EE 511 Problem Set 5

Due on 17 October 2007

1. An experiment has four equally likely outcomes 0, 1, 2, and 3, i. e., $S = \{0, 1, 2, 3\}$. If a random process X_t is defined as $X_t = \cos(2\pi st)$ for all $s \in S$, then

   (a) Sketch all the possible sample functions.

   (b) Sketch the marginal CDF's of the random variables X_0, $X_{0.25}$ and $X_{0.5}$.

   (c) Determine the conditional pmf of $X_{0.25}$ given that $X_{0.5} = -1$.

   (d) Determine the conditional pmf of $X_{0.25}$ given that $X_{0.5} = 1$.

2. a) Determine the auto-correlation function of the random process $X_t = A\cos(2\pi f_c t + \Theta)$, where A and f_c are constants, and Θ is uniformly distributed in $[0, 2\pi]$. b) Can you come up with a different pdf for Θ such that X_t remains wide-sense stationary (W.S.S)?

3. A random process Y_t is defined as $Y_t = X_t \cos(2\pi f_c t + \Theta)$ where X_t is a W.S.S random process, f_c is a constant, and Θ is a random variable independent of X_t and uniform in $[0, 2\pi]$. a) Is Y_t W.S.S? b) If Y_t is defined as $Y_t = X_t \cos(2\pi f_c t)$, is Y_t W.S.S?

4. A random process X_t is defined in terms of random variables X_1 and X_2 as follows:

   $$X_t = X_1 \cos 2\pi f_c t + X_2 \sin 2\pi f_c t$$

   where f_c is a constant. Determine the necessary and sufficient conditions on X_1 and X_2 such that X_t is wide-sense stationary.

5. Show that if X_t is a complex W. S. S. process, then

   $$E[|X_{t+\tau} - X_t|^2] = 2\ \mathrm{Re}(R_X(0) - R_X(\tau)).$$

   Clearly explain each step.

6. $\{W_n\}_{n=-\infty}^{\infty}$ (for all integer n) is a sequence of i. i. d. random variables such that $E[W_n] = 0$ and $E[W_n^2] = 1$ for all n. A sequence of random variables $\{X_n\}_{n=0}^{\infty}$ is defined as

   $$X_0 = 0$$

   and

   $$X_n = \rho X_{n-1} + W_n \quad \text{for } n = 1, 2, ...$$

   where $0 < \rho < 1$.

   (a) Find $E[X_n]$ and $Var(X_n)$.

   (b) Find $E[X_n X_{n+k}]$ for $k \geq 1$.

   (c) Is the discrete random process X_n W. S. S.? Why?

7. If X_t and Y_t are two jointly wide-sense stationary processes, show that $|R_{XY}(\tau)| \leq 0.5[R_X(0) + R_Y(0)]$.

8. Let Y_t and Z_t be independent W.S.S. random processes. If $X_t = Y_t Z_t$, determine the auto-correlation function of X_t in terms of the auto-correlation functions of Y_t and Z_t.

9. X_t is a W.S.S. random process. Let $Y_t = X_t - X_{t-T}$ be a new process obtained by filtering X_t. a) Determine the power spectral density of Y_t in terms of the power spectral density of X_t. b) Using the approximation that $\sin\theta = \theta$ for small θ, write $S_Y(f)$ in terms of $S_X(f)$ when $S_X(f)$ has only low frequency components, i.e., $f \ll 1/T$. Show that the filter defined above as $Y_t = X_t - X_{t-T}$ acts as a differentiator for low-frequency inputs.

10. Let X_t and Y_t be individually and jointly W.S.S random processes. a) Is $Z_t = X_t + Y_t$ W.S.S? b) Express $S_Z(f)$ in terms of the power spectral densities and cross power spectral densities of X_t and Y_t. c) Specialize the result in part b) to the case when X_t and Y_t are uncorrelated random processes.

11. Consider two jointly W. S. S. random processes X_t and Y_t, with known $R_X(\tau)$, $R_Y(\tau)$, and $R_{XY}(\tau)$. Let $S_t = X_t + Y_t$ and $D_t = X_t - Y_t$. (a) Find the auto-correlation functions of S_t and D_t. (b) Find the cross-correlation function $R_{XS}(t, s)$. (c) Find the cross-correlation function $R_{SD}(t, s)$.

12. X_t and Y_t are jointly W.S.S. random processes. Express $S_{ZW}(f)$ in terms of $S_{XY}(f)$ and the filter transfer functions. See Figure 1.

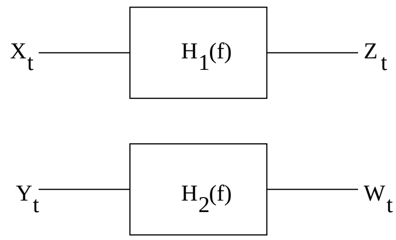

Figure 1:

13. X_n is an i. i. d. discrete random process where each X_n is Poisson distributed, i. e.,

$$P[X_n = k] = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k \geq 0.$$

The above process is filtered using a 3-tap FIR filter with taps $\{1, 1, 1\}$ to get a random process Y_n. (a) Determine $P[Y_n = k]$. (b) Repeat (a) if the input X_n is an independent sequence of Poisson distributed random variables with parameter λ_n (i. e., not necessarily identical for all n).