EE5130: Detection and Estimation Theory
Problem Set 5

1. (Poor III.F.8) Suppose we have observations $Y_k = N_k + \theta S_k$, $k = 1, \ldots, n$, where $\underline{N} \sim \mathcal{N}(\underline{0}, \mathbf{I}$ and where $S_1, \ldots, S_n$ are i.i.d. random variables, independent of $\underline{N}$, and each taking on the values $+1$ and -1 with equal probabilities of $1/2$.

   (a) Find the likelihood ratio for testing $H_0 : \theta = 0$ versus $H_1 : \theta = A$, where A is a known constant.

   (b) For the case $n = 1$, find the Neyman-Pearson rule and corresponding detection probability for false-alarm probability $\alpha \in (0, 1)$, for the hypotheses of (a).

   (c) Is there a UMP test of $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ in this model? If so, why and what is it? If not, why not? Consider the cases $n = 1$ and $n > 1$ separately.

2. (Poor III.F.13) Consider the model $Y_k = \theta^{1/2} s_k R_k + N_k$, $k = 1, \ldots, n$, where $s_1, s_2, \ldots, s_n$ is a known signal sequence, $\theta \geq 0$ is a constant, and $R_1, R_2, \ldots, R_n, N_1, N_2, \ldots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

   (a) Consider the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta = A$, where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.

   (b) Consider now the hypothesis pair $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Under what conditions on $s_1, s_2, \ldots, s_n$ does a UMP test exist?

   (c) For the hypothesis pair in part (b) with $s_1, s_2, \ldots, s_n$ general, is there a locally optimum detector? If so, find it. If not, describe the generalized likelihood ratio test.

3. (Poor III.F.14 (a)) Consider the following hypothesis about a sequence $Y_1, Y_2, \ldots, Y_n$, of real observations: $H_0 : Y_k = N_k$, $k = 1, \ldots, n$, versus $H_1 : Y_k = N_k + \Theta s_k$, $k = 1, 2, \ldots, n$, where $N_1, N_2, \ldots, N_n$ is a sequence of i.i.d. $\mathcal{N}(0, \sigma^2)$ random variables; where $s_1, s_2, \ldots, s_n$ is a known signal sequence satisfying $\underline{s}^T \underline{s} = 1$; and where Θ is a $\mathcal{N}(\mu, v^2)$ random variable, independent of $N_1, N_2, \ldots, N_n$. Show that the critical region for Neyman-Pearson testing between these two hypotheses is of the form

$$\Gamma_1 = \left\{ \mu \underline{s}^T \underline{y} + \frac{v^2}{2\sigma^2} |\underline{s}^T \underline{y}|^2 > \tau' \right\},$$

   where τ' is an appropriately chosen threshold. [*Hint:* The covariance matrix of $\underline{Y}$ equals $\sigma^2 \mathbf{I} + v^2 \underline{s}\underline{s}^T$ under hypothesis H_1.]

4. (Poor III.F.21) Compute the Chernoff bound for the binary symmetric channel with equal priors, and compare it to the actual minimum error probability.