EE 511 Solutions to Problem Set 6

1. The power spectral density of the output noise process is equal to $N_0/2$ for $|f| \le B$ and equal to 0 otherwise. Therefore, the variance of the output noise process (zero-mean) is the area under the PSD $= (2B)(N_0/2) = N_0 B$.

2. a) $m_Y(t) = E[Y_t] = E[X_{t+D}] - E[X_t] = m(t+D) - mt = mD$.

$$\begin{aligned} R_Y(t,s) &= E[(X_{t+D} - X_t)(X_{s+D} - X_s)] \\ &= R_X(t+D, s+D) - R_X(t+D, s) - R_X(t, s+D) + R_X(t,s) \\ &= \sigma^2[\min(t+D, s+D) - \min(t+D, s) - \min(t, s+D) + \min(t,s)] + m^2D^2 \end{aligned}$$

For $0 \le t - s \le D$, $R_Y(t,s) = m^2D^2 + \sigma^2(s + D - s - t + s) = m^2D^2 + \sigma^2(s-t) + \sigma^2 D$.

For $t - s \ge D$, $R_Y(t,s) = m^2D^2 + \sigma^2(s + D - s - (s+D) + s) = m^2D^2$.

For $-D \le t - s \le 0$, $R_Y(t,s) = m^2D^2 + \sigma^2(t + D - s - t + t) = m^2D^2 + \sigma^2(t-s) + \sigma^2 D$.

For $t - s \le -D$, $R_Y(t,s) = m^2D^2 + \sigma^2(t + D - (t+D) - t + t) = m^2D^2$.

Therefore, defining $\tau = t - s$ we have

$$R_Y(\tau) = \begin{cases} m^2D^2 + \sigma^2(D - |\tau|) & |\tau| \le D \\ m^2D^2 & |\tau| > D \end{cases}$$

b) Since Y_t is W.S.S. (from part (a)) and Y_t is a Gaussian random process, Y_t is also strictly stationary.

$$S_Y(f) = m^2D^2\delta(f) + \sigma^2 D^2 \mathrm{sinc}^2(fD).$$

3. a) $E[Y_t] = E[X_t^2] = R_X(0)$.

b)

$$\begin{aligned} C_y(t, t+\tau) &= E[(Y_t - R_X(0))(Y_{t+\tau} - R_X(0))] \\ &= R_Y(t, t+\tau) - R_X^2(0) \\ &= E[X_t^2 X_{t+\tau}^2] - R_X^2(0) \\ &= R_X^2(0) + 2R_X^2(\tau) - R_X^2(0) \\ &= 2R_X^2(\tau) \end{aligned}$$

In the above solution, $E[X_t^2 X_{t+\tau}^2]$ can be shown to be $R_X^2(0) + 2R_X^2(\tau)$ in the following manner.

(i) The joint characteristic function of X_t and $X_{t+\tau}$ is

$$\phi_{X_t, X_{t+\tau}}(s_1, s_2) = \exp\left\{\frac{1}{2}\begin{bmatrix} s_1 & s_2 \end{bmatrix} C \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right\}$$

since X_t and $X_{t+\tau}$ are jointly Gaussian random variables with zero-mean and covariance matrix C given by

$$C = \begin{bmatrix} R_X(0) & R_X(\tau) \\ R_X(\tau) & R_X(0) \end{bmatrix}$$

Therefore, we have

$$\phi_{X_t,X_{t+\tau}}(s_1,s_2) = \exp\left\{R_X(0)s_1^2 + 2R_X(\tau)s_1s_2 + R_X(0)s_2^2\right\}$$

(ii) Then,

$$E[X_t^2X_{t+\tau}^2] = \frac{\partial^4}{\partial^2 s_1 \partial^2 s_2}\phi_{X_t,X_{t+\tau}}(s_1,s_2)\bigg|_{s_1=0,s_2=0} = R_X^2(0) + 2R_X^2(\tau).$$

4. a) Y_t is also a zero-mean W.S.S. Gaussian random process. We have

$$|H(f)|^2 = \text{sinc}^2(fT).$$

Therefore, we have $S_Y(f) = S_X(f)|H(f)|^2 = S_X(f)\text{sinc}^2(fT)$.

$$E[Y^2] = R_Y(0) = \int_{-\infty}^{\infty} S_X(f)\text{sinc}^2(fT)df.$$

b) Y is a Gaussian random variable with zero-mean and variance given by

$$\int_{-\infty}^{\infty} S_X(f)\text{sinc}^2(fT)df.$$

5. (a) $S_Z(f) = S_X(f)|H(f)|^2$ and $S_Y(f) = S_X(f)|H(f)|^2|E(f)|^2$.
   (b) $S_{XZ}(f) = S_X(f)H^*(f)$,
   $S_{ZY}(f) = S_Z(f)E^*(f) = S_X(f)|H(f)|^2E^*(f)$, and
   $S_{XY}(f) = S_X(f)H^*(f)E^*(f)$.
   (c) $H(f) = E(f) = 1/(1+j2\pi f)$. Therefore, we have

$$S_Z(f) = \frac{S_X(f)}{1+4\pi^2f^2},$$

$$S_Y(f) = \frac{S_X(f)}{(1+4\pi^2f^2)^2},$$

$$S_{XZ}(f) = \frac{S_X(f)}{1-j2\pi f},$$

$$S_{ZY}(f) = \frac{S_X(f)}{(1+4\pi^2f^2)(1-j2\pi f)}, \quad \text{and}$$

$$S_{XZ}(f) = \frac{S_X(f)}{(1-j2\pi f)^2}.$$

6.

$$\begin{aligned}
R_{YZ}(t_1,t_2) &= E[Y_{t_1}Z_{t_2}] \\
&= E\left[\int_{-\infty}^{\infty} h_1(\tau_1)X_{t_1-\tau_1}d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2)X_{t_2-\tau_2}d\tau_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)E[X_{t_1-\tau_1}X_{t_2-\tau_2}]d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_X(t_1-t_2-\tau_1+\tau_2)d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} h_1(\tau_1)\left[\int_{-\infty}^{\infty} h_2(\tau_2)R_X(t_1-t_2-\tau_1+\tau_2)d\tau_2\right]d\tau_1
\end{aligned}$$

From the above result, we see that $R_{YZ}(t,s)$ is a function of $\tau = t_1 - t_2$ and is the convolution of $R_X(\tau)$, $h_1(\tau)$ and $h_2(-\tau)$. Therefore, we have

$$S_{YZ}(f) = S_X(f)H_1(f)H_2{}^*(f).$$

Y_t and Z_t are Gaussian random processes. They are independent, if they are uncorrelated, i.e., $R_{YZ}(t_1-t_2) = m_Y(t_1)m_Z(t_2) = m_Y m_Z$. We have

$$m_Y = m_X \int_{-\infty}^{\infty} h_1(\tau)d\tau = m_X H_1(0) = 0$$

and

$$m_Z = m_X H_2(0) = 0.$$

Therefore, we need $R_{YZ}(t_1-t_2) = m_Y(t_1)m_Z(t_2) = m_Y m_Z = 0$. Equivalently, we need

$$S_X(f)H_1(f)H_2^*(f) = 0,$$

i.e., we need the frequency response of the filters $H_1(f)$ and $H_2(f)$ to be non-overlapping in the region where $S_X(f)$ is non-zero.

7. (a) $E[Y_t^2] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df = \int_{-\infty}^{\infty} S_X(f)|H_1(f)|^2 df = 2.$
$E[Z_t^2] = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)df = \int_{-\infty}^{\infty} S_X(f)|H_2(f)|^2 df = 3.5.$

(b)

$$S_{YZ}(f) = S_X(f)H_1(f)H_2^*(f) = \begin{cases} \frac{1}{W} & |f| \le W \\ 0 & \text{else} \end{cases}$$

Therefore, we have $R_{YZ}(\tau) = 2\text{sinc}(2W\tau)$.

(c) Since X_t is a Gaussian process and Y_t and Z_t are obatined from X_t using linear filters, Y_{t_1} and Z_{t_2} are jointly Gaussian. The elements of the mean vector and covariance matrix are as follows:

$$E[Y_{t_1}] = E[Z_{t_2}] = 0.$$

$$E[Y_{t_1} Z_{t_2}] = R_{YZ}\left(\frac{1}{2W}\right) = 0.$$

$$E[Y_{t_1}^2] = 2 \quad \text{and} \quad E[Z_{t_2}^2] = 3.5.$$

8. $Y_t = \frac{1}{c} X_{c^2 t}$. Since X_t is a Gaussian process, Y_t is also Gaussian.

$$E[Y_t] = \frac{1}{c} E[X_{c^2 t}] = \frac{1}{c} m c^2 t = mct.$$

$$\begin{aligned} R_Y(t,s) &= E[Y_t Y_s] \\ &= E\left[\frac{1}{c} X_{c^2 t} \frac{1}{c} X_{c^2 s}\right] \\ &= \frac{1}{c^2} R_X(c^2 t, c^2 s) \\ &= \frac{1}{c^2}\left[c^4 m^2 ts + \sigma^2 \min(c^2 t, c^2 s)\right] \\ &= c^2 m^2 ts + \sigma^2 \min(t,s). \end{aligned}$$

Therefore, Y_t is also a Wiener process (with parameters mc and σ^2).