\pi_0) = \pi_0 \int_0^{\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)-1} \frac{2}{3}(y+1)\,dy + \pi_1 \int_{\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)-1}^{1} 1 \cdot dy$$

$$= \pi_0 \quad \frac{2}{3}\left(\frac{y^2}{2}+y\right)\Bigg|_0^{\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)-1} + (1-\pi_0)\left[1-\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)+1\right]$$

$$= \pi_0\left[\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)-1\right]\frac{2}{3} + \pi_0\left[\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)-1\right]^2\frac{1}{3} + (1-\pi_0)\left[2-\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)\right]$$

$$= 1-\pi_0 - \frac{2\pi_0}{3} + \frac{\pi_0}{3}\left[\frac{9}{4}\left(\frac{1-\pi_0}{\pi_0}\right)^2+1-3\left(\frac{1-\pi_0}{\pi_0}\right)\right] + 2(1-\pi_0) - \frac{3}{2}\frac{(1-\pi_0)^2}{\pi_0}$$

$$= 3 - \frac{11\pi_0}{3} + \frac{3}{4}\frac{(1-\pi_0)^2}{\pi_0} + \frac{\pi_0}{3} - (1-\pi_0) - \frac{3}{2}\frac{(1-\pi_0)^2}{\pi_0}$$

$$= 2 - \frac{7\pi_0}{3} - \frac{3}{4}\frac{(1-\pi_0)^2}{\pi_0}$$

$$\pi_L = \arg\max_{\pi_0} V(\pi_0)$$

$$\frac{d}{d\pi_0}V(\pi_0) = -\frac{7}{3} - \frac{3}{4}\left[\frac{2\pi_0(1-\pi_0)(-1)-(1-\pi_0)^2}{\pi_0^2}\right] = 0$$

$$\Rightarrow \pi_0^2 = 9/37 \Rightarrow \pi_L = \sqrt{\frac{9}{37}} = 0.49320.$$

Minimax rule is the rule corresponding to π_L.

$$\delta_{minmax}(y) = \begin{cases} 1 & y < \frac{3}{2}\left(\frac{1-\pi_L}{\pi_L}\right) - 1 = \underline{0.5414} \\ 0 & \text{else} \end{cases}$$

(2)

2.) $P_j(y) = \frac{j+1}{2} e^{-(j+1)|y|}$

$$C_{ij} = \begin{cases} 0 & j = i \\ 1 & i = 1,\ j = 0 \\ 3/4 & i = 0,\ j = 1 \end{cases}$$

$$\delta_B(y) = \begin{cases} 1 & , 2e^{-|y|} > \frac{1}{3/4} \frac{\pi_0}{\pi_1} \\ 0 & , 2e^{-|y|} < \frac{4}{3}\frac{\pi_0}{\pi_1} \end{cases}$$

$$\Gamma_1 : \left\{ 2e^{-|y|} > \frac{2}{3} \frac{\pi_0}{1-\pi_0} \right\} \text{ or } \left\{ |y| < \ln \frac{3}{2}\frac{(1-\pi_0)}{\pi_0} \right\}$$

if $\frac{3}{2}\frac{(1-\pi_0)}{\pi_0} < 1 \Rightarrow 3 - 3\pi_0 < 2\pi_0 \Rightarrow \frac{3}{5} < \pi_0$

then $V(\pi_0) = 1 - \pi_0$

else
$$V(\pi_0) = \int_{\Gamma_1} \frac{\pi_0}{2} e^{-|y|} dy + \frac{3}{4}\pi_1 - \int_{\Gamma_1} \frac{3}{4}\pi_1 e^{-2|y|} dy$$

$$= 2 \times \int_0^{\ln\left(\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)\right)} \frac{\pi_0}{2} e^{-y} + \frac{3}{4}\pi_1 - 2\int_0^{\ln\left(\frac{3}{2}\left(\frac{1-\pi_0}{\pi_0}\right)\right)} 2\frac{3}{4}\pi_1 e^{-2y} dy$$

$$= -\pi_0 \left[\frac{2\pi_0}{3(1-\pi_0)} - 1 \right] + \frac{3}{4}\pi_1 + \frac{3}{4}\pi_1 \left[\frac{4}{3}\frac{\pi_0}{(1-\pi_0)} - 1 \right]$$

$$= \frac{-2\pi_0^2}{3(1-\pi_0)} + \pi_0 + \pi_0$$

~~$= \frac{-2\pi_0^2}{3(1-\pi_0)} + \frac{2\pi_0(3(1-\pi_0))}{3(1-\pi_0)}$~~

$$= \frac{-2\pi_0^2}{3(1-\pi_0)} + 2\pi_0$$

$$\frac{dV}{d\pi_0} = \frac{+1}{(1-\pi_0)^2} \times \frac{2\pi_0^2}{3} - \frac{4\pi_0}{3(1-\pi_0)} + 2 = 0$$

$$\Rightarrow 2\pi_0^2 - 4\pi_0(1-\pi_0) + 6(1-\pi_0)^2 = 0$$

$$2\pi_0^2 + 4\pi_0^2 - 4\pi_0 + 6\left(1 + \pi_0^2 - 2\pi_0\right) = 0$$

$$\Rightarrow 6\pi_0^2 - 8\pi_0 + 3 = 0$$

$$\Rightarrow \pi_0 = \frac{8 \pm \sqrt{64 - 48}}{12}$$

$$= \frac{8 \pm 4}{12} = \frac{1}{3} \text{ or } 1$$

$\pi_0 = 1$ is greater $3/5$ thus, is not of interest

$V(0) = 0$, $V(1/3) = \dfrac{-4 \times \frac{1}{9}}{3 \times \frac{2}{3}} + 2 \times \frac{1}{3} = 5/9$

$V(3/5) = \dfrac{2}{5}$

$\therefore$ Max. Bayes Risk $= 5/9$

for $\pi_0 = 1/3$

3. $L(y) = \sqrt{\frac{2}{\pi}}\, e^{y - y^2/2} = \sqrt{\frac{2e}{\pi}}\, e^{-\frac{(y-1)^2}{2}}, \quad y \geq 0.$

Bayes critical regions are of the form

$$\Gamma_1 = \{ y \geq 0 \mid (y-1)^2 \leq \tau' \}.$$

where $\tau' = -\sqrt{\frac{\pi}{2e}} \log\left(\frac{\pi_0}{1-\pi_0}\right).$

There will be three cases:

$$\Gamma_1 = \begin{cases} \phi & \text{if } \tau' < 0 \\ [1-\sqrt{\tau'}, 1+\sqrt{\tau'}] & \text{if } 0 \leq \tau' \leq 1. \\ [0, 1+\sqrt{\tau'}] & \text{if } \tau' > 1 \end{cases}$$

$\tau' < 0$ is equivalent to $\dfrac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}} \leq \pi_0 \leq 1.$

$0 \leq \tau' \leq 1 \Rightarrow \dfrac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}} \leq \pi_0 \leq \dfrac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}}.$

$\tau' > 1 \Rightarrow 0 \leq \pi_0 < \dfrac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}}$

$$V(\pi_0) = \begin{cases} 1 - \pi_0 & \text{when } \dfrac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}} < \pi_0 \le 1. \\[2em] \pi_0 \displaystyle\int_{1-\sqrt{\tau'}}^{1+\sqrt{\tau'}} e^{-y}\, dy + (1-\pi_0)\sqrt{\frac{2}{\pi}} \left[\int_0^{1-\sqrt{\tau'}} e^{-y^2/2} dy + \int_{1+\sqrt{\tau'}}^{\infty} e^{-y^2/2} dy \right] & \text{when. } \dfrac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}} \le \pi_0 \le \dfrac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}} \\[2em] \pi_0 \displaystyle\int_0^{1+\sqrt{\tau'}} e^{-y} dy + (1-\pi_0)\sqrt{\frac{2}{\pi}} \int_{1+\sqrt{\tau'}}^{\infty} e^{-y^2/2} dy & \text{when } 0 \le \pi_0 < \dfrac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}} \end{cases}$$

The minimax rule can be found by equating conditional risks.

This equality occurs in the intermediate region

$$\frac{\sqrt{\frac{2}{\pi}}}{1+\sqrt{\frac{2}{\pi}}} \le \pi_0 \le \frac{\sqrt{\frac{2e}{\pi}}}{1+\sqrt{\frac{2e}{\pi}}}$$

The threshold τ_L' can thus be found by solving,

$$e^{\sqrt{\tau_L'}} - e^{-\sqrt{\tau_L'}} = 2e\left(1 + Q(1+\sqrt{\tau_L'}) - Q(1-\sqrt{\tau_L'})\right)$$

The minimax risk is then either of the equal conditional risks. eg:

$$V(\pi_L) = e^{-1+\sqrt{\tau_L'}} - e^{-1-\sqrt{\tau_L'}}$$

4. $p_0(y) = p_N(y+s)$

$p_1(y) = p_N(y-s)$.

$$L(y) = \frac{1+(y+s)^2}{1+(y-s)^2}.$$

With equal priors and equal costs, the Bayes tests is.

$$\delta_B(y) = \begin{cases} 1 & \text{if } y \geq 0. \\ 0 & \text{if } y < 0. \end{cases}$$

The minimum Bayes risk is then given by.

$$\frac{1}{2}\int_0^{\infty} p_0\,dy + \frac{1}{2}\int_{-\infty}^{0} p_1\,dy.$$

$$= 2 \times \frac{1}{2} \int_0^{\infty} p_0\,dy$$

$$= A \text{ (say)}.$$

For generic priors,

$$\Gamma_1 : \left\{ y : L(y) > \frac{\pi_0}{\pi_1} \right\}$$

$$\Gamma_0 : \left\{ y : L(y) < \frac{\pi_0}{\pi_1} \right\}.$$

Now the Bayes risk will be given by.

$$\pi_0 \int_{\Gamma_1} p_0\,dy + \pi_1 \int_{\Gamma_0} p_1\,dy.$$

Since this is the Bayes risk, it must be minimum.

$$\therefore \pi_0 \int_{\Gamma_1} P_0(y)dy + \pi_1 \int_{\Gamma_0} P_1(y)dy \le \pi_0 \int_0^{\infty} P_0(y)dy + \pi_1 \int_{-\infty}^{0} P_1(y)dy.$$

$$\le (\pi_0 + \pi_1) A.$$

$$\le A.$$

But we know that when the priors are equal, ie, $\pi_0 = \pi_1 = 1/2$, the Bayes risk $= A$.

$\therefore$ The $\pi_0 = \pi_1 = 1/2$ will give us the case of the least favourable priors.

$\therefore$ The minimax risk is given by

$$\frac{1}{2} \int_0^{\infty} \frac{1}{\pi[1+(y+s)^2]} dy + \frac{1}{2} \int_{-\infty}^{0} \frac{1}{\pi[1+(y-s)^2]} dy.$$

$$= \frac{1}{2} - \frac{\tan^{-1}(s)}{\pi}.$$