EE 511 Solutions to Problem Set 7

1. (i) $S_{\hat{X}}(f) = |H(f)|^2 S_X(f)$ where $H(f) = -j\text{sgn}(f)$. Therefore, $S_{\hat{X}}(f) = S_X(f)$.
(ii) Suppose $h(\tau)$ is the impulse response of the Hilbert transformer, $-h(\tau) = h(-\tau)$ is the impulse response of the inverse Hilbert transformer. Therefore, we have

$$R_{\hat{X}X}(\tau) = R_X(\tau) \star h(\tau)$$

and

$$R_{X\hat{X}}(\tau) = R_{\hat{X}}(\tau) \star (-h(\tau)) = R_X(\tau) \star (-h(\tau)) = -R_{\hat{X}X}(\tau),$$

where $\star$ denotes convolution. From the above result, we have

$$S_{\hat{X}X}(f) = -S_{X\hat{X}}(f).$$

(iii)$E[X_t\hat{X}_t] = R_{X\hat{X}}(0)$ and $E[X_t\hat{X}_t] = R_{\hat{X}X}(0)$. Therefore, we have $E[X_t\hat{X}_t] = 0$.
(iv)

$$\begin{aligned} R_Z(\tau) &= E[(X_{t+\tau} + j\hat{X}_{t+\tau})(X_t - j\hat{X}_t)] \\ &= R_X(\tau) - jR_{X\hat{X}}(\tau) + jR_{\hat{X}X}(\tau) + R_{\hat{X}}(\tau) \\ &= 2R_X(\tau) + 2jR_{\hat{X}X}(\tau) \end{aligned}$$

Therefore

$$\begin{aligned} S_Z(f) &= 2S_X(f) + 2jS_{\hat{X}X}(f) \\ &= 2S_X(f) + 2jS_X(f)[-j\text{sgn}(f)] \\ &= 2S_X(f)[1 + \text{sgn}(f)] \\ &= \begin{cases} 4S_X(f) & f > 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

2. See Figure 1. $Re[S_{X^I X^Q}(f)] = 0$.

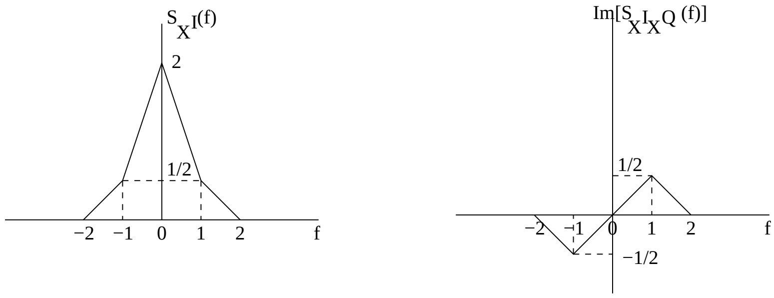

Figure 1:

3. See Figure 2. $Re[S_{X^I X^Q}(f)] = 0$.

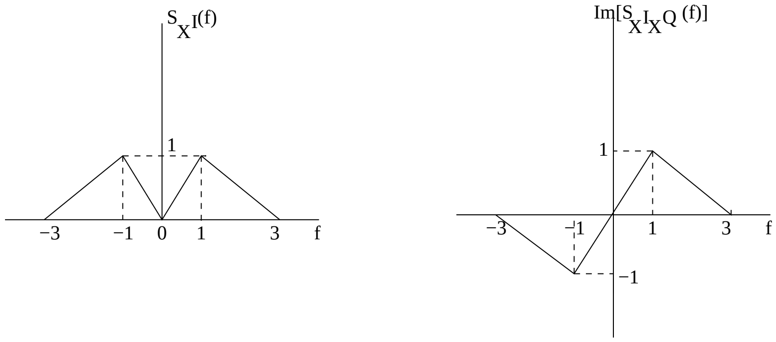

Figure 2:

4.

$$
\begin{aligned}
R_{N^I}(\tau) &= E[(N_{t+\tau}\cos 2\pi f_c(t+\tau) + \hat{N}_{t+\tau}\sin 2\pi f_c(t+\tau))(N_t\cos 2\pi f_c t + \hat{N}_t \sin 2\pi f_c t)] \\
&= R_N(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t + R_{N\hat{N}}(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t \\
&\quad + R_{\hat{N}N}(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t + R_{\hat{N}}(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t
\end{aligned}
$$

Since $R_{\hat{N}}(\tau) = R_N(\tau)$ and $R_{\hat{N}N}(\tau) = -R_{N\hat{N}}(\tau)$, we get

$$
R_{N^I}(\tau) = R_N(\tau)\cos 2\pi f_c\tau + R_{\hat{N}N}(\tau)\sin 2\pi f_c\tau = R_N(\tau)\cos 2\pi f_c\tau + \hat{R}_N(\tau)\sin 2\pi f_c\tau
$$

Similarly, we have

$$
\begin{aligned}
R_{N^Q}(\tau) &= E[(\hat{N}_{t+\tau}\cos 2\pi f_c(t+\tau) - N_{t+\tau}\sin 2\pi f_c(t+\tau))(\hat{N}_t\cos 2\pi f_c t - N_t \sin 2\pi f_c t)] \\
&= R_{\hat{N}}(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t - R_{\hat{N}N}(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t \\
&\quad - R_{N\hat{N}}(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t + R_N(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t \\
&= R_N(\tau)\cos 2\pi f_c\tau + R_{\hat{N}N}(\tau)\sin 2\pi f_c\tau \\
&= R_N(\tau)\cos 2\pi f_c\tau + \hat{R}_N(\tau)\sin 2\pi f_c\tau
\end{aligned}
$$

$$
\begin{aligned}
R_{N^I N^Q}(\tau) &= E[(N_{t+\tau}\cos 2\pi f_c(t+\tau) + \hat{N}_{t+\tau}\sin 2\pi f_c(t+\tau))(\hat{N}_t\cos 2\pi f_c t - N_t \sin 2\pi f_c t)] \\
&= R_{N\hat{N}}(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t - R_N(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t \\
&\quad + R_{\hat{N}}(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t - R_{\hat{N}N}(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t \\
&= R_N(\tau)\sin 2\pi f_c\tau - R_{\hat{N}N}(\tau)\cos 2\pi f_c\tau \\
&= R_N(\tau)\sin 2\pi f_c\tau - \hat{R}_N(\tau)\cos 2\pi f_c\tau
\end{aligned}
$$

$$
\begin{aligned}
R_{N^Q N^I}(\tau) = R_{N^I N^Q}(-\tau) &= -R_N(-\tau)\sin 2\pi f_c\tau - R_{\hat{N}N}(-\tau)\cos 2\pi f_c\tau \\
&= -R_N(\tau)\sin 2\pi f_c\tau + R_{\hat{N}N}(\tau)\cos 2\pi f_c\tau = -R_{N^I N^Q}(-\tau)
\end{aligned}
$$

5. $N_t = W_t \cos(2\pi f_c t + \Theta) - W_t \sin(2\pi f_c t + \Theta)$.

$$
\begin{aligned}
R_N(\tau) &= E[(W_{t+\tau}\cos(2\pi f_c(t+\tau)+\Theta) - W_{t+\tau}\sin(2\pi f_c(t+\tau)+\Theta)) \\
&\quad (W_t\cos(2\pi f_c t+\Theta) - W_t\sin(2\pi f_c t+\Theta))] \\
&= R_W(\tau)E[\cos(2\pi f_c(t+\tau)+\Theta)\cos(2\pi f_c t+\Theta)] \\
&\quad -R_W(\tau)E[\cos(2\pi f_c(t+\tau)+\Theta)\sin(2\pi f_c t+\Theta)] \\
&\quad -R_W(\tau)E[\sin(2\pi f_c(t+\tau)+\Theta)\cos(2\pi f_c t+\Theta)] \\
&\quad +R_W(\tau)E[\sin(2\pi f_c(t+\tau)+\Theta)\sin(2\pi f_c t+\Theta)] \\
&= R_W(\tau)\cos 2\pi f_c\tau - R_W(\tau)E[\sin(2\pi f_c(2t+\tau)+2\Theta)] \\
&= R_W(\tau)\cos 2\pi f_c\tau
\end{aligned}
$$

$$S_N(f) = \frac{1}{2}\left[S_W(f-f_c) + S_W(f+f_c)\right].$$

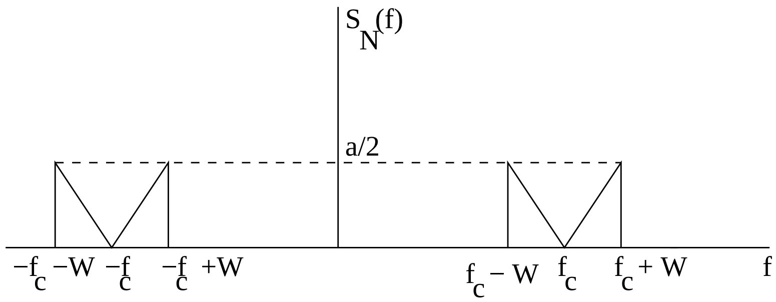

Figure 3:

6. (a) $S_{X^Q}(f) = S_{X^I}(f)$ and $S_{X^Q X^I}(f) = -S_{X^I X^Q}(f)$.
   (b) See Figure 6c.
   (c) See Figure 6c.

7. (a) See Figure 4.
   (b) See Figure 4.
   (c) $S_{X^Q}(f) = S_{X^I}(f)$ and $S_{X^Q X^I}(f) = -S_{X^I X^Q}(f)$.

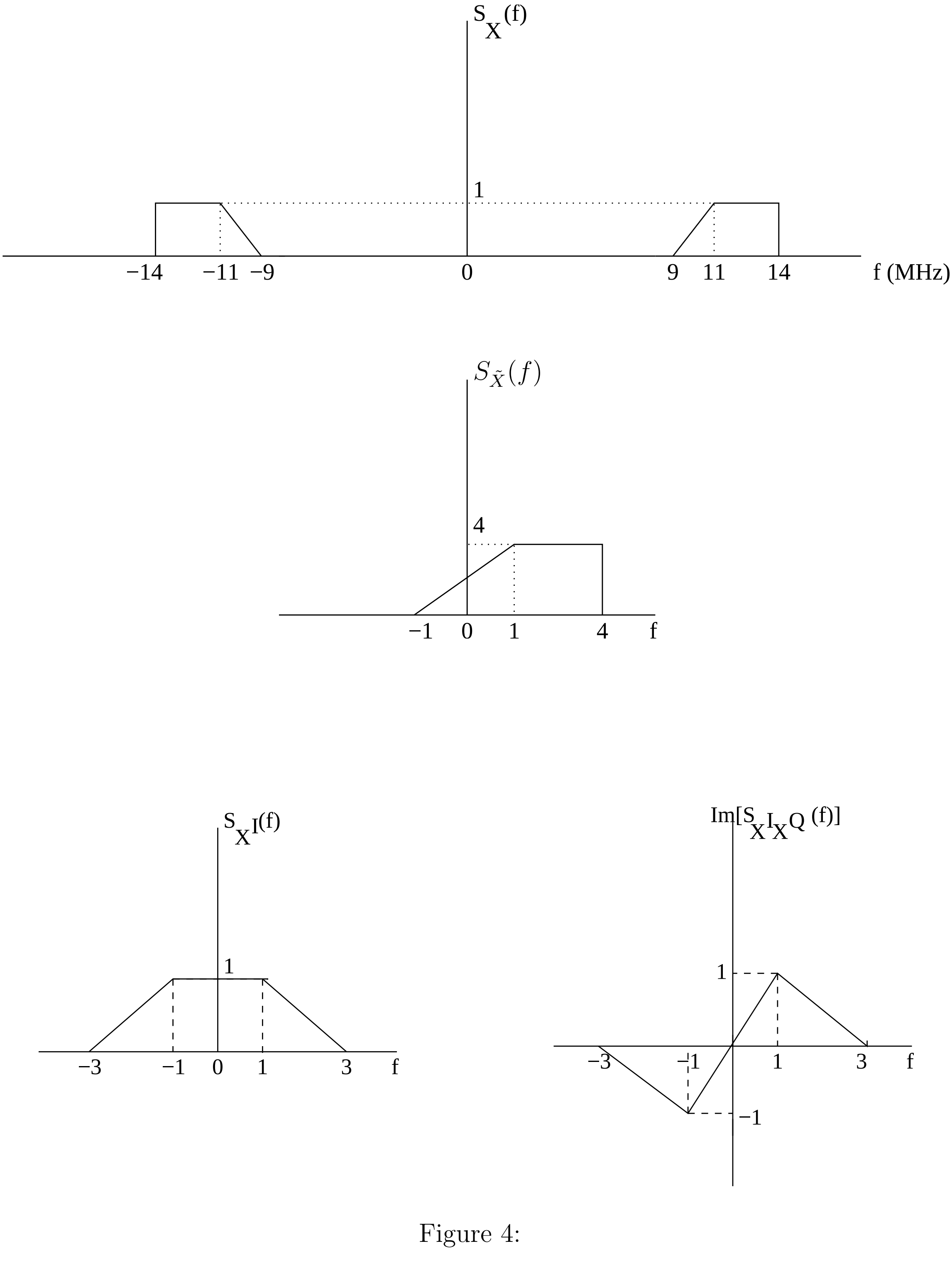
S_X(f)
1
−14
−11
−9
0
9
11
14
f (MHz)
$S_{\tilde{X}}(f)$
4
−1
0
1
4
f
S_XI(f)
1
−3
−1
0
1
3
f
Im[S_XI_XQ (f)]
1
−3
−1
1
3
f
−1

Figure 4: