Discussion in "An Introduction to Random Signals and Communication Theory" by B.P. Lathi, International Textbook Company, 1968.

Example: Traffic lights in a city.

Assume
- large number of traffic lights in the city
- Each light stays green for $\frac{3}{4}$ s in the E-W direction & $\frac{1}{4}$ s in the N-S direction
- Switching instants of all the traffic lights are random & independent.

If we consider a certain person driving a car arriving at any traffic light randomly in the E-W direction, the probability that he will have a green signal is $\frac{3}{4}$, i.e., on the average 75% of the time he will observe a green light.

(sample-function statistics)

↳ Need to look at a single traffic light?

same statistical information.

If we consider a large number of drivers each of them arriving at any random traffic light in the E-W direction simultaneously, then 75% of the drivers will observe green lights & 25% will observe red lights.

(ensemble statistics at one instant).

~~The~~ Moreover, if a large number of drivers arrive randomly many times at their chosen traffic light, the experience of each driver will have the same

Transmission of a random process through a linear filter:

$$X_t \longrightarrow \boxed{h(t,\tau)} \longrightarrow Y_t$$

$$Y_t = \int_{-\infty}^{\infty} h(t,\tau)\, X_{t-\tau}\, d\tau$$

Linear time-varying system.

$$m_y(t) = E[Y_t] = E\left[\int_{-\infty}^{\infty} h(t,\tau)\, X_{t-\tau}\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(t,\tau)\, m_x(t-\tau)\, d\tau$$

(Time-invariant channel) $$= \int_{-\infty}^{\infty} h(\tau)\, m_x(t-\tau)\, d\tau$$

(W.S.S) $$= \int_{-\infty}^{\infty} h(\tau)\, m_x\, d\tau = m_x \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$= m_y$$

If X_t is W.S.S. and $h(t,\tau) = h(\tau)$, Y_t has a constant mean.

$$R_{YX}(t,s) = E[Y_t X_s]$$

$$= E\left[\int_{-\infty}^{\infty} X_s\, h(t,\gamma)\, X_{t-\gamma}\, d\gamma\right]$$

(Time-invariant $= \int^{\infty} h(\gamma)\, E[X_t X_{\ldots}]\, d\gamma$

$$= \int_{-\infty}^{\infty} h(\tau)\, R_X(s, t-\tau)\, d\tau$$

(X is W.S.S) $$= \int_{-\infty}^{\infty} h(\tau)\, R_X(t-s-\tau)\, d\tau$$

define $t-s=\tau$. (i.e $R_{YX}(t,s) = R_{YX}(t-s) = R_{YX}(\tau)$)

$$R_{YX}(t,s) = R_{YX}(t-s) = R_{YX}(\tau)$$

$$= \int_{-\infty}^{\infty} h(\nu)\, R_X(\tau-\nu)\, d\nu .$$

$$R_{YX}(\tau) = R_X(\tau) * h(\tau)$$

convolution of $R_X(\cdot)$ & $h(\cdot)$.

(X_t & Y_t are jointly W.S.S.)

$$R_Y(t,s) = E[Y_t Y_s]$$

$$= E\left[Y_t \int_{-\infty}^{\infty} h(t,\nu)\, X_{s-\nu}\, d\nu \right]$$

$$= \int_{-\infty}^{\infty} h(t,\nu)\, E[Y_t X_{s-\nu}]\, d\nu$$

($h(t,\nu)$ is time-invariant) $$= \int_{-\infty}^{\infty} h(\nu)\, R_{YX}(t-s+\nu)\, d\nu$$

$$= \int_{-\infty}^{\infty} h(\nu)\, R_{YX}(\tau+\nu)\, d\nu$$

($\tau+\nu=\alpha$) $$= \int_{-\infty}^{\infty} h(\alpha-\tau)\, R_{YX}(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} h(-(\tau-\alpha))\, R_{YX}(\alpha)\, d\alpha$$

Alternate simplification:
Let $\alpha=-\nu$

$$\int_{-\infty}^{\infty} h(-\alpha) R_{YX}(\tau-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \tilde{h}(\alpha) R_{YX}(\tau-\alpha)\, d\alpha$$

$$= R_{YX}(\tau) * \tilde{h}(\tau)$$

$$R_Y(t,s) = R_Y(t-s) = R_Y(\tau) = R_{YX}(\tau) * h(-\tau)$$

$$R_X(\cdot) \longrightarrow \boxed{h(\cdot)} \xrightarrow{R_{YX}(\cdot)} \boxed{\tilde{h}(\cdot)} \longrightarrow R_Y(\cdot)$$

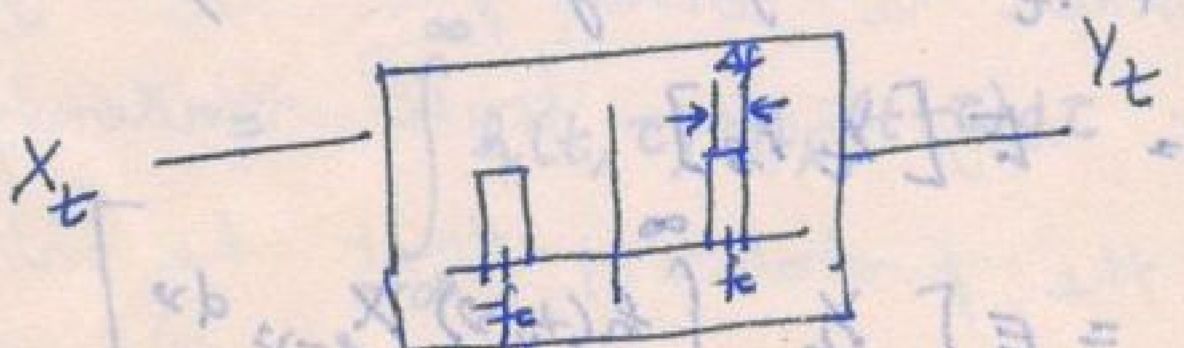
$$\tilde{h}(t) = h(-t).$$

Power Spectral density (PSD):

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau.$$

Fourier transform of the autocorrelation function.

Interpretation:



$$E[Y_t^2] \approx 2\Delta f\, S_X(f_c).$$

$S_X(f_c)$ represents the frequency density of the average power in the random process X_t evaluated at the frequency $f = f_c$. The dimensions of the power spectral density are therefore in W/Hz.

Properties of the Power Spectral Density:

1) $\int_{-\infty}^{\infty} R_X(\tau)\, d\tau = S_X(0)$

3) $S_X(f)$ is real & $S_X(f) \geq 0$ for all f.
($R_X(\tau)$ is even)

Suppose it is ~~not~~ negative for some interval $[f_0, f_0+\Delta f]$. Filter the process using an ideal BPF in $[f_0, f_0+\Delta f]$.

$$E[Y_t^2] = \int_{f_0}^{f_0+\Delta f} S_X(f)\,df < 0$$ not possible. since $E[Y_t^2] \geq 0$.

4) $S_X(-f) = S_X(f)$ for a real random process.

$\boxed{\text{Lecture 25 :}}$

Examples:

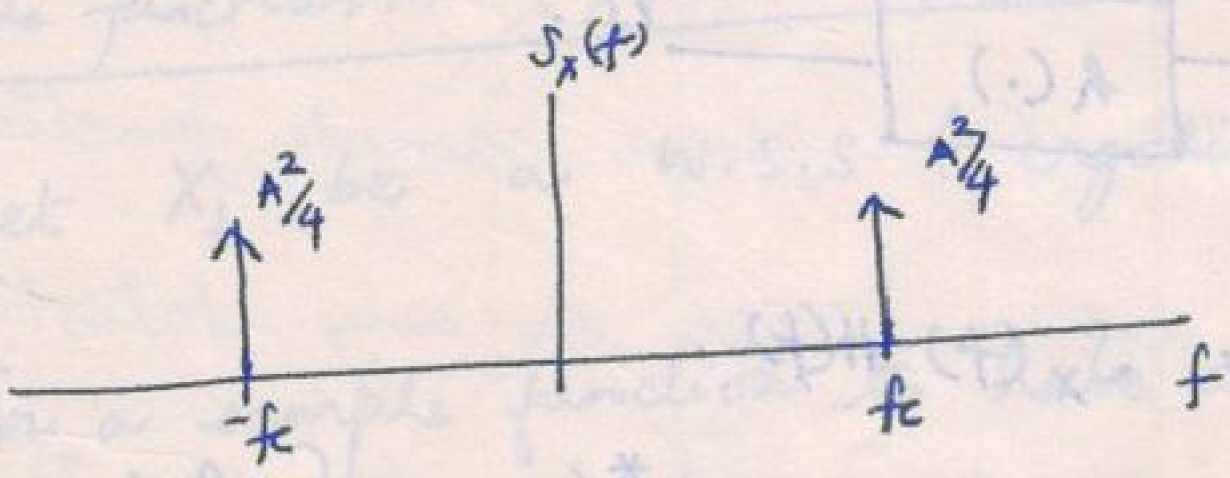
① Sine wave with random phase

$$X_t = A\cos(2\pi f_c t + \theta)$$

A, f_c - constants $\quad \theta \sim \text{uniform}[0, 2\pi]$

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

$$S_X(f) = \frac{A^2}{4}\left[\delta(f-f_c) + \delta(f+f_c)\right].$$



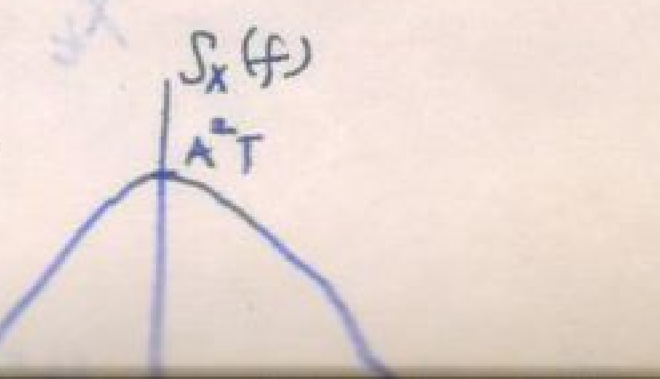
② Random binary wave.

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right) & \text{for } |\tau| \leq T \\ 0 & \text{else} \end{cases}$$

$A\,\text{rect}(T) \rightleftharpoons AT\,\text{sinc}\,fT.$

$R_X(\tau) = \frac{1}{T}\left(A\,\text{rect}(\tau) * A\,\text{rect}(\tau)\right)$

$$S_X(f) = A^2 T\,\text{sinc}^2(fT)$$



3) Mixing of a random process with a sine wave.

$$Y_t = X_t \cos(2\pi f_c t + \theta)$$

$\theta \sim$ uniform $[0, 2\pi]$. indep. of X_t for all t.

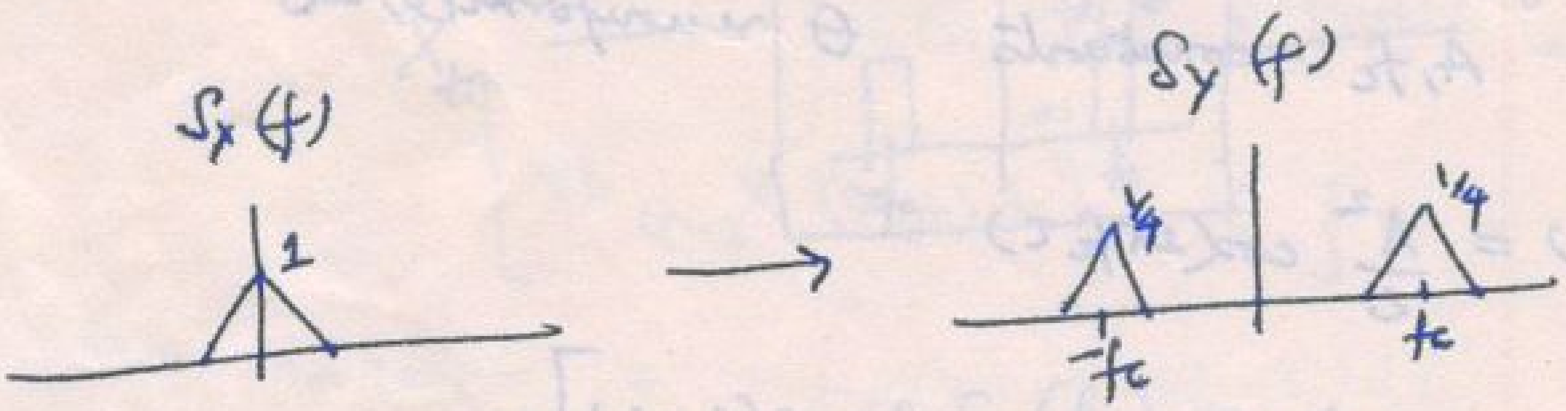
$$R_Y(\tau) = E[Y_{t+\tau} Y_t]$$

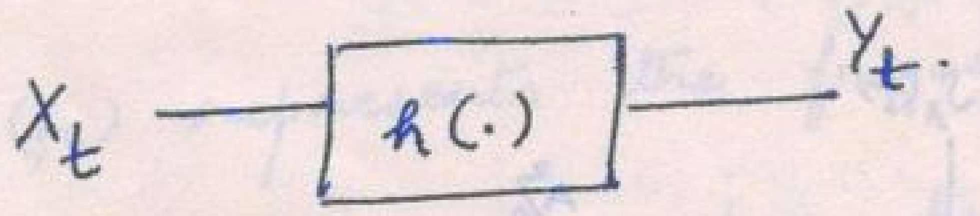
$$= E[X_{t+\tau} \cos(2\pi f_c(t+\tau)+\theta)\, X_t \cos(2\pi f_c t+\theta)]$$

$$= E[X_{t+\tau} X_t]\, E\left[\tfrac{1}{2}\left(\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau\right)\right]$$

$$= \frac{1}{2} R_X(\tau) \cos 2\pi f_c \tau.$$

$$S_Y(f) = \frac{1}{4}\left[S_X(f-f_c) + S_X(f+f_c)\right]$$



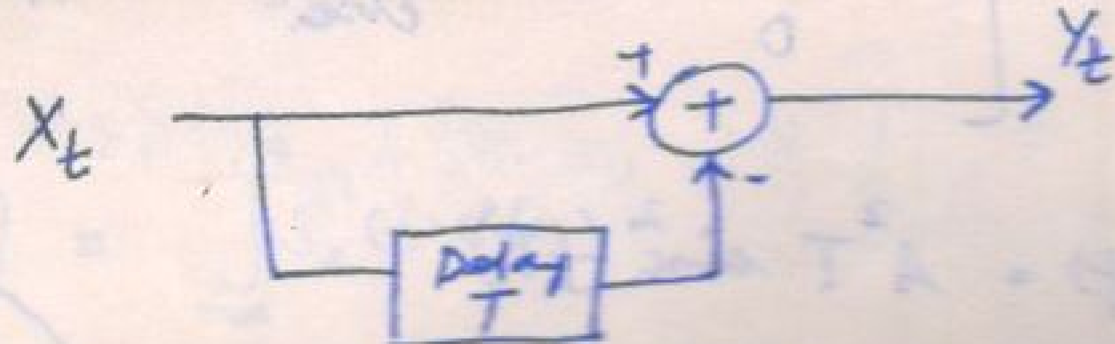


$$S_{YX}(f) = S_X(f) H(f).$$

$$S_Y(f) = S_X(f) H(f) H^*(f) \qquad \left\{ \text{where } H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt \right\}$$

$$= S_X(f) |H(f)|^2.$$

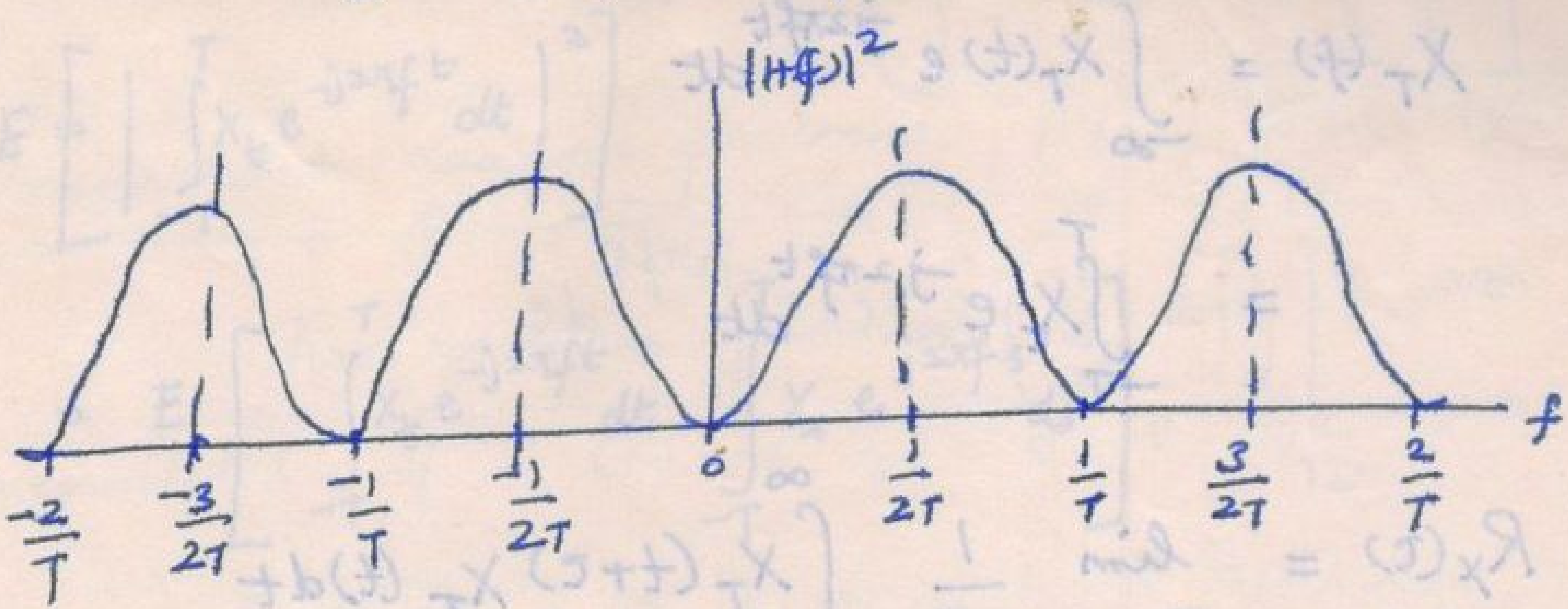
Example: Comb filter. (Skip this)



$-j2\pi f T$

$$|H(f)|^2 = (1-\cos 2\pi f T)^2 + (\sin 2\pi f t)^2$$

$$= 2(1-\cos 2\pi f T) = 4 \sin^2 \pi f T.$$



$$S_y(f) = 4 \sin^2(\pi f T)\, S_x(f).$$

For small frequencies such that $\pi f \ll \frac{1}{T}$, we have

$$\sin \pi f T \approx \pi f T.$$

$$\Rightarrow \quad S_y(f) \approx 4\pi^2 f^2 T^2 S_x(f)$$

(Differentiator for low frequency inputs).

→ <u>Relation among the PSD and amplitude spectrum of a sample functions.</u> (Do this after bandlimited r.p & theorem).

Let X_t be a W.S.S & ergodic random process.

For a sample function to have a Fourier transform, it must be absolutely integrable, i.e,

$$\int_{-\infty}^{\infty} |X_t|\, dt < \infty \quad \text{— (A)}$$

This is never satisfied for a W.S.S. process.

Therefore, we define

$$X_T(t) = \begin{cases} X_t & -T \le t \le T \\ 0 & \text{else.} \end{cases}$$

provided T is finite and X_t is a W.S.S. process with finite mean square value.

$$X_T(f) = \int_{-\infty}^{\infty} X_T(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-T}^{T} X_t\, e^{-j2\pi f t}\, dt$$

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} X_T(t+\tau)\, X_T(t)\, dt$$
(ergodic)

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 e^{j2\pi f \tau}\, df$$

$$\left(\text{since} \quad \int_{-\infty}^{\infty} X_T(t+\tau)\, X_T(t)\, dt \overset{F^{-1}}{\underset{F}{\rightleftharpoons}} |X_T(f)|^2 \right).$$

$\dfrac{|X_T(f)|^2}{2T}$ has the same units as PSD and is called the <u>periodogram.</u>

For a fixed frequency f, the periodogram is a random variable. For a given sample function, $\dfrac{|X_T(f)|^2}{2T}$ does not converge in any statistical sense to a limiting value as $T\to\infty$. Therefore, it would be incorrect to interchange the order of the integration & the limit. However, if we take expectation of both sides, we get

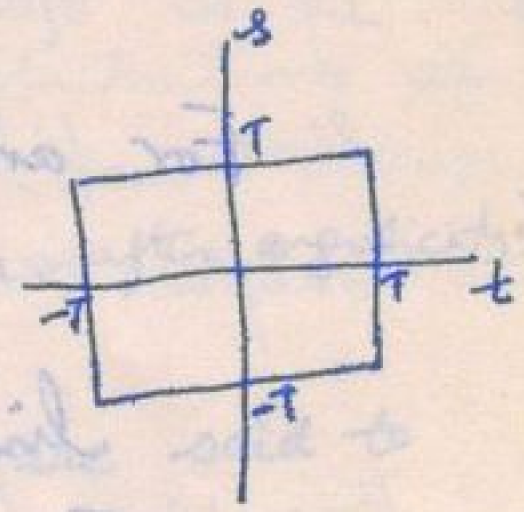
$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} E\left[|X_T(f)|^2\right] e^{j2\pi f \tau}\, df.$$

$$\Rightarrow S_X(f) = \lim_{T\to\infty} \frac{1}{2T} E\left[\left|\int_{-T}^{T} X_t e^{-j2\pi f t}\,dt\right|^2\right]$$

$$E\left[\left|\int_{-T}^{T} X_t e^{-j2\pi f t}\,dt\right|^2\right]$$

$$= E\left[\int_{-T}^{T} X_t e^{-j2\pi f t}\,dt \int_{-T}^{T} X_s e^{j2\pi f s}\,ds\right]$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_X(t,s)\, e^{-j2\pi f(t-s)}\,dt\,ds$$

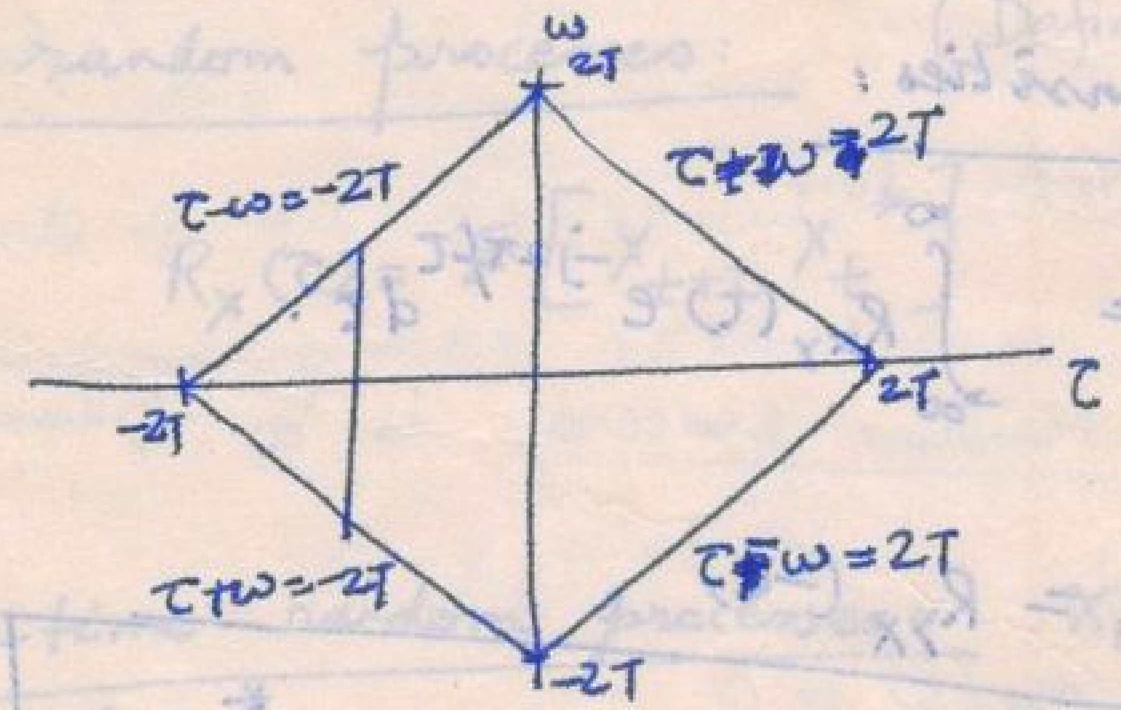


$$= \int_{-T}^{T}\int_{-T}^{T} R_X(t-s)\, e^{-j2\pi f(t-s)}\,dt\,ds$$

Let $\omega = t+s$

$\tau = t-s$

$$|J| = \left|\det\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right| = 2.$$



$t=T$

$\Rightarrow \tau = T-s$

$s = \omega - T$

$\Rightarrow \tau = 2T-\omega$

$t=-T$

$\Rightarrow \tau = -T-s$

$s = \omega + T$

$\tau = -2T-\omega$

for $\tau<0$

$$= \frac{1}{2}\int_{-2T}^{2T}\int_{-2T-\tau}^{+2T+\tau} R_X(\tau)\, e^{-j2\pi f\tau}\,d\omega\,d\tau$$

for $\tau>0$

$$= \frac{1}{2}\int_{-2T}^{2T}\int_{-2T+\tau}^{2T-\tau} R_X(\tau)\, e^{-j2\pi f\tau}\,d\omega\,d\tau$$

$$= \frac{1}{2}\int^{2T} 2(2T-|\tau|)\, R_X(\tau)\, e^{-j2\pi f\tau}\,d\tau$$

$$\frac{1}{2T} E\left[\left|\int_{-T}^{T} X_t e^{-j2\pi f t} dt\right|^2\right]$$

$$= \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) R_X(\tau) e^{-j2\pi f \tau} d\tau$$

For an integrable $|\tau R_X(\tau)|$,

$$\lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau = \underline{\underline{S_X(f)}}.$$

<u>Cross Spectral densities:</u>

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f \tau} d\tau.$$

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

$$\Rightarrow \boxed{S_{XY}(f) = S_{YX}(-f) = S_{YX}^{*}(f)}$$

(Assuming X_t & Y_t are real r.p.s)

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(-\tau) e^{-j2\pi f \tau} d\tau$$

$$\int_{-\infty}^{\infty} \ldots e^{j2\pi f \tau}$$

Assuming X_t & Y_t are real r.p's, $R_{YX}(\tau) = R_{YX}^{*}(\tau)$.

$$\rightarrow S_{YX}^{*}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{j2\pi f\tau} d\tau = S_{YX}(-f)$$

Uncorrelated random processes: (Done earlier: after ergodicity)

Two random processes X_t and Y_t are uncorrelated if $Cov(X_t, Y_s) = 0 \;\forall\; t, s$ (or)

$$R_{XY}(t,s) = m_X(t)\, m_Y(s) \;\forall\; t, s.$$

Independent random processes: (Done earlier: after ergodicity).

Two random processes X_t and Y_t are said to be independent if random vectors $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(Y_{s_1}, Y_{s_2}, \ldots, Y_{s_m})$ are independent for all $t_i \in T$, $s_i \in T$, m, n.

Complex random processes: (Defined earlier)

$$R_X(\tau) = E\left[X_{t+\tau} X_t^{*}\right]$$

Discrete-time random processes: X_n, $n = \ldots, -3, -1, 0, 1, 2, \ldots$

$$m_X(n) = E[X_n]$$

$$R_X(n,k) = E\left[X_n X_k\right]$$

For a W.S.S. process, $R_X(n,k)$ is a fn. of $(n-k)$ only.

$$S_X(\omega) = \sum_{m=-\infty}^{\infty} R_X(m)\, e^{-j\omega m} \quad \text{(DTFT)}$$

Lecture 26:

Gaussian random processes:

A random process is said to be a Gaussian random process if all finite dimensional distributions of the process are jointly Gaussian.

* For a Gaussian process, W.S.S $\Rightarrow$ stationarity

* If a Gaussian process X_t is applied to a stable linear filter, then the random process Y_t at the output of the filter is also Gaussian.

Examples: (Do these examples after noise)

① A Wiener process is a Gaussian r.p. with

$$m_X(t) = mt$$

$$R_X(t,s) = \sigma^2 \min(t,s) + m^2 t s \quad \text{for } t,s \geq 0.$$

Not W.S.S.

② A Gauss-Markov process is a Gaussian r.p with

$$m_X(t) = 0$$

$$R_X(t,s) = \sigma^2 e^{-\beta|t-s|} \quad \sigma^2 > 0, \beta > 0$$

W.S.S ($\Rightarrow$ stationarity)

$$X_{t_1}, X_{t_2}, X_{t_3}, \ldots \quad t_1 \leq t_2 < t_3 \ldots$$

$$f_{X_{t_3}|X_{t_2}, X_{t_1}} = f_{X_{t_3}|X_{t_2}}.$$

Noise:

The term <u>noise</u> is used customarily to designate unwanted waves that tend to disturb the transmission and processing of signals in communication systems, and over which we have incomplete control.

Sources of noise: External or Internal

e.g. atmospheric, galactic, man-made — shot noise, thermal noise (due to spontaneous fluctuations of current & voltage in electrical circuits).

White noise:

The noise analysis of communication systems is customarily based on an idealized form of noise called white noise, the psd of which is Independent of the operating frequency. The adjective white is used in the sense that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation.

$$S_W(f) = \frac{N_0}{2} \quad \text{for all } f$$

(W_t : white noise process)

The factor $\frac{1}{2}$ has been included to indicate that half the power is associated with +ve frequency & half with -ve frequency.

$$N_0 = k\,T_e \qquad (\text{Watts}/\text{Hz})$$

T_e: equivalent noise temperature of the sys

The <u>equivalent noise temperature</u> of a system is defined as the temperature at which a noisy resistor has to be maintained such that, by connecting the resistor to the input of a noiseless version of the system, it produces the same available noise power at the output of the system as that produced by all the sources of noise in the actual system.

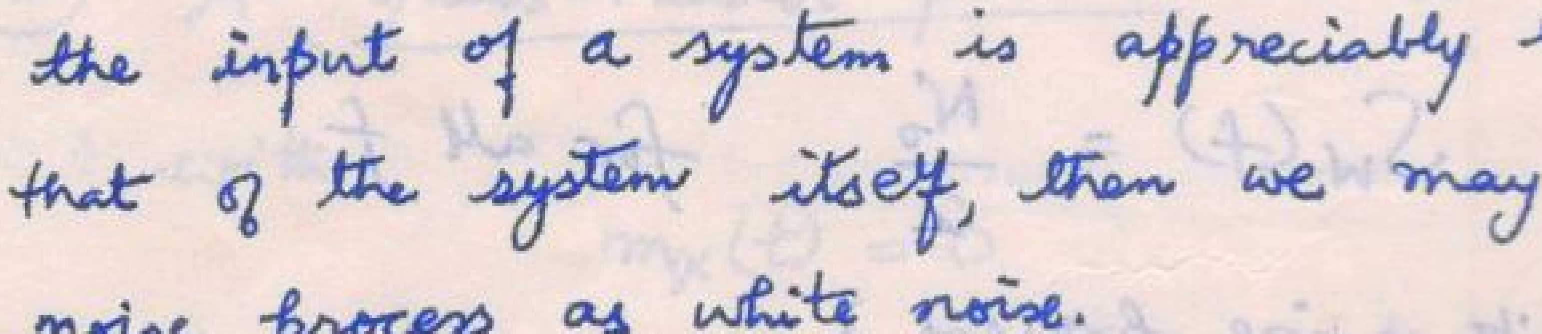
$$R_W(\tau) = \frac{N_0}{2}\,\delta(\tau)$$

* Any two different samples of white noise, no matter how closely together in time they are taken, are uncorrelated. If the white noise is also Gaussian, then the samples are statistically independent.

* Strictly speaking, white noise has infinite variance and is not physically realizable. However, it is a convenient mathematical tool for system analysis.

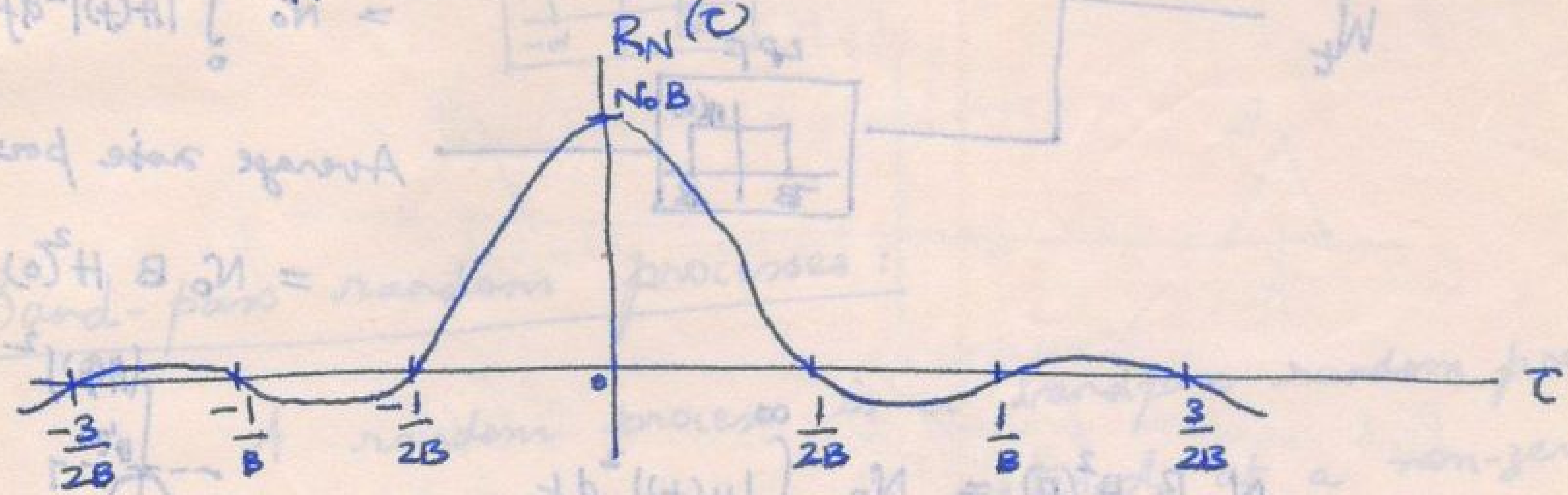
* As long as the bandwidth of the noise process at the input of a system is appreciably larger than that of the system itself, then we may model the noise process as white noise.

<u>Example</u>: Ideal low-pass filtered white noise



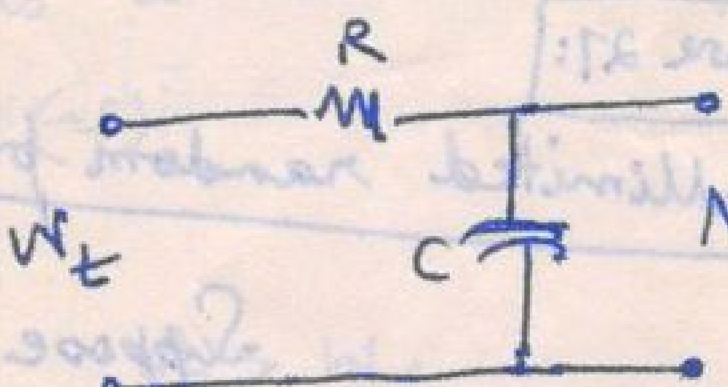
$$S_N(f) = \begin{cases} \frac{N_0}{2} & -B \le f \le B \\ 0 & \text{else} \end{cases}$$

$$R_N(\tau) = N_0 B \, \text{sinc}(2B\tau).$$



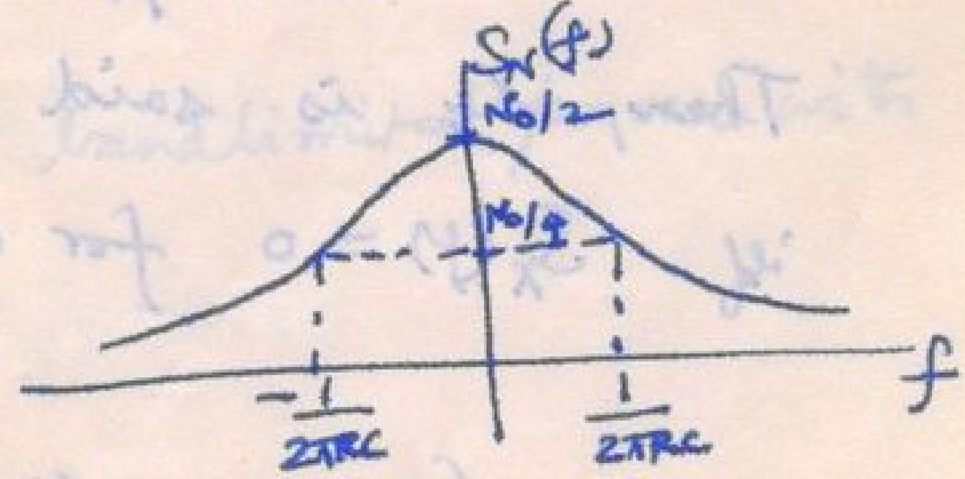
Suppose N_t is sampled at $2B$ samples/sec, the resulting noise samples are uncorrelated and independent. Each such noise sample has zero-mean and variance N_0B. (Discrete white noise)

<u>Example:</u> RC low-pass filtered white noise



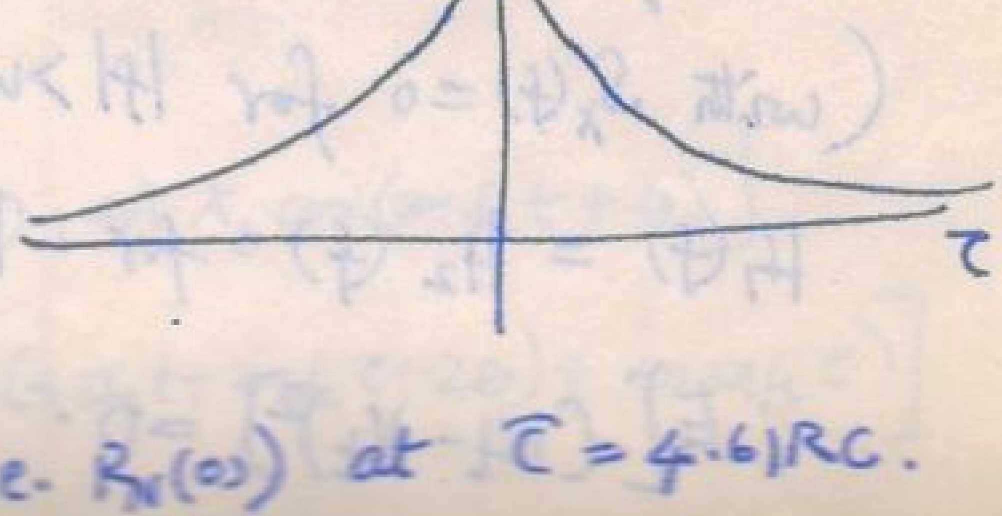
$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$S_N(f) = \frac{N_0/2}{1 + (2\pi fRC)^2}$$



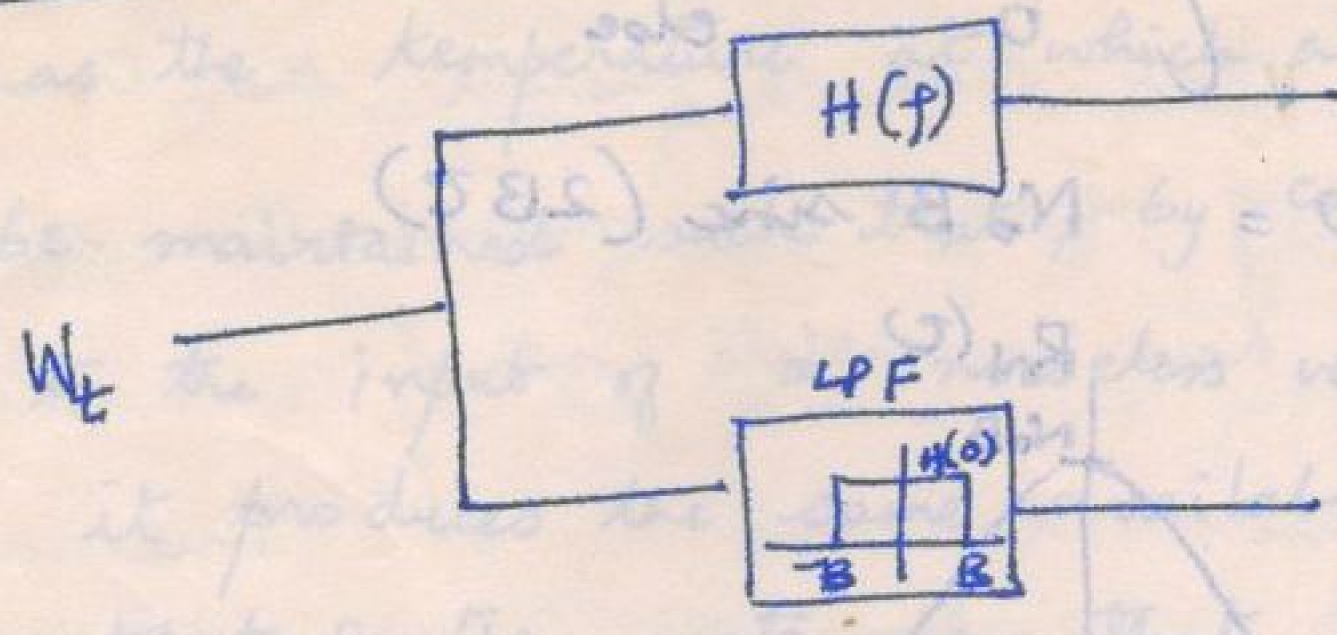
$$e^{-|\tau|} \rightleftharpoons \frac{2}{1+(2\pi f)^2}$$

$$\Rightarrow R_N(\tau) = \frac{N_0}{4RC} e^{-|\tau|/RC}$$



$R_N(\tau)$ drops to 1% of $\frac{N_0}{4RC}$ (i.e. $R_N(0)$) at $\tau = 4.61RC$.

Noise equivalent bandwidth :



Average noise power

$= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$

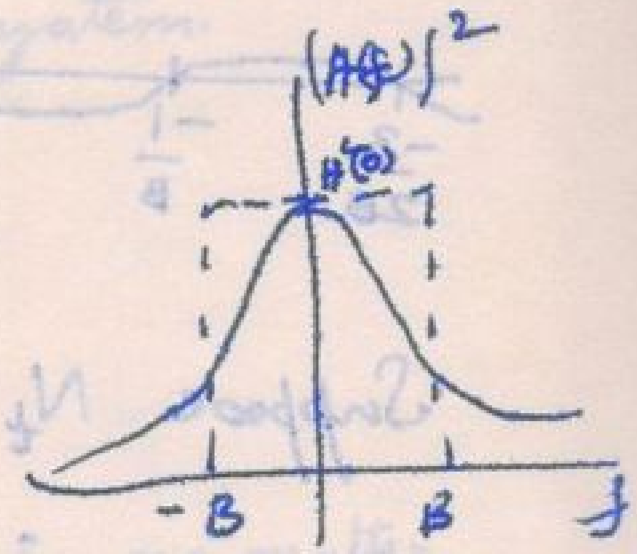
$= N_0 \int_{0}^{\infty} |H(f)|^2 df$

Average noise power

$= N_0 B H^2(0)$

$$N_0 B H^2(0) = N_0 \int_0^{\infty} |H(f)|^2 df$$

$$\Rightarrow B = \frac{\int_0^{\infty} |H(f)|^2 df}{H^2(0)}$$



Similar definition of noise equivalent bandwidth for a bandpass filter is also possible.

$\boxed{\text{Lecture 27:}}$

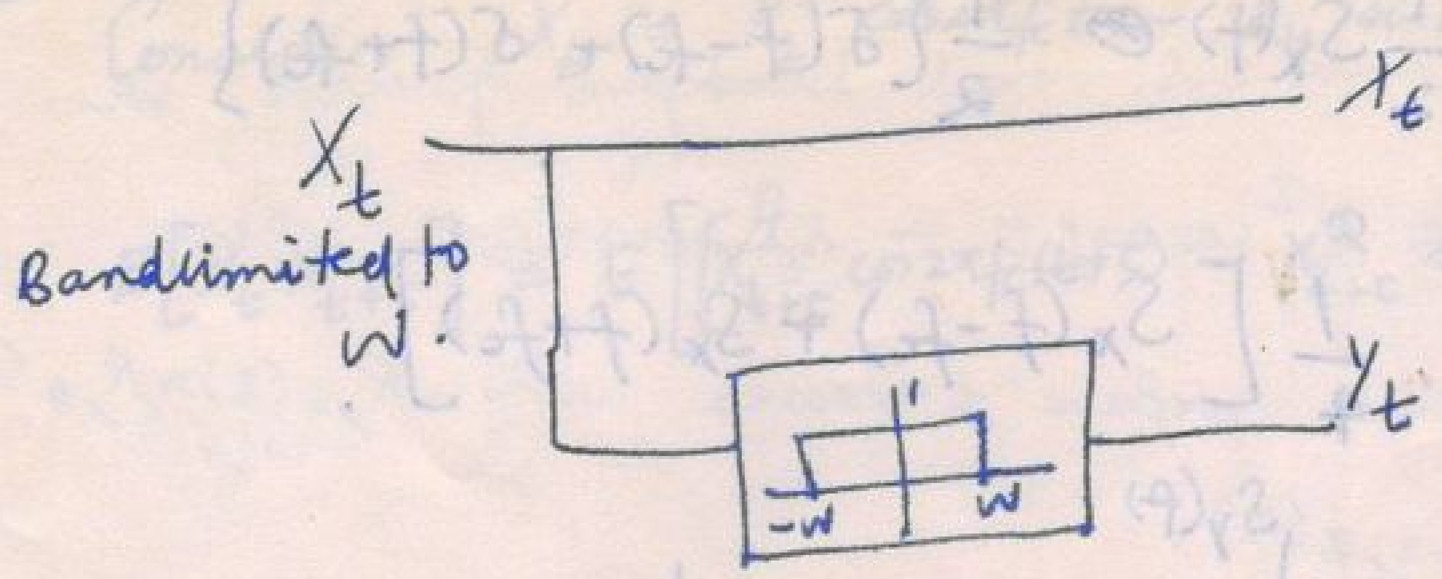
Bandlimited random processes: (Do this after doing properties of PSD)

Suppose X_t is a W.S.S. random process. Then, X_t is said to be bandlimited with bandwidth W if $S_X(f) = 0$ for all $|f| > W$.

Theorem (from Papoulis p. 376)

Suppose that W_t and V_t are the responses of the systems $H_1(f)$ and $H_2(f)$ to a bandlimited process X_t (with $S_X(f) = 0$ for $|f| > W$). We can show that if $H_1(f) = H_2(f)$ for $|f| \le W$, then $W_t \overset{ms}{=} V_t$ i.e.,

$E[(W_t - V_t)^2] = 0$.

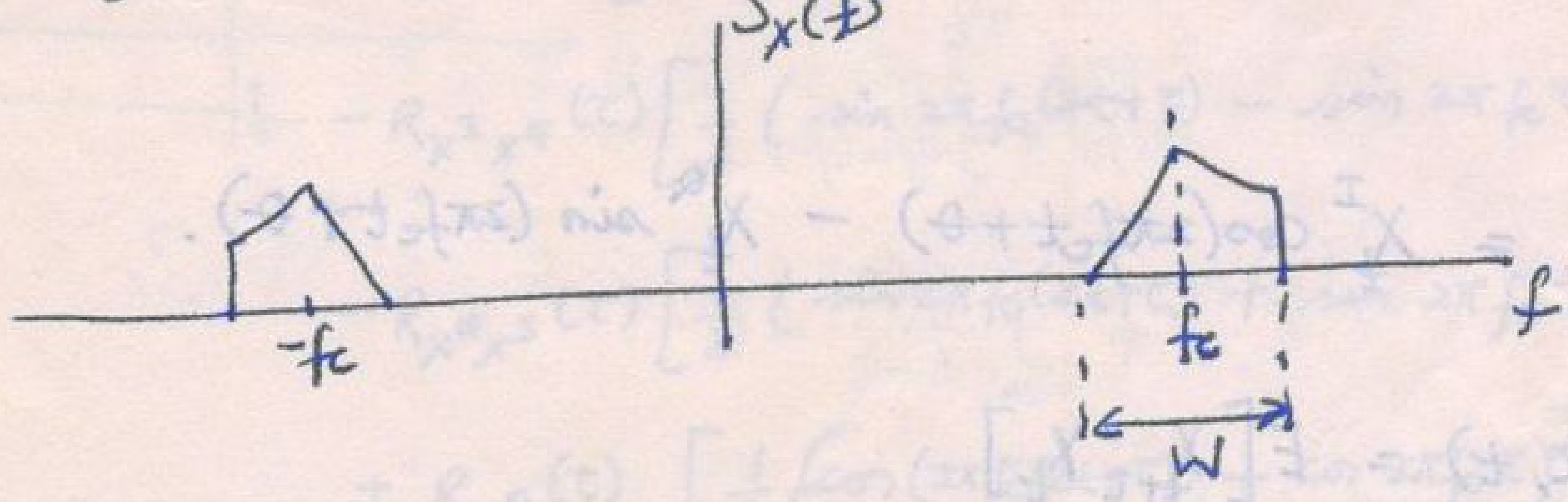


$$E\left[(X_t - Y_t)^2\right] = 0.$$

$$X_t \overset{m.s}{=} Y_t$$

Band-pass random processes:

A random process is a bandpass random process if it's one-sided spectrum is centered at a non-zero frequency f_c and it does not extend in frequency to 0.

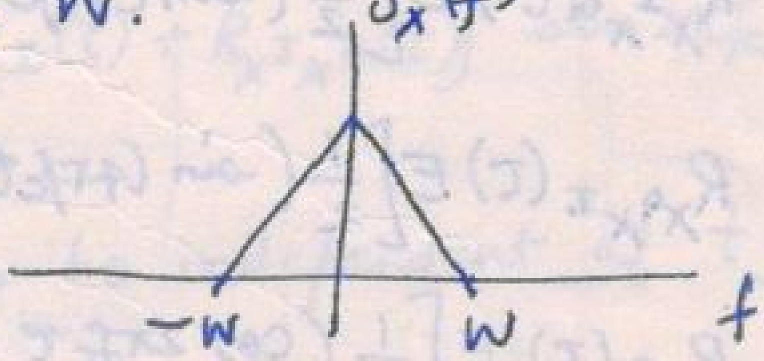


If $W \ll f_c$, then the process is a narrowband process.

Real random process $\Rightarrow S_X(f) = S_X(-f)$.

Eg: (1) $Y_t = X_t \cos(2\pi f_c t + \theta)$. $f_c > W$.

where X_t is a w.s.s. lowpass bandlimited process with bandwidth W.



$\theta \sim$ uniform$[0, 2\pi]$ & independent of X_t for all t.

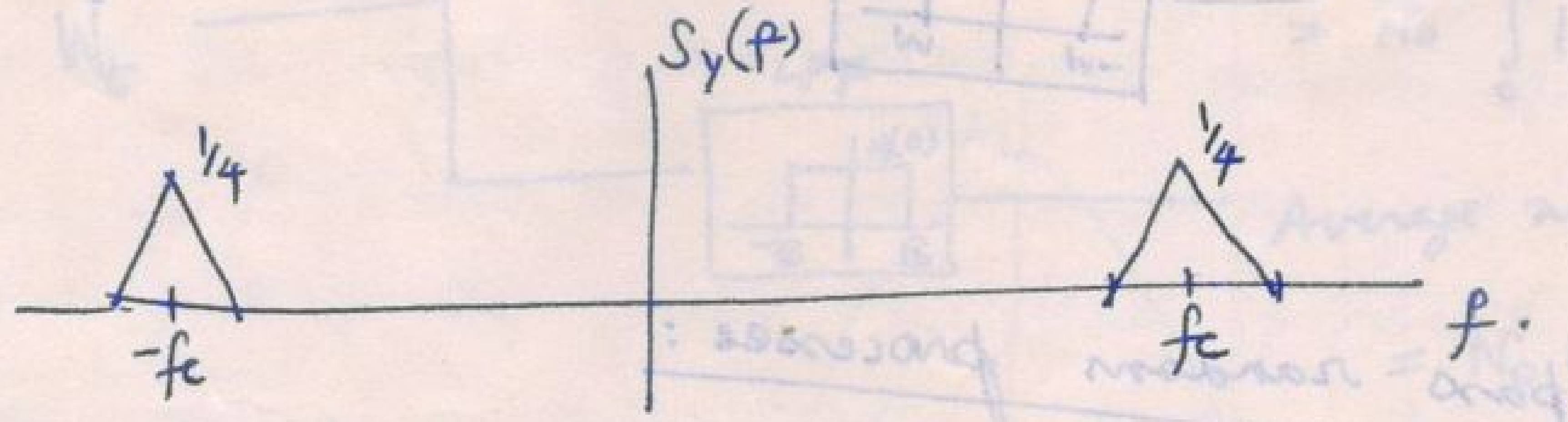
$$R_Y(\tau) = E\left[Y_{t+\tau} Y_t\right]$$

$$= E\left[X_{t+\tau} \cos(2\pi f_c(t+\tau) + \theta) X_t \cos(2\pi f_c t + \theta)\right]$$

$$= R_X(\tau) E\left[\frac{1}{2}\left(\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau\right)\right]$$

$$\Rightarrow S_Y(f) = \frac{1}{2} S_X(f) \circledast \frac{1}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right]$$

$$= \frac{1}{4}\left[S_X(f-f_c) + S_X(f+f_c)\right].$$



Real random process $\Rightarrow$ $S_X(f) = S_X(-f)$

Modulated real lowpass process $\Rightarrow$ $S_X(f)$ for $f \geq f_c - W$, $f \leq f_c + W$ symmetric about f_c.

(2) $$X_t = X_t^I \cos(2\pi f_c t + \theta) - X_t^Q \sin(2\pi f_c t + \theta).$$

$$R_X(t+\tau, t) = E\left[X_{t+\tau} X_t\right]$$

$$= E\Big[\left(X_{t+\tau}^I \cos(2\pi f_c(t+\tau) + \theta) - X_{t+\tau}^Q \sin(2\pi f_c(t+\tau)+\theta)\right)$$
$$\left(X_t^I \cos(2\pi f_c t + \theta) - X_t^Q \sin(2\pi f_c t + \theta)\right)\Big]$$

$$= R_{X^I}(\tau)\, E\left[\tfrac{1}{2}\left(\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau\right)\right]$$

$$- R_{X^I X^Q}(\tau)\, E\left[\tfrac{1}{2}\left(\sin(4\pi f_c t + 2\pi f_c \tau + 2\theta) - \sin 2\pi f_c \tau\right)\right]$$

$$- R_{X^Q X^I}(\tau)\, E\left[\tfrac{1}{2}\left(\sin(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \sin 2\pi f_c \tau\right)\right]$$

$$+ R_{X^Q}(\tau)\, E\left[\tfrac{1}{2}\left(\cos 2\pi f_c \tau - \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right)\right]$$

$$= \frac{1}{2}\left(R_{X^I}(\tau) + R_{X^Q}(\tau)\right)\cos 2\pi f_c \tau$$

$$+ \frac{1}{2}\left(R_{X^I X^Q}(\tau) - R_{X^Q X^I}(\tau)\right)\sin 2\pi f_c \tau$$

Consider $X_t = X_t^I \cos 2\pi f_c t - X_t^Q \sin 2\pi f_c t$

$$E[X_t X_{t+\tau}] = E\Big[\big(X_{t+\tau}^I \cos 2\pi f_c(t+\tau) - X_{t+\tau}^Q \sin 2\pi f_c(t+\tau)\big)\big(X_t^I \cos 2\pi f_c t - X_t^Q \sin 2\pi f_c t\big)\Big]$$

$$= R_{X^I}(\tau) \cos 2\pi f_c t \cos 2\pi f_c(t+\tau)$$
$$- R_{X^I X^Q}(\tau) \sin 2\pi f_c t \cos 2\pi f_c(t+\tau)$$
$$- R_{X^Q X^I}(\tau) \cos 2\pi f_c t \sin 2\pi f_c(t+\tau)$$
$$+ R_{X^Q}(\tau) \sin 2\pi f_c t \sin 2\pi f_c \tau.$$

$$= R_{X^I}(\tau)\Big[\tfrac{1}{2}\big(\cos 2\pi f_c(2t+\tau) + \cos 2\pi f_c \tau\big)\Big]$$
$$- R_{X^I X^Q}(\tau)\Big[\tfrac{1}{2}\big(\sin 2\pi f_c(2t+\tau) - \sin 2\pi f_c \tau\big)\Big]$$
$$- R_{X^Q X^I}(\tau)\Big[\tfrac{1}{2}\big(\sin 2\pi f_c(2t+\tau) + \sin 2\pi f_c \tau\big)\Big]$$
$$+ R_{X^Q}(\tau)\Big[\tfrac{1}{2}\big(\cos(2\pi f_c(2t+\tau)) + \cos 2\pi f_c \tau\big)\Big]$$

$$= \tfrac{1}{2}\big[R_{X^I}(\tau) + R_{X^Q}(\tau)\big] \cos 2\pi f_c \tau$$
$$+ \tfrac{1}{2}\big[R_{X^I X^Q}(\tau) - R_{X^Q X^I}(\tau)\big] \sin 2\pi f_c \tau$$
$$+ \tfrac{1}{2}\big[R_{X^I}(\tau) - R_{X^Q}(\tau)\big] \cos 2\pi f_c(2t+\tau)$$
$$- \tfrac{1}{2}\big[R_{X^Q X^I}(\tau) + R_{X^I X^Q}(\tau)\big] \sin 2\pi f_c(2t+\tau)$$

For $E[X_{t+\tau} X_t]$ to be independent of t, we need

$$R_{X^I}(\tau) = R_{X^Q}(\tau)$$
$$R_{X^Q X^I}(\tau) = -R_{X^I X^Q}(\tau).$$

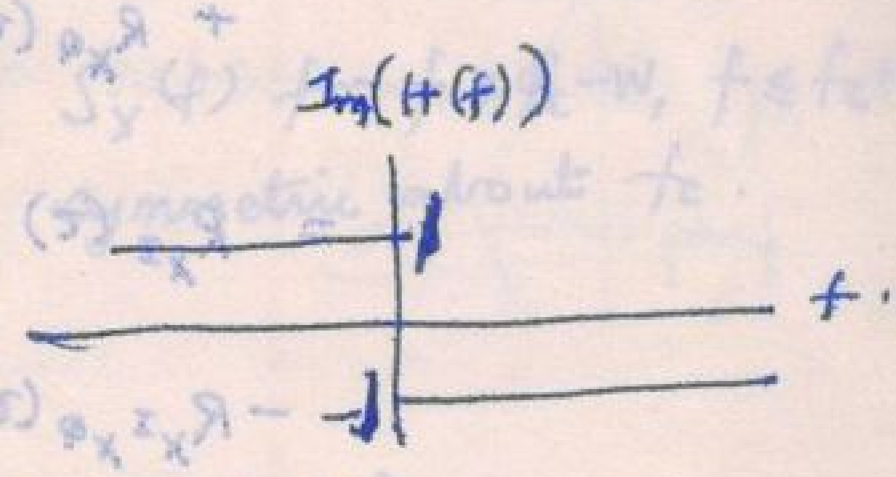
$$R_X(\tau) = R_{X^I}(\tau) \cos 2\pi f_c \tau + R_{X^I X^Q}(\tau) \sin 2\pi f_c \tau.$$

Lecture 28:

Suppose X_t is a W.S.S. real bandpass process, we can express $X_t = X_t^I \cos 2\pi f_c t - X_t^Q \sin 2\pi f_c t$ where X_t^I & X_t^Q are real lowpass processes with $R_{X^I}(\tau) = R_{X^Q}(\tau)$ & $R_{X^I X^Q}(\tau) = -R_{X^Q X^I}(\tau)$.

Hilbert transform:

$$g(t) \longrightarrow \boxed{H\cdot T} \longrightarrow \hat{g}(t)$$



$$h(t) = \frac{1}{\pi t}$$

$$H(f) = -j\,\mathrm{sgn}(f).$$

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau.$$

Inverse Hilbert transform:

$$g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{t-\tau}\, d\tau.$$

$$\hat{\hat{g}}(t) = -g(t)$$

$$X_t \longrightarrow \boxed{H\cdot T.} \longrightarrow \hat{X}_t$$

$$h(\tau) = \frac{1}{\pi\tau}, \quad h(\tau) = -h(-\tau)$$

$$* \quad R_{\hat{X}}(\tau) = R_X(\tau) * \underset{H\cdot T}{h(\tau)} * \underset{I.H\cdot T}{h(-\tau)} = R_X(\tau)$$

$$* \quad R_{X\hat{X}}(\tau) = -R_{\hat{X}X}(\tau)$$

$$* \quad S_{\hat{X}X}(f) = -j\, S_X(f)\, \mathrm{sgn}(f) \quad , \quad S_{X\hat{X}}(f) = j\, S_X(f)\, \mathrm{sgn}(f).$$

$$S_X(f) = S_{\hat{X}}(f)$$

Define $X_t^+ = X_t + j\hat{X}_t$

$$R_{X^+}(\tau) = E\left[X_{t+\tau}^+ \, X_t^{+*}\right]$$

$$= E\left[\left(X_{t+\tau} + j\hat{X}_{t+\tau}\right)\left(X_t - j\hat{X}_t\right)\right]$$

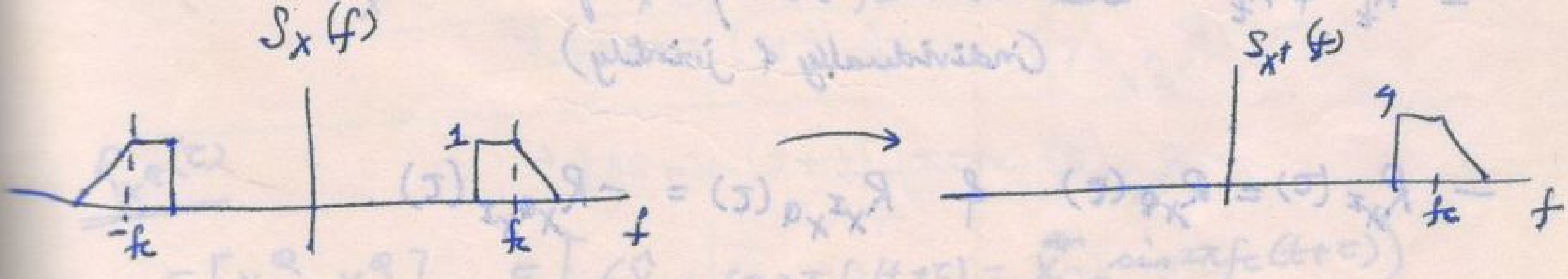
$$= R_X(\tau) - j R_{X\hat{X}}(\tau) + j R_{\hat{X}X}(\tau) + R_{\hat{X}}(\tau)$$

$$= 2R_X(\tau) + 2j R_{\hat{X}X}(\tau)$$

$$S_{X^+}(f) = 2S_X(f) + 2j\left(-j\, S_X(f)\, \mathrm{sgn}(f)\right)$$

$$= 2S_X(f)\left(1 + \mathrm{sgn}(f)\right)$$

$$= \begin{cases} 4\, S_X(f) & f > 0 \\ 0 & \text{else.} \end{cases}$$

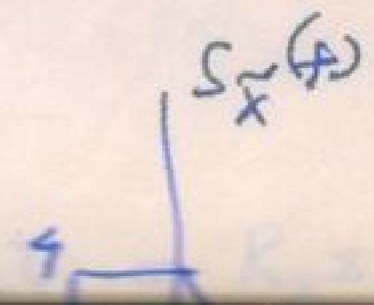


Define $\tilde{X}_t = X_t^+ e^{-j2\pi f_c t}$

$$R_{\tilde{X}}(\tau) = E\left[\tilde{X}_{t+\tau}\, \tilde{X}_t^*\right]$$

$$= E\left[X_{t+\tau}^+ e^{-j2\pi f_c(t+\tau)}\, X_t^{+*} e^{j2\pi f t}\right]$$

$$= R_{X^+}(\tau)\, e^{-j2\pi f_c \tau}$$



$$\Rightarrow S_{\tilde{X}}(f) = S_{X^+}(f + f_c)$$

$$\tilde{X}_t = (X_t + j\hat{X}_t)\, e^{-j2\pi f_c t}$$

$$= (X_t + j\hat{X}_t)(\cos 2\pi f_c t - j \sin 2\pi f_c t)$$

$$= \underbrace{(X_t \cos 2\pi f_c t + \hat{X}_t \sin 2\pi f_c t)}_{X_t^I} + j\,\underbrace{(\hat{X}_t \cos 2\pi f_c t - X_t \sin 2\pi f_c t)}_{X_t^Q}.$$

$$X_t = Re\left[\tilde{X}_t e^{j2\pi f_c t}\right]$$

$$= Re\left[(X_t^I + jX_t^Q)(\cos 2\pi f_c t + j\sin 2\pi f_c t)\right]$$

$$= X_t^I \cos 2\pi f_c t - X_t^Q \sin 2\pi f_c t.$$

$$\hat{X}_t = Im\left[\tilde{X}_t e^{j2\pi f_c t}\right]$$

$$= X_t^Q \cos 2\pi f_c t + X_t^I \sin 2\pi f_c t.$$

We can show

- X_t^I & X_t^Q are W.S.S low pass processes (individually & jointly)

- $R_{X^I}(\tau) = R_{X^Q}(\tau)$ & $R_{X^I X^Q}(\tau) = -R_{X^Q X^I}(\tau)$.

- $$S_{X^I}(f) = S_{X^Q}(f) = \begin{cases} S_X(f-f_c) + S_X(f+f_c) & |f| \le W \\ 0 & \text{else.} \end{cases}$$

$$S_{X^I X^Q}(f) = -S_{X^Q X^I}(f) = \begin{cases} j(S_X(f+f_c) - S_X(f-f_c)) & |f| \le W \\ 0 & \text{else.} \end{cases}$$

$\boxed{\text{Lecture 29:}}$

$$- \; S_{X^I}(f) = \frac{1}{4}\left[S_{\tilde{X}}(f) + S_{\tilde{X}}(-f) \right] = S_{X^Q}(f)$$

$$S_{X^I X^Q}(f) = \frac{j}{4}\left[S_{\tilde{X}}(f) - S_{\tilde{X}}(-f) \right] = - S_{X^Q X^I}(f)$$

$\underline{\underline{R_{X^I}(\tau)}}$

$$E\left[X^I_{t+\tau} X^I_t \right] = E\Big[\left(X_{t+\tau} \cos 2\pi f_c(t+\tau) + \hat{X}_{t+\tau} \sin 2\pi f_c(t+\tau) \right) \left(X_t \cos 2\pi f_c t + \hat{X}_t \sin 2\pi f_c t \right) \Big]$$

$$= R_X(\tau) \cos 2\pi f_c(t+\tau) \cos 2\pi f_c t$$
$$+ R_{X\hat{X}}(\tau) \cos 2\pi f_c(t+\tau) \sin 2\pi f_c t$$
$$+ R_{\hat{X}X}(\tau) \sin 2\pi f_c(t+\tau) \cos 2\pi f_c t$$
$$+ R_{\hat{X}}(\tau) \sin 2\pi f_c(t+\tau) \cos 2\pi f_c t$$

We know $R_X(\tau) = R_{\hat{X}}(\tau)$ & $R_{X\hat{X}}(\tau) = - R_{\hat{X}X}(\tau)$

Therefore,

$$R_{X^I}(\tau) = R_X(\tau) \cos 2\pi f_c \tau + R_{\hat{X}X}(\tau) \sin 2\pi f_c \tau .$$

$\underline{\underline{R_{X^Q}(\tau)}}$

$$E\left[X^Q_{t+\tau} X^Q_t \right] = E\Big[\left(\hat{X}_{t+\tau} \cos 2\pi f_c(t+\tau) - X_{t+\tau} \sin 2\pi f_c(t+\tau) \right) \left(\hat{X}_t \cos 2\pi f_c t - X_t \sin 2\pi f_c t \right) \Big]$$

$$= R_{\hat{X}}(\tau) \cos 2\pi f_c(t+\tau) \cos 2\pi f_c t$$
$$- R_{\hat{X}X}(\tau) \cos 2\pi f_c(t+\tau) \sin 2\pi f_c t$$
$$- R_{X\hat{X}}(\tau) \sin 2\pi f_c(t+\tau) \cos 2\pi f_c t$$
$$+ R_X(\tau) \sin 2\pi f_c(t+\tau) \sin 2\pi f_c t$$

$$= R_X(\tau) \cos 2\pi f_c \tau + R_{\hat{X}X}(\tau) \sin 2\pi f_c \tau = R_{X^I}(\tau)$$

$\underline{\underline{R_{X^I X^Q}(\tau)}}$

$$E\left[X^I_{t+\tau}X^Q_t\right] = E\left[\left(X_{t+\tau}\cos 2\pi f_c(t+\tau) + \hat{X}_{t+\tau}\sin 2\pi f_c(t+\tau)\right)\left(\hat{X}_t\cos 2\pi f_c t - X_t \sin 2\pi f_c t\right)\right]$$

$$= R_{X\hat{X}}(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t$$
$$- R_X(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t$$
$$+ R_{\hat{X}}(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t$$
$$- R_{\hat{X}X}(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t$$

$$= -R_{\hat{X}X}(\tau)\cos 2\pi f_c\tau + R_X(\tau)\sin 2\pi f_c\tau.$$

$\underline{\underline{R_{X^Q X^I}(\tau)}}$

$$E\left[X^Q_{t+\tau}X^I_t\right] = E\left[\left(\hat{X}_{t+\tau}\cos 2\pi f_c(t+\tau) - X_{t+\tau}\sin 2\pi f_c(t+\tau)\right)\left(X_t\cos 2\pi f_c t + \hat{X}_t \sin 2\pi f_c t\right)\right]$$

$$= R_{\hat{X}X}(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t$$
$$+ R_{\hat{X}}(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t$$
$$- R_X(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t$$
$$- R_{X\hat{X}}(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t$$

$$= R_{\hat{X}X}(\tau)\cos 2\pi f_c\tau - R_X(\tau)\sin 2\pi f_c\tau$$

$$= -R_{X^I X^Q}(\tau).$$

Since $R_{X^I}(\tau) = R_{X^Q}(\tau)$, $S_{X^I}(f) = S_{X^Q}(f)$

$$R_{X^I X^Q}(\tau) = -R_{X^Q X^I}(\tau) \Rightarrow S_{X^I X^Q}(f) = -S_{X^Q X^I}(f).$$

$\underline{\underline{S_{X^I}(f)}}$

$$R_{X^I}(\tau) = R_X(\tau)\cos 2\pi f_c \tau + R_{\hat{X}X}(\tau)\sin 2\pi f_c \tau.$$

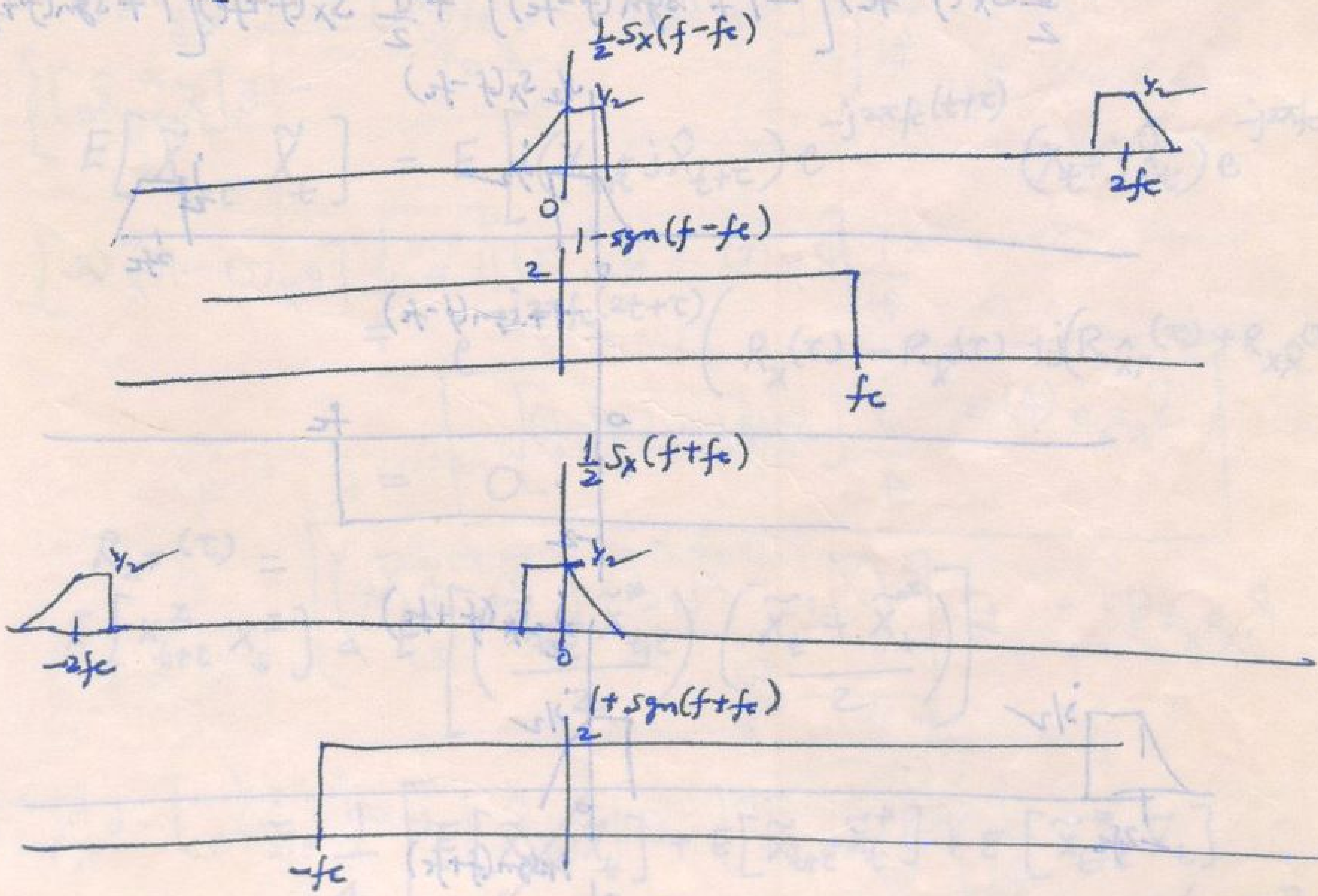
$$\Rightarrow S_{X^I}(f) = S_X(f) * \left[\frac{1}{2}\left(\delta(f-f_c)+\delta(f+f_c)\right)\right]$$

$$+ \left(-j\,\mathrm{sgn}(f)\, S_X(f)\right) \circledast \left[\frac{1}{2j}\left(\delta(f-f_c)-\delta(f+f_c)\right)\right]$$

$$= \frac{1}{2}\left[S_X(f-f_c)+S_X(f+f_c)\right]$$

$$+ \frac{1}{2}\left[-S_X(f-f_c)\,\mathrm{sgn}(f-f_c) + S_X(f+f_c)\,\mathrm{sgn}(f+f_c)\right]$$

$$= \frac{1}{2}S_X(f-f_c)\left[1-\mathrm{sgn}(f-f_c)\right] + \frac{1}{2}S_X(f+f_c)\left[1+\mathrm{sgn}(f+f_c)\right].$$



$$S_{X^I}(f) = \begin{cases} S_X(f-f_c)+S_X(f+f_c) & \text{for } |f|< \\ 0 & \text{else} \end{cases}$$

$S_{x^Ix^Q}(f)$ $\quad R_{x^Ix^Q}(\tau) = R_x(\tau)\sin 2\pi f_c\tau - R_{\hat{x}x}(\tau)\cos 2\pi f_c\tau$.

$$S_{x^Ix^Q}(f) = S_x(f)\circledast\left(\frac{1}{2j}\left(\delta(f-f_c)-\delta(f+f_c)\right)\right)$$

$$-\left(-j\,\mathrm{sgn}(f)\,S_x(f)\right)\circledast\left(\frac{1}{2}\left(\delta(f-f_c)+\delta(f+f_c)\right)\right)$$

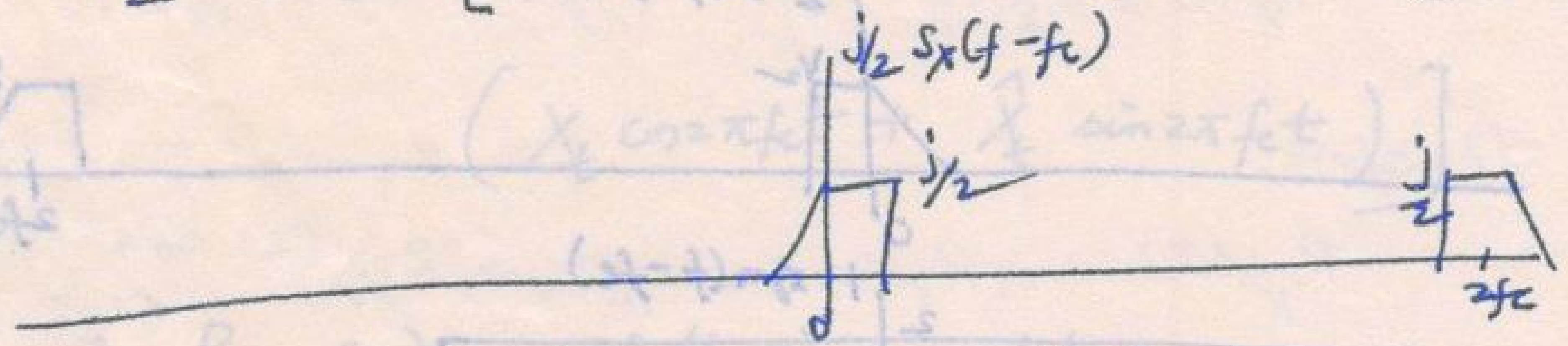
$$= -\frac{j}{2}S_x(f)\circledast\left(\delta(f-f_c)-\delta(f+f_c)\right)$$

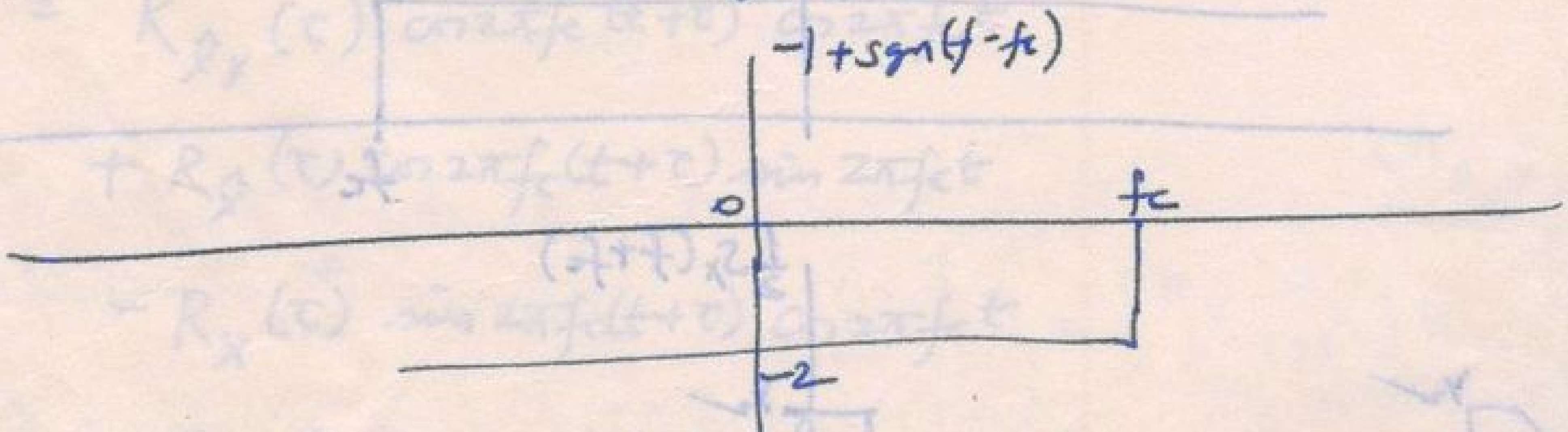
$$+\left(\frac{j}{2}\,\mathrm{sgn}(f)\,S_x(f)\right)\circledast\left(\delta(f-f_c)+\delta(f+f_c)\right)$$

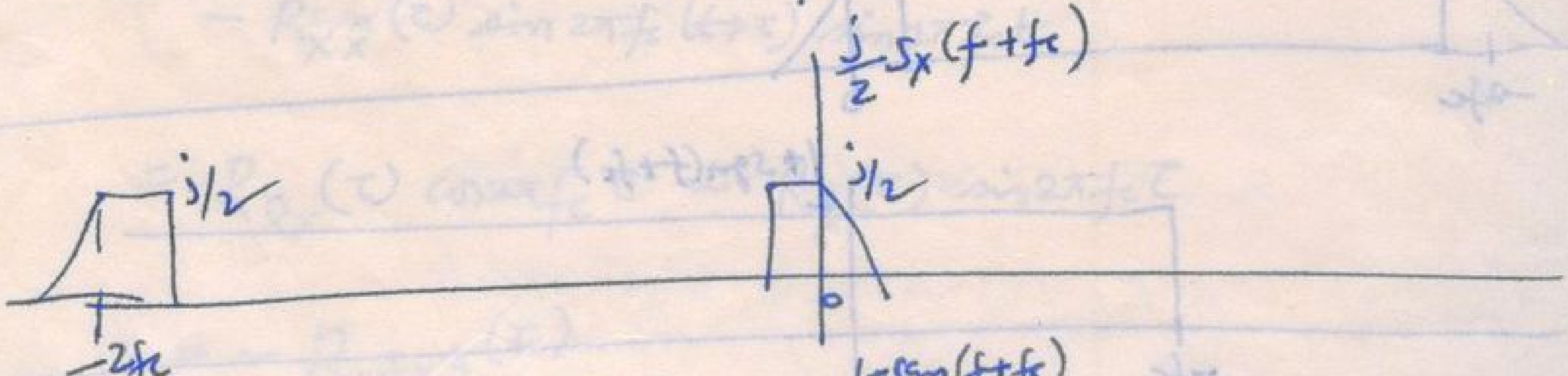
$$= -\frac{j}{2}\left[S_x(f-f_c)-S_x(f+f_c)\right]$$

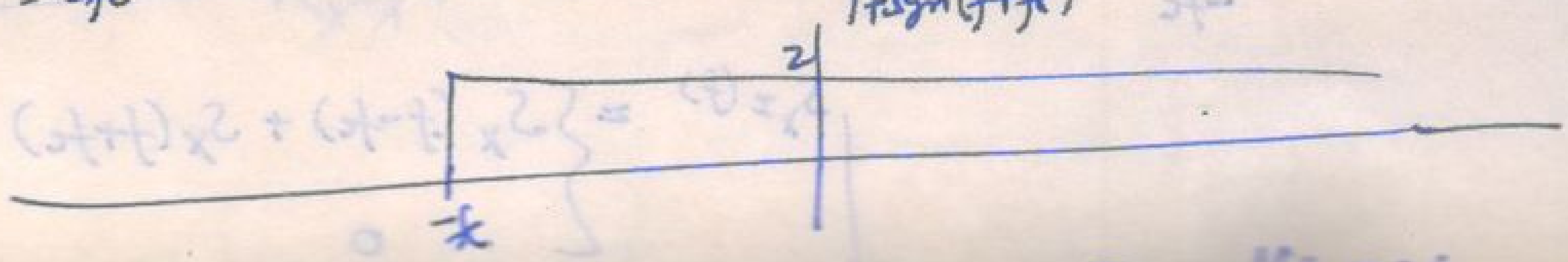
$$+\frac{j}{2}\left[\mathrm{sgn}(f-f_c)\,S_x(f-f_c)+\mathrm{sgn}(f+f_c)\,S_x(f+f_c)\right]$$

$$= \frac{j}{2}S_x(f-f_c)\left[-1+\mathrm{sgn}(f-f_c)\right]+\frac{j}{2}S_x(f+f_c)\left[1+\mathrm{sgn}(f+f_c)\right]$$









Alternative method :

$$\tilde{X}_t = X_t^I + j X_t^Q.$$

$$\Rightarrow X_t^I = \frac{\tilde{X}_t + \tilde{X}_t^*}{2} \quad \& \quad X_t^Q = \frac{\tilde{X}_t - \tilde{X}_t^*}{2j}$$

$$R_{\tilde{X}}(\tau) = E[\tilde{X}_{t+\tau}\tilde{X}_t^*] \quad \& \quad R_{\tilde{X}^*}(\tau) = E[\tilde{X}^*_{t+\tau}\tilde{X}_t] = R^*_{\tilde{X}}(\tau)$$

$$S_{\tilde{X}}(f) = \int_{-\infty}^{\infty} R_{\tilde{X}}(\tau) e^{-j2\pi f\tau} d\tau \quad \& \quad S_{\tilde{X}^*}(f) = \int_{-\infty}^{\infty} R^*_{\tilde{X}}(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \left[\int_{-\infty}^{\infty} R_{\tilde{X}}(\tau) e^{j2\pi f\tau} d\tau\right]^*$$

$$= S^*_{\tilde{X}}(-f)$$

Since $S_{\tilde{X}}(f)$ is real, we have $\boxed{S_{\tilde{X}^*}(f) = S_{\tilde{X}}(-f)}$

$$E[\tilde{X}_{t+\tau}\tilde{X}_t] = E\left[(X_{t+\tau} + j\hat{X}_{t+\tau}) e^{-j2\pi f_c(t+\tau)} (X_t + j\hat{X}_t) e^{-j2\pi f_c t}\right]$$

$$= e^{-j2\pi f_c(2t+\tau)}\left(R_X(\tau) - R_{\hat{X}}(\tau) + j\left(R_{\hat{X}X}(\tau) + R_{X\hat{X}}(\tau)\right)\right)$$

$$= 0.$$

$$R_{X^I}(\tau) =$$

$$E[X^I_{t+\tau} X^I_t] = E\left[\left(\frac{\tilde{X}_{t+\tau} + \tilde{X}^*_{t+\tau}}{2}\right)\left(\frac{\tilde{X}_t + \tilde{X}_t^*}{2}\right)\right]$$

$$= \frac{1}{4}\left[\underbrace{E[\tilde{X}_{t+\tau}\tilde{X}_t]}_{0} + E[\tilde{X}_{t+\tau}\tilde{X}_t^*] + E[\tilde{X}^*_{t+\tau}\tilde{X}_t] + \underbrace{E[\tilde{X}^*_{t+\tau}\tilde{X}_t^*]}_{0}\right]$$

$$= \frac{1}{4}\left[R_{\tilde{X}}(\tau) + R_{\tilde{X}^*}(\tau)\right]$$

$$R_{x^Q}(\tau) = E\left[\left(\frac{\tilde{x}_{t+\tau} - \tilde{x}^*_{t+\tau}}{2j}\right)\left(\frac{\tilde{x}^*_t - \tilde{x}^*_t}{-2j}\right)\right]$$

$$= \frac{1}{4}\left[-\underbrace{\cancel{E[\tilde{x}_{t+\tau}\tilde{x}_t]}}_{0} + E[\tilde{x}_{t+\tau}\tilde{x}^*_t] + E[\tilde{x}^*_{t+\tau}\tilde{x}_t] - \underbrace{\cancel{E[\tilde{x}^*_{t+\tau}\tilde{x}^*_t]}}_{0}\right]$$

$$= \frac{1}{4}\left[+R_{\tilde{x}}(\tau) + R^*_{\tilde{x}}(\tau)\right]$$

$$\boxed{S_{x^Q}(f) = \frac{S_{\tilde{x}}(f) + S_{\tilde{x}}(-f)}{4}.}$$

$$R_{x^I x^Q}(\tau) = E\left[\left(\frac{\tilde{x}_{t+\tau} + \tilde{x}^*_{t+\tau}}{2}\right)\left(\frac{\tilde{x}^*_t - \tilde{x}_t}{-2j}\right)\right]$$

$$= -\frac{1}{4j}\left\{E[\tilde{x}_{t+\tau}\tilde{x}^*_t] - \underbrace{\cancel{E[\tilde{x}_{t+\tau}\tilde{x}_t]}}_{0} + \underbrace{\cancel{E[\tilde{x}^*_{t+\tau}\tilde{x}^*_t]}}_{0} - E[\tilde{x}^*_{t+\tau}\tilde{x}_t]\right\}$$

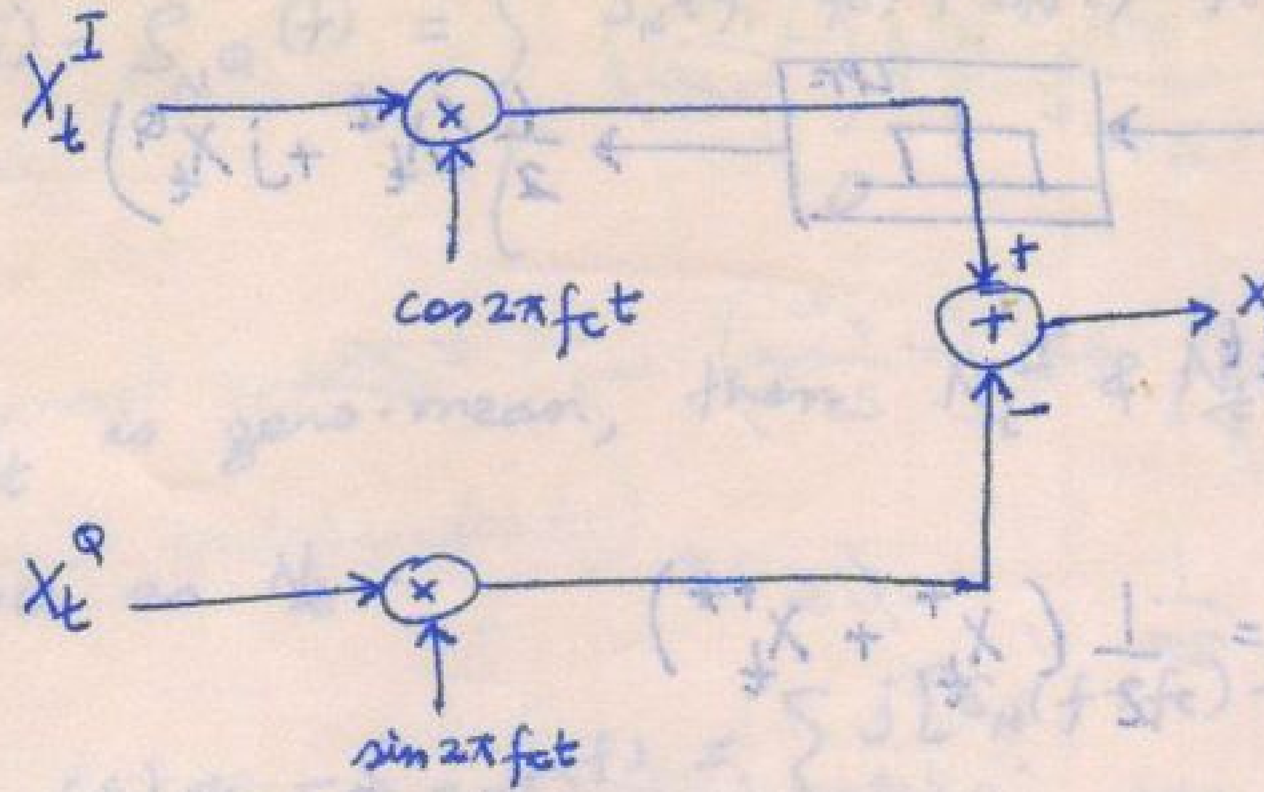
$$= -\frac{1}{4j}\left[R_{\tilde{x}}(\tau) - R^*_{\tilde{x}}(\tau)\right] = \frac{j}{4}\left[R_{\tilde{x}}(\tau) - R^*_{\tilde{x}}(\tau)\right]$$

$$\boxed{S_{x^I x^Q}(f) = \frac{j}{4}\left[S_{\tilde{x}}(f) - S_{\tilde{x}}(-f)\right].}$$

$$R_{x^Q x^I}(f) = E\left[\left(\frac{\tilde{x}_{t+\tau} - \tilde{x}^*_{t+\tau}}{2j}\right)\left(\frac{\tilde{x}^*_t + \tilde{x}_t}{2}\right)\right]$$

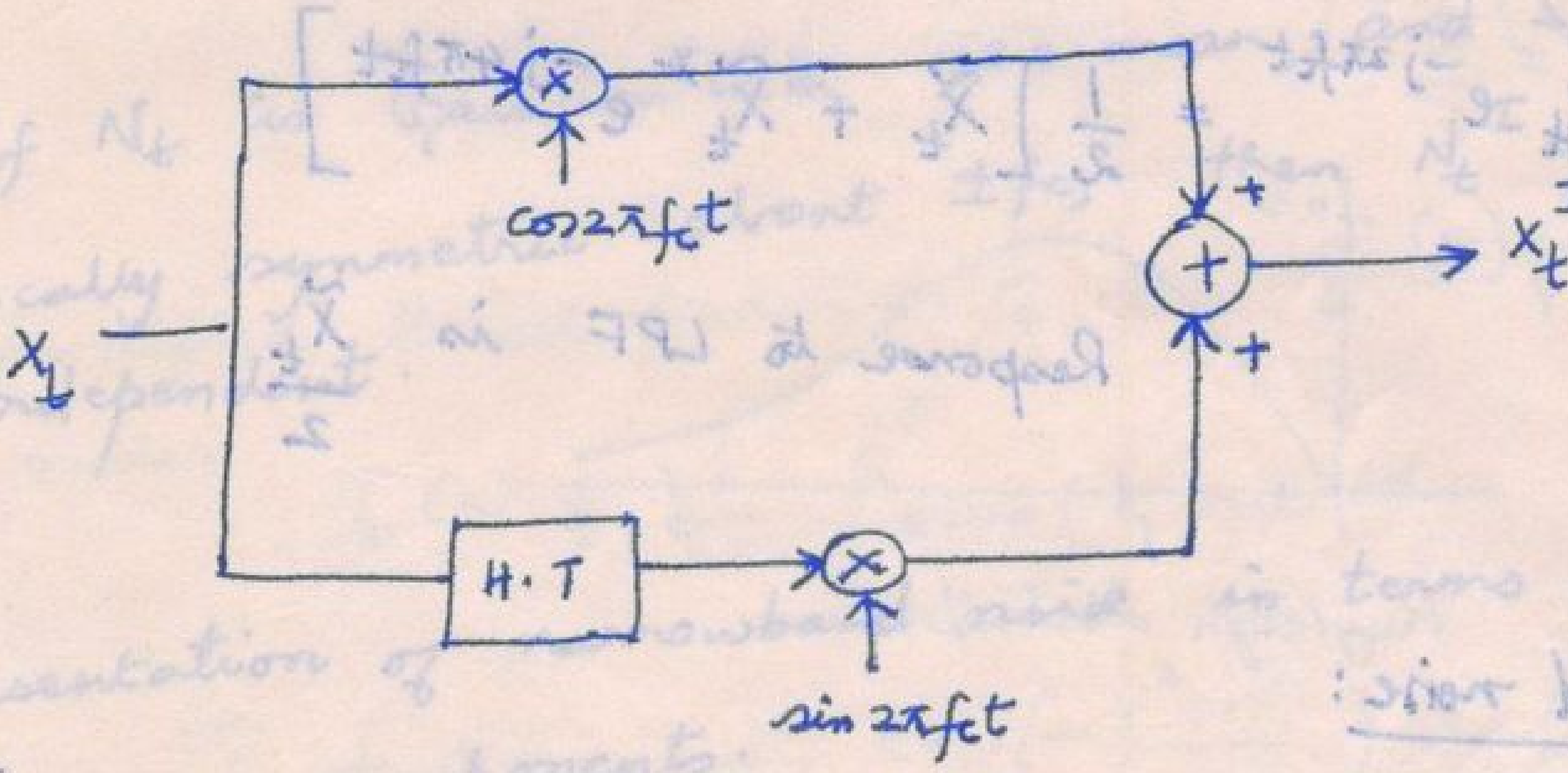
$$= \frac{1}{4j}\left\{E[\tilde{x}_{t+\tau}\tilde{x}^*_t] + \underbrace{\cancel{E[\tilde{x}_{t+\tau}\tilde{x}_t]}}_{0} - \underbrace{\cancel{E[\tilde{x}^*_{t+\tau}\tilde{x}^*_t]}}_{0} - E[\tilde{x}^*_{t+\tau}\tilde{x}_t]\right\}$$

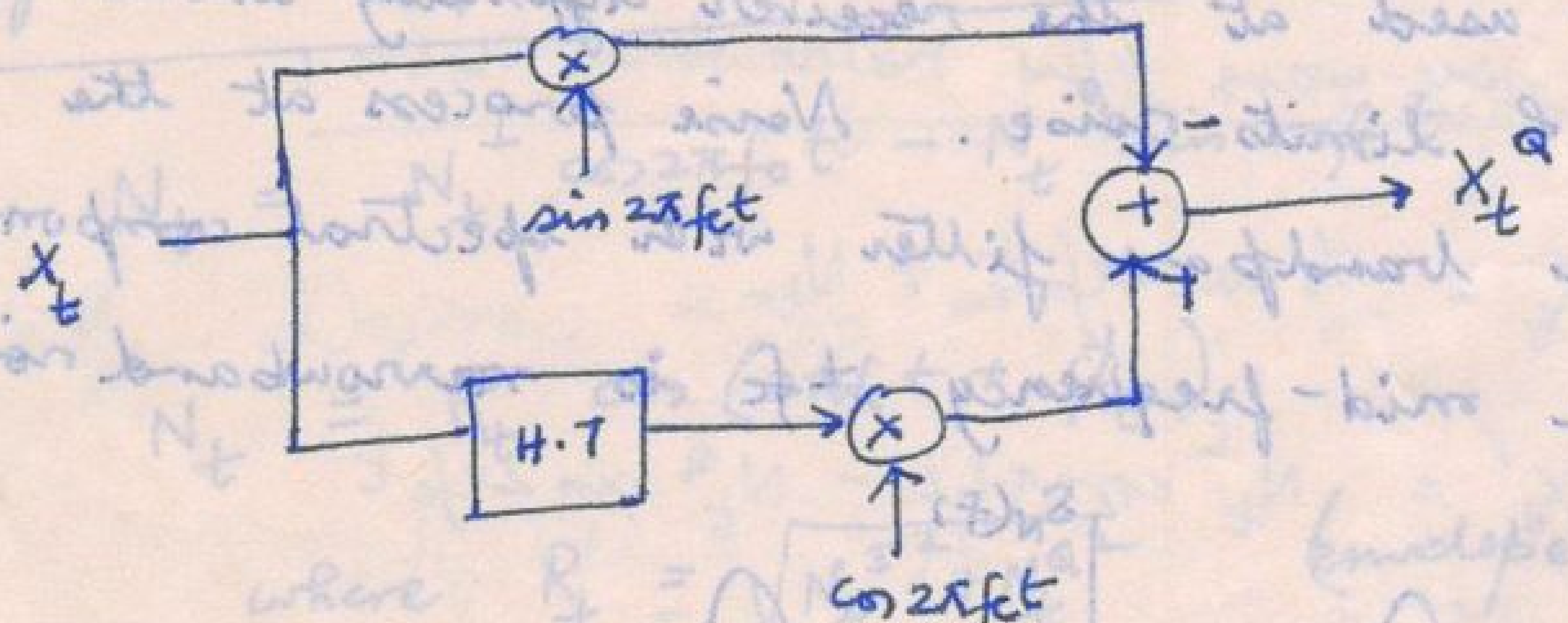
$$= -\frac{j}{4}\left[R_{\tilde{x}}(\tau) - R^*_{\tilde{x}}(\tau)\right]$$



$$R_{X^I}(\tau) = R_{X^Q}(\tau)$$

$$R_{X^I X^Q}(\tau) = -R_{X^Q X^I}(\tau).$$

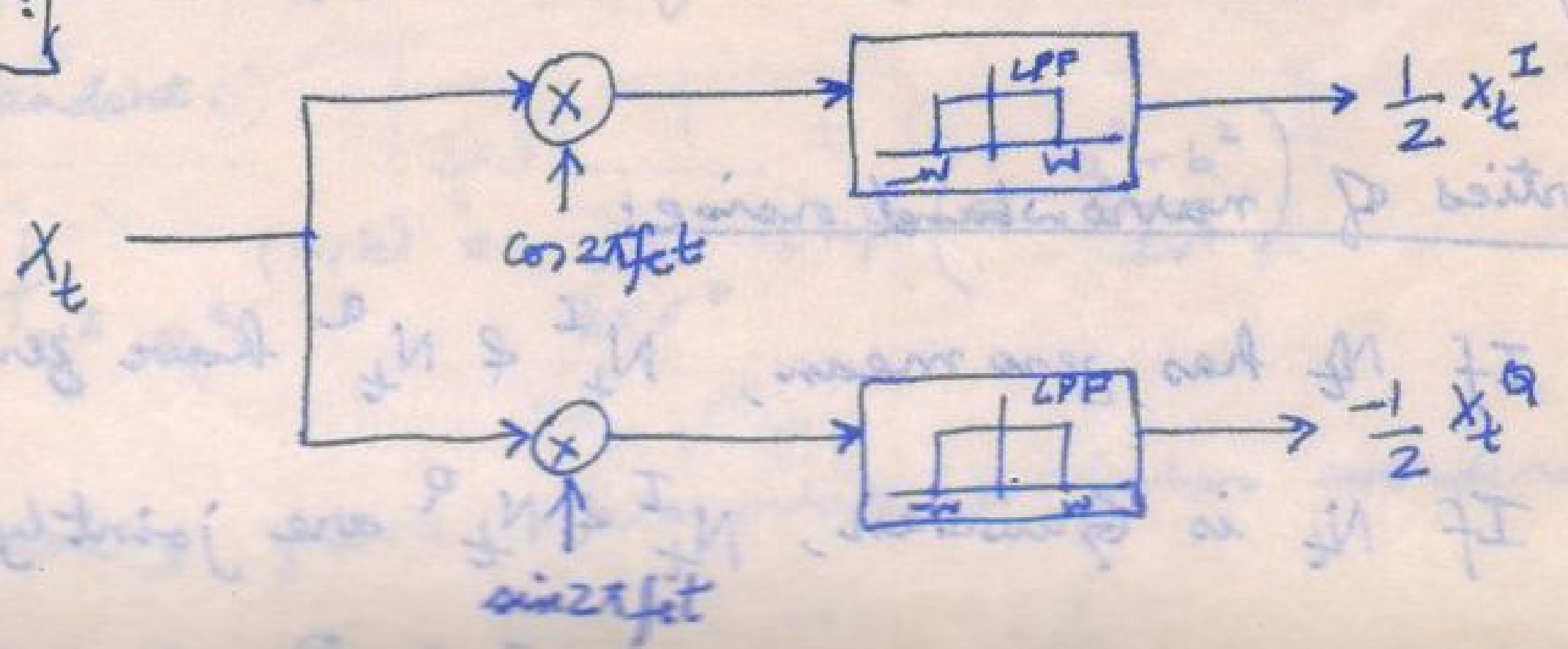


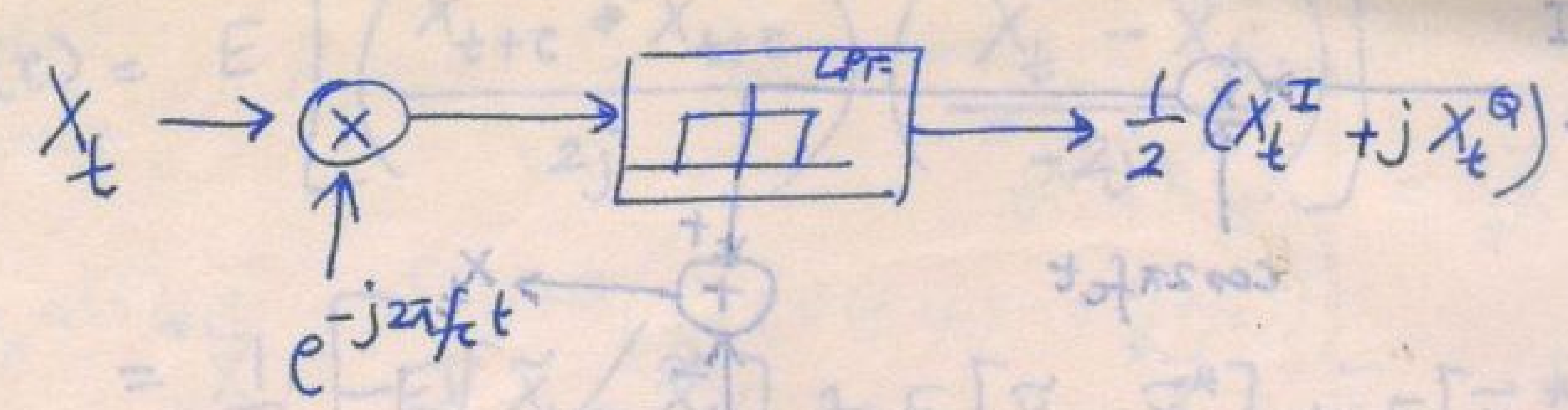


$$X_t^I = X_t \cos 2\pi f_c t + \hat{X}_t \sin 2\pi f_c t$$

$$X_t^Q = -X_t \sin 2\pi f_c t + \hat{X}_t \cos 2\pi f_c t.$$

Lecture 30:





$$X_t = \frac{1}{2}\left(X_t^+ + X_t^{+*}\right)$$

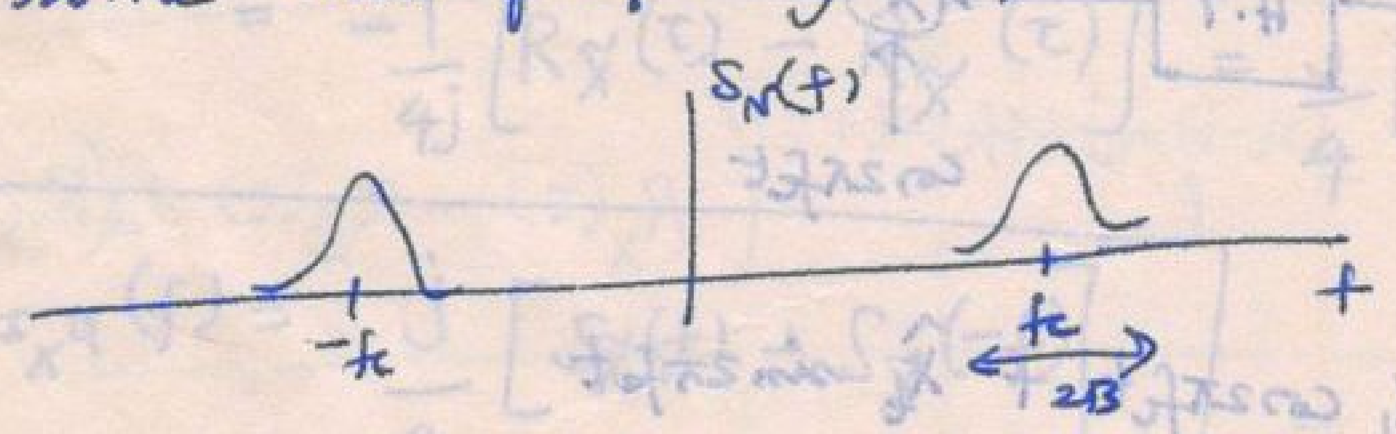
$$\tilde{X}_t = X_t^+ e^{-j2\pi f_c t} \qquad \tilde{X}_t^* = X_t^{+*} e^{j2\pi f_c t}$$

$$X_t e^{-j2\pi f_c t} = \frac{1}{2}\left[\tilde{X}_t + \tilde{X}_t^* e^{-j4\pi f_c t}\right]$$

Response to LPF is $\dfrac{\tilde{X}_t}{2}$

Narrowband noise:

BPF used at the receiver typically allows signals of interest & limits noise. Noise process at the output of a bandpass filter with spectral components about some mid-frequency $\pm f_c$ is narrowband noise.



Any narrowband noise may be modelled as filtered white noise by choosing a suitable filter.

Properties of narrowband noise:

1. If N_t has zero mean, N_t^I & N_t^Q have zero mean too.
2. If N_t is Gaussian, N_t^I & N_t^Q are jointly Gaussian

4. $S_{N^I}(f) = S_{N^Q}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c) & |f| < B \\ 0 & \text{else} \end{cases}$

5. If N_t is zero-mean, then N_t^I & N_t^Q have the same variance as N_t.

6. $S_{N^I N^Q}(f) = -S_{N^Q N^I}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] & |f| \le B \\ 0 & \text{else} \end{cases}$

7. If N_t is Gaussian with zero-mean, and $S_N(f)$ is locally symmetric about $\pm f_c$, then N_t^I & N_t^Q are independent.

Representation of narrowband noise in terms of envelope and phase components.

$$N_t = N_t^I \cos 2\pi f_c t - N_t^Q \sin 2\pi f_c t$$

$$N_t = R_t \cos(2\pi f_c t + \phi_t)$$

where $R_t = \sqrt{{N_t^I}^2 + {N_t^Q}^2}$ (envelope of N_t) — (I)

$$\phi_t = \tan^{-1}\left[\frac{N_t^Q}{N_t^I}\right] \quad \text{(Phase of } N_t\text{)}. \text{ — (II)}$$

Let us assume $S_N(f)$ is symmetric about $\pm f_c$. $\Rightarrow N_t^I$ & N_t^Q are independent.

$$f_{N_t^I, N_t^Q}(a,b) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{a^2+b^2}{2\sigma^2}\right)$$

Transforming from rectangular to polar coordinates

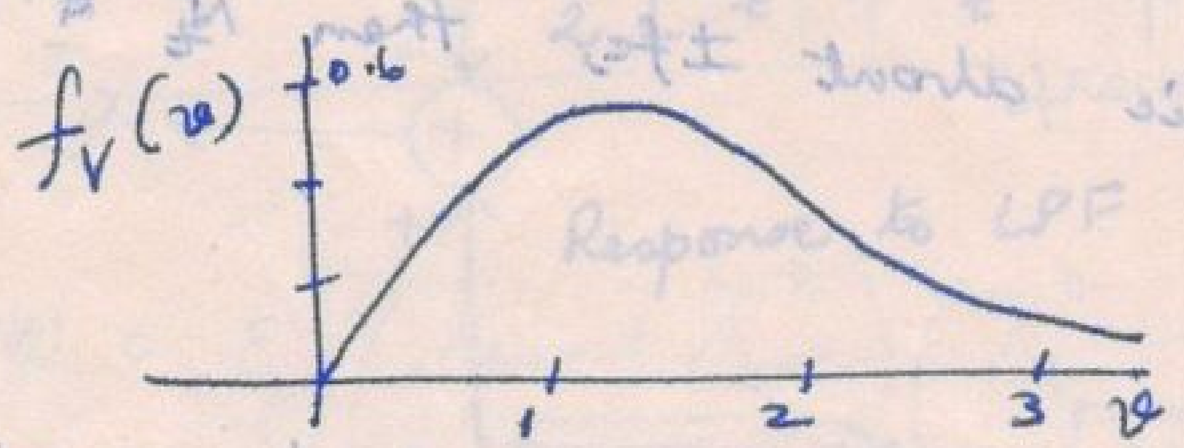
$$f_{R_t, \Phi_t}(r, \phi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

$$f_{R_t}(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r \geq 0$$

$$f_{\Phi_t}(\phi) = \frac{1}{2\pi}, \quad 0 \leq \phi \leq 2\pi$$

R_t & Φ_t are independent.

Define $V = \frac{R}{\sigma}$.



Rayleigh distribution.

Envelope of sine-wave plus narrowband noise:

$$X_t = A \cos 2\pi f_c t + N_t$$

$$X_t' = N_t^{I'} \cos 2\pi f_c t - N_t^{Q} \sin 2\pi f_c t$$

where $N_t^{I'} = A + N_t^{I}$.

$N_t^{I'}$ & N_t^{Q} are statistically independent.

If N_t is Gaussian,

$$f_{N_t^{I'}, N_t^{Q}}(a, b) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(a-A)^2 + b^2}{2\sigma^2}\right]$$

$$R_t = \sqrt{N_t^{I'2} + N_t^{Q2}}$$

$$\Phi_t = \tan^{-1}\left[\frac{N_t^{Q}}{N_t^{I'}}\right]$$

$$f_{R_t,\phi_t}(r,\phi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2+A^2-2Ar\cos\phi}{2\sigma^2}\right)$$

We cannot express this as $f_{R_t}(r)\cdot f_{\phi_t}(\phi)$ (i.e., R_t & ϕ_t are not independent).

$$f_{R_t}(r) = \int_0^{2\pi} f_{R_t,\phi_t}(r,\phi)\,d\phi$$

$$= \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2+A^2}{2\sigma^2}\right) \int_0^{2\pi} \exp\left(\frac{Ar}{\sigma^2}\cos\phi\right) d\phi$$

Modified Bessel function of the first kind of zero order $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\cos\phi)\,d\phi$

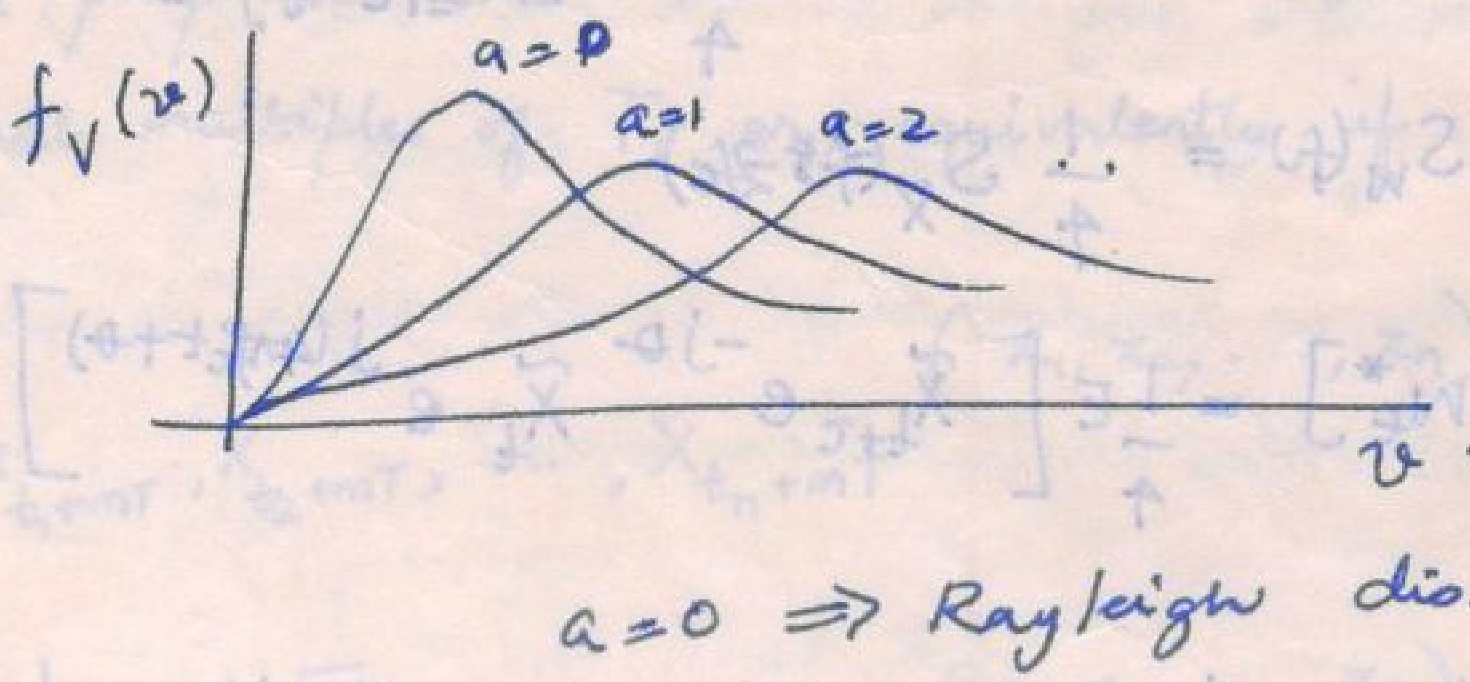
$$\Rightarrow f_{R_t}(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2+A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right).$$

Rician distribution.

Let $V = \frac{R}{\sigma}$. Define $a = \frac{A}{\sigma}$ and $v = \frac{r}{\sigma}$.

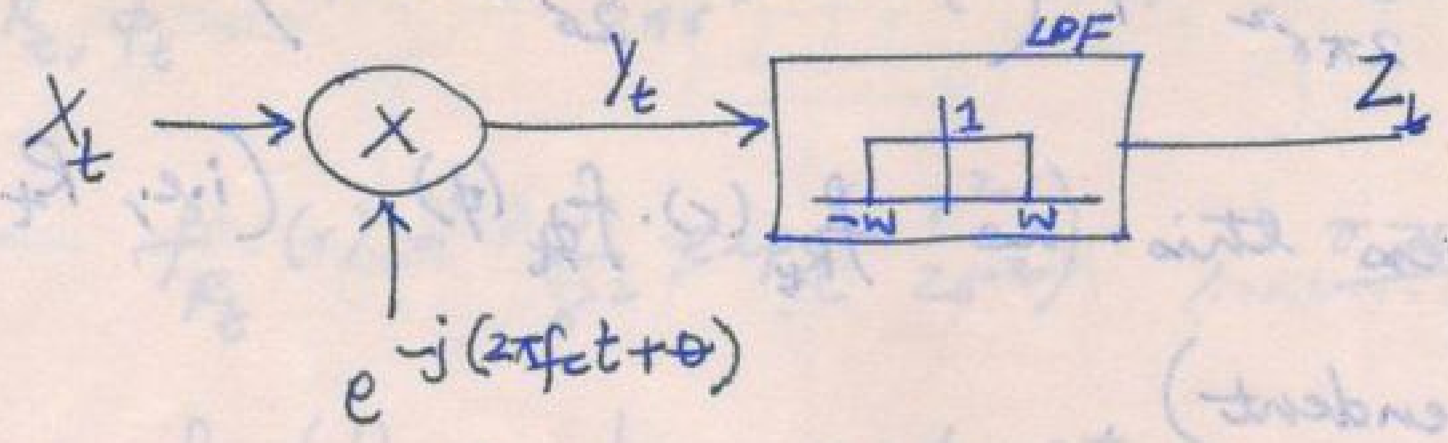
$$f_V(v) = \sigma f_R(r)$$

$$= v \exp\left(-\frac{v^2+a^2}{2}\right) I_0(av)$$



$a=0 \Rightarrow$ Rayleigh distribution.

Continuation from page 130. (effect of carrier phase offset)



$$Y_t = X_t\, e^{-j(2\pi f_c t+\theta)}$$

$$E\left[Y_{t+\tau} Y_t^*\right] = E\left[X_{t+\tau}\, e^{-j(2\pi f_c (t+\tau)+\theta)}\, X_t^*\, e^{j(2\pi f_c t+\theta)}\right]$$

$$= R_X(\tau)\, e^{-j2\pi f_c \tau}.$$

$$S_Y(f) = S_X(f+f_c).$$

$$Y_t = X_t\, e^{-j(2\pi f_c t+\theta)}$$

$$= \frac{1}{2}\left(X_t^+ + X_t^{+*}\right) e^{-j(2\pi f_c t+\theta)} \qquad \left(\tilde{X}_t = X_t^+ e^{-j2\pi f_c t}\right) \Rightarrow X_t^+ = \tilde{X}_t\, e^{j2\pi f_c t}$$

$$= \frac{1}{2}\left(\tilde{X}_t\, e^{-j\theta} + \tilde{X}_t^*\, e^{-j(4\pi f_c t+\theta)}\right)$$

$$= V_t + W_t.$$

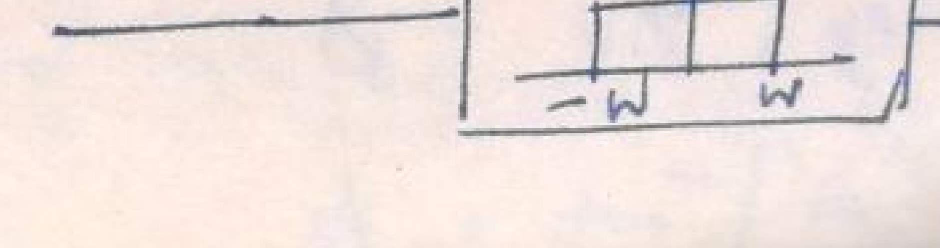
$$R_V(\tau) = E\left[V_{t+\tau} V_t^*\right] = \frac{1}{4} E\left[\tilde{X}_{t+\tau} \tilde{X}_t^*\right] \Rightarrow S_V(f) = \frac{1}{4} S_{\tilde{X}}(f).$$

$$R_W(\tau) = E\left[W_{t+\tau} W_t^*\right] = \frac{1}{4} E\left[\tilde{X}_{t+\tau}^*\, e^{-j(4\pi f_c (t+\tau)+\theta)}\, \tilde{X}_t\, e^{j(4\pi f_c t+\theta)}\right]$$

$$= \frac{1}{4} E\left[\tilde{X}_{t+\tau}^* \tilde{X}_t\right] e^{-j(4\pi f_c \tau)}$$

$$\Rightarrow S_W(f) = \frac{1}{4} S_{\tilde{X}}(-f-2f_c).$$

$$E\left[V_{t+\tau} W_t^*\right] = \frac{1}{4} E\left[\tilde{X}_{t+\tau}\, e^{-j\theta}\, \tilde{X}_t\, e^{j(4\pi f_c t+\theta)}\right] = 0.$$



$$Z_t = V_t = \frac{1}{2}\tilde{X}_t\, e^{-j\theta}.$$

$$Z_t = \int_{-\infty}^{\infty} (W_{t-\tau} + V_{t-\tau})\, h(\tau)\, d\tau$$

We can show that

$$E\left[|Z_t - V_t|^2\right] = 0, \quad \text{i.e.,}$$

$$\underbrace{E\left[|Z_t|^2\right]}_{R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)\,df \,=\, R_V(0)} + \underbrace{E\left[|V_t|^2\right]}_{R_V(0)} - \underbrace{E\left[Z_t V_t^*\right]}_{E\left[\int_{-\infty}^{\infty}(W_{t-\tau}+V_{t-\tau})h(\tau)d\tau \; V_t^*\right]} - \underbrace{E\left[V_t Z_t^*\right]}_{R_V(0)} = 0$$

$$E\left[\int_{-\infty}^{\infty}(W_{t-\tau}+V_{t-\tau})h(\tau)d\tau \; V_t^*\right] = \int_{-\infty}^{\infty} E\left[W_{t-\tau} V_t^*\right] h(\tau)\,d\tau + \int_{-\infty}^{\infty} E\left[V_{t-\tau} V_t^*\right] h(\tau)\,d\tau$$

$$= 0 + \int_{-\infty}^{\infty} R_V(-\tau)\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} S_V(f)\, df = R_V(0).$$

$$\Rightarrow Z_t \overset{m.s}{=} V_t = \frac{1}{2} \tilde{X}_t e^{-j\theta}.$$

where θ : carrier phase offset at the receiver.

$\boxed{\text{Lecture 31:}}$

Cyclostationary random processes:

A process X_t is cyclostationary with period T if its statistical properties are invariant to a shift of the origin by integral multiples of T, or, equivalently, if

$$F_{X_{t_1+mT}, X_{t_2+mT}, \ldots, X_{t_n+mT}}(x_1, x_2, \ldots, x_n) = F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n)$

A process X_t is <u>wide-sense cyclostationary</u> with period T if

$$m_X(t+mT) = m_X(t) \quad \left(\text{i.e., } E[X_{t+mT}] = E[X_t]\right)$$

for every integer m.

$$\& \quad R_X(t+mT, s+mT) = R_X(t,s)$$

for every integer m.

$$\Big(\text{or} \quad R_X(t+mT, \tau) = R_X(t,\tau)$$

for every integer m $\Big)$.

It follows from the definition that if a random process X_t is cyclostationary it is also wide-sense cyclostationary.

Connection between stationary and cyclostationary processes (see following theorem).

Theorem:

If X_t is a cyclostationary process with period T, and θ is a random variable uniform in the interval $(0,T]$ and independent of X_t, then the process $\bar{X}_t = X_{t-\theta}$ obtained by a random shift of the origin is stationary and its n^{th} order distribution equals

$$F_{\bar{X}_{t_1}, \bar{X}_{t_2}, \ldots, \bar{X}_{t_n}}(x_1, x_2, \ldots, x_n) = \frac{1}{T}\int_0^T F_{X_{t_1-\alpha}, X_{t_2-\alpha}, \ldots X_{t_n-\alpha}}(x_1, x_2, \ldots, x_n)\, d\alpha$$

Proof:

To show that $\bar{X}_t$ is stationary, it is sufficient to show that

$$F_{\bar{X}_{t_1+c}, \bar{X}_{t_2+c}, \ldots, \bar{X}_{t_n+c}}(x_1, x_2, \cdots, x_n)$$

$$= P\left[\bar{X}_{t_1+c} \le x_1, \bar{X}_{t_2+c} \le x_2, \ldots, \bar{X}_{t_n+c} \le x_n\right]$$

is independent of c.

Define A to be the event $\left\{\bar{X}_{t_1+c} \le x_1, \bar{X}_{t_2+c} \le x_2, \ldots, \bar{X}_{t_n+c} \le x_n\right\}$.

We know that

$$P(A) = \int_{-\infty}^{\infty} P(A|\theta = a) f_{\theta}(a)\, da = \frac{1}{T}\int_0^T P(A|\theta = a)\, da$$

$$P(A|\theta=a) = P\left[\bar{X}_{t_1+c} \le x_1, \ldots, \bar{X}_{t_n+c} \le x_n \,\middle|\, \theta = a\right]$$

$$= P\left[X_{t_1+c-\theta} \le x_1, \ldots, X_{t_n+c-\theta} \le x_n \,\middle|\, \theta = a\right]$$

$$= P\left[X_{t_1+c-a} \le x_1, \ldots, X_{t_n+c-a} \le x_n \,\middle|\, \theta = a\right]$$

(Since θ is independent of X_t) $$= P\left[X_{t_1+c-a} \le x_1, \ldots, X_{t_n+c-a} \le x_n\right]$$

$$= F_{X_{t_1+c-a}, X_{t_2+c-a}, \ldots, X_{t_n+c-a}}(x_1, x_2, \cdots, x_n).$$

$$F_{\bar{X}_{t_1+c}, \bar{X}_{t_2+c}, \ldots, \bar{X}_{t_n+c}}(x_1, x_2, \ldots, x_n) = P(A)$$

$$= \frac{1}{T}\int_0^T F_{X_{t_1+c-a}, X_{t_2+c-a}, \ldots, X_{t_n+c-a}}(x_1, x_2, \cdots, x_n)\, da$$

Define $\alpha = a - c$.

Now, $$P(A) = \frac{1}{T}\int_{-c}^{T-c} F_{X_{t_1-\alpha}, X_{t_2-\alpha}, \ldots, X_{t_n-\alpha}}(x_1, x_2, \cdots, x_n)\, d\alpha$$

Since X_t is a cyclostationary random process

$$F_{X_{t_1-\alpha}, X_{t_2-\alpha}, \ldots, X_{t_n-\alpha}}(x_1, x_2, \ldots, x_n) = F_{X_{t_1-\alpha+mT}, X_{t_2-\alpha+mT}, \ldots X_{t_n-\alpha+mT}}(x_1, x_2, \ldots, x_n)$$

$$\Rightarrow P[A] = \frac{1}{T}\int_0^T F_{X_{t_1-\alpha}, X_{t_2-\alpha}, \ldots, X_{t_n-\alpha}}(x_1, x_2, \ldots, x_n)\, d\alpha$$

We have shown that $P[A]$ is independent of c and

$$F_{\bar{X}_{t_1}, \bar{X}_{t_2}, \ldots, \bar{X}_{t_n}}(x_1, x_2, \ldots, x_n) = \frac{1}{T}\int_0^T F_{X_{t_1-\alpha}, X_{t_2-\alpha}, \ldots X_{t_n-\alpha}}(x_1, x_2, \ldots, x_n)\, d\alpha.$$

Theorem:

If X_t is a wide-sense cyclostationary process with period T, then the shifted process $\bar{X}_t = X_{t-\theta}$ (where θ is a uniform r.v. in $[0,T]$ and independent of X_t) is wide-sense stationary with mean

$$m_{\bar{X}} = \frac{1}{T}\int_0^T m_X(t)\, dt$$

and autocorrelation

$$R_{\bar{X}}(\tau) = \frac{1}{T}\int_0^T R_X(t+\tau, t)\, dt$$

Proof:

$$m_{\bar{X}} = E[\bar{X}_t] = E[X_{t-\theta}] = E[E[X_{t-\theta}|\theta]]$$

(Since θ is independent of X_t for all t) $= E[E[X_{t-\theta}]]$

$$= E[m_X(t-\theta)]$$

$$= \frac{1}{T}\int_{0}^{T} m_X(t-\theta)\, d\theta$$

(Since X_t is W.S.C.S) $= \frac{1}{T}\int_0^T m_X(t')\,dt'$

$$R_{\bar{X}}(\tau) = E\left[\bar{X}_{t+\tau}\bar{X}_t\right] = E\left[X_{t+\tau-\theta}X_{t-\theta}\right]$$

$$= E\left[E\left[X_{t+\tau-\theta}X_{t-\theta} \mid \theta=\theta\right]\right]$$

(Since θ is independent of X_t for all t) $= E\left[E\left[X_{t+\tau-\theta}X_{t-\theta}\right]\right]$

$$= E\left[R_X(t+\tau-\theta,\ t-\theta)\right]$$

$$= \frac{1}{T}\int_0^T R_X(t+\tau-\theta,\ t-\theta)\,d\theta$$

$(t'=t-\theta)$ $= \frac{1}{T}\int_{-\theta}^{T-\theta} R_X(t'+\tau,\ t')\,dt'$

(Since X_t is W.S.C.S) $= \frac{1}{T}\int_0^T R_X(t'+\tau,\ t')\,dt'$

$\Rightarrow \bar{X}_t$ is W.S.S.

Lecture 32:

Pulse Amplitude Modulation (PAM): Example of a cyclostationary process.

$$X_t = \sum_{n=-\infty}^{\infty} A_n\, p(t-nT)$$

where A_n is a stationary sequence of random variables and $p(t)$ is a given pulse shape.

Let $R_A(k) = E\left[A_{n+k}A_n\right]$.

$$R_X(t+\tau, t) = E\left[X_{t+\tau} X_t\right]$$

$$= E\left[\left(\sum_{n=-\infty}^{\infty} A_n\, p(t+\tau-nT)\right)\left(\sum_{m=-\infty}^{\infty} A_m\, p(t-mT)\right)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[A_n A_m\right] p(t+\tau-nT)\, p(t-mT)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_A(n-m)\, p(t+\tau-nT)\, p(t-mT).$$

Let $k = n-m$.

$$R_X(t+\tau, t) = \sum_{k=-\infty}^{\infty} R_A(k) \sum_{m=-\infty}^{\infty} p(t+\tau-(m+k)T)\, p(t-mT).$$

$$m_X(t) = E\left[X_t\right] = E\left[\sum_{n=-\infty}^{\infty} A_n\, p(t-nT)\right]$$

$$= \sum_{n=-\infty}^{\infty} m_A\, p(t-nT) = m_A\left[\sum_{n=-\infty}^{\infty} p(t-nT)\right]$$

X_t is W.S.C.S if

$$m_X(t+lT) = m_X(t) \quad \text{for all integer } l \text{ \& for all } t.$$

$$\& \quad R_X(t+\tau, t) = R_X(t+lT+\tau, t+lT) \quad \text{for all integer } l \text{ \& for all } t, \tau.$$

$$m_X(t+lT) = m_A \sum_{n=-\infty}^{\infty} p(t+lT-nT) = m_A \sum_{n'=-\infty}^{\infty} p(t-n'T)$$

$$(n' = n-l)$$

$$= m_X(t).$$

$$R_X(t+lT+\tau,\ t+lT) = \sum_{k=-\infty}^{\infty} R_A(k) \sum_{m=-\infty}^{\infty} p(t+lT+\tau-(m+k)T)\,p(t+lT-mT)$$

$$(m'=m-l) \qquad = \sum_{k=-\infty}^{\infty} R_A(k) \sum_{m'=-\infty}^{\infty} p(t+\tau-(m'+k)T)\,p(t-m'T)$$

$$= R_X(t+\tau, t)$$

$\Rightarrow X_t$ is W.S.C.S.

Define a new process $\bar{X}_t = X_{t-T_d}$ where T_d is uniform in $[0,T]$ and independent of X_t for all t.

$\bar{X}_t$ is W.S.S. and

$$R_{\bar{X}}(\tau) = \frac{1}{T}\int_0^T R_X(t+\tau, t)\,dt$$

and $m_{\bar{X}} = \dfrac{1}{T}\displaystyle\int_0^T m_X(t)\,dt.$

$$m_{\bar{X}} = \frac{1}{T}\int_0^T m_X(t)\,dt = \frac{1}{T}\int_0^T m_A \sum_{n=-\infty}^{\infty} p(t-nT)\,dt$$

$$(t'=t-nT) \qquad = \frac{m_A}{T}\sum_{n=-\infty}^{\infty}\int_{-nT}^{-nT+T} p(t')\,dt'$$

$$\Rightarrow \quad \boxed{m_{\bar{X}} = \frac{m_A}{T}\int_{-\infty}^{\infty} p(t')\,dt'}$$

$$R_{\bar{X}}(\tau) = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} R_A(k) \sum_{m=-\infty}^{\infty} p(t+\tau-(k+m)T)\, p(t-mT)\, dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k) \sum_{m=-\infty}^{\infty} \int_0^T p(t+\tau-kT-mT)\, p(t-mT)\, dt$$

$(t' = t - mT)$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k) \sum_{m=-\infty}^{\infty} \int_{-mT}^{-mT+T} p(t'+\tau-kT)\, p(t')\, dt'$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k) \left[\int_{-\infty}^{\infty} p(t'+\tau-kT)\, p(t')\, dt'\right]$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k)\, R_p(\tau-kT)$$

where $R_p(\tau) = \int_{-\infty}^{\infty} p(t+\tau)\, p(t)\, dt$

$$\Rightarrow \boxed{R_{\bar{X}}(\tau) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k)\, R_p(\tau-kT)}$$

Power spectral density of $\bar{X}_t$:

$$S_{\bar{X}}(f) = \int_{-\infty}^{\infty} R_{\bar{X}}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k)\, R_p(\tau-kT)\right] e^{-j2\pi f\tau}\, d\tau$$

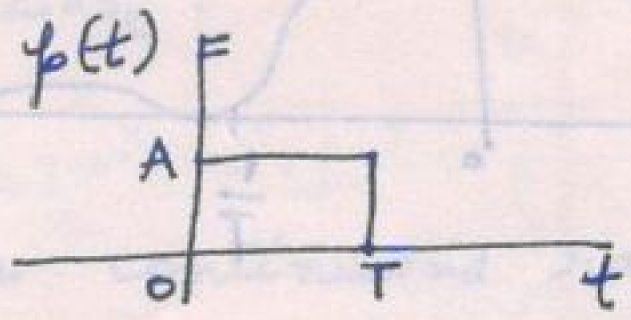
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A(k) \left[\int_{-\infty}^{\infty} R_p(\tau-kT)\, e^{-j2\pi f\tau}\, d\tau\right]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_A(k)\, |P(f)|^2\, e^{-j2\pi f k T}$$

$$= \frac{|P(f)|^2}{T} \underbrace{\sum_{k=-\infty}^{\infty} R_A(k)\, e^{-j2\pi f k T}}_{\text{DTFT of } R_A(k)}.$$

Examples:

1. Unipolar NRZ A_n i.i.d 0 or 1 with equal prob.



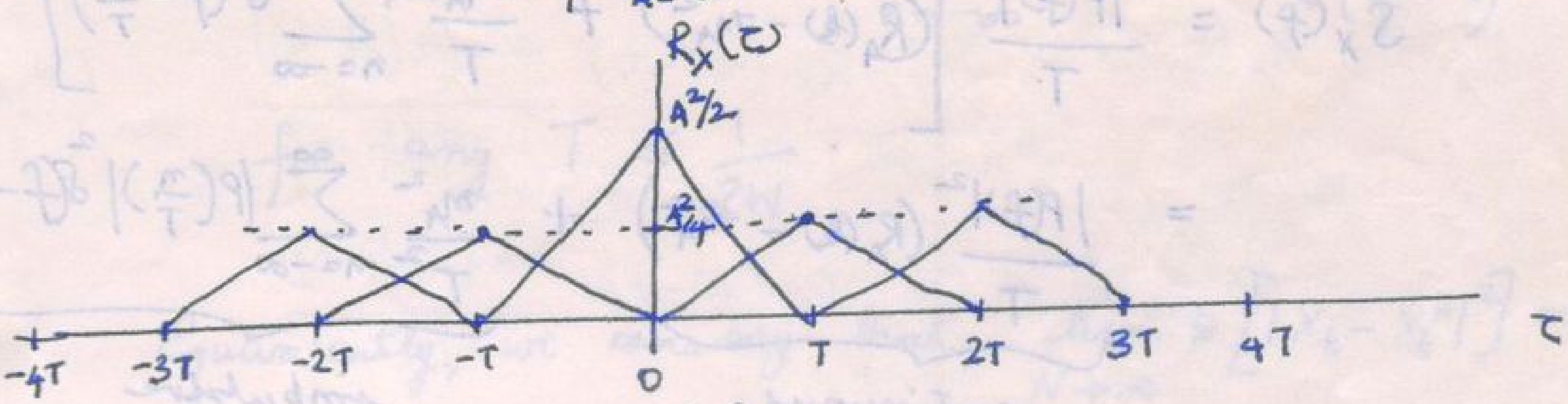
$E[A_n] = \frac{1}{2}$

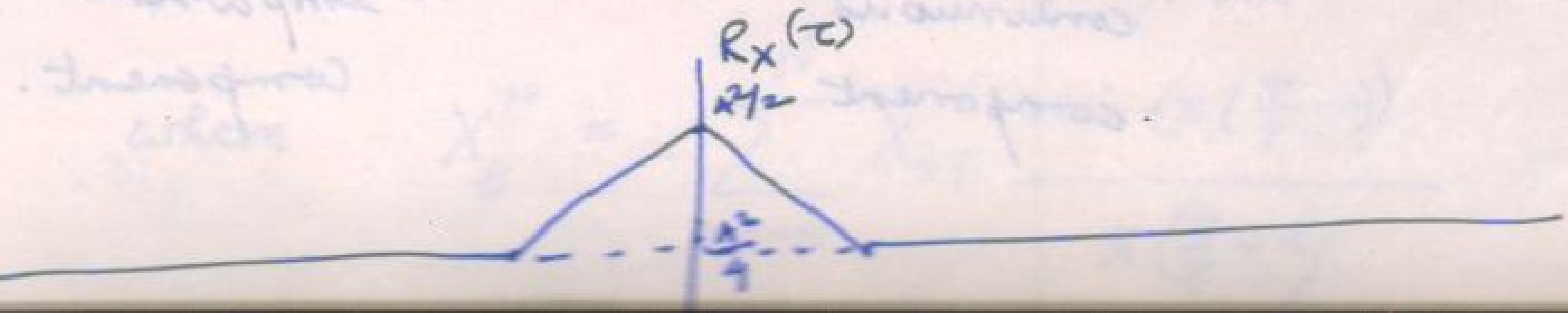
$$R_A(k) = E[A_n A_{n+k}] = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{4} & k \neq 0. \end{cases}$$

$$R_p(\tau) = \begin{cases} A^2(T - |\tau|) & \text{for } |\tau| \le T \\ 0 & \text{else} \end{cases}$$

$$X_t = \sum_{n=-\infty}^{\infty} A_n\, p(t - nT - T_d).$$

$$R_X(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_A(k)\, R_p(\tau - kT)$$

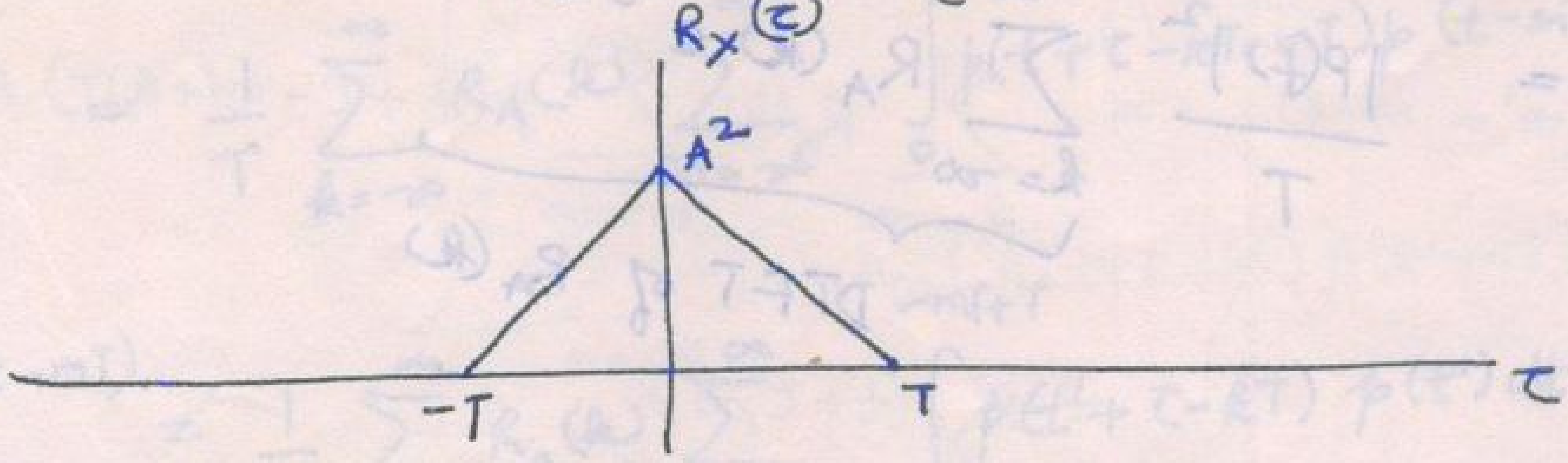




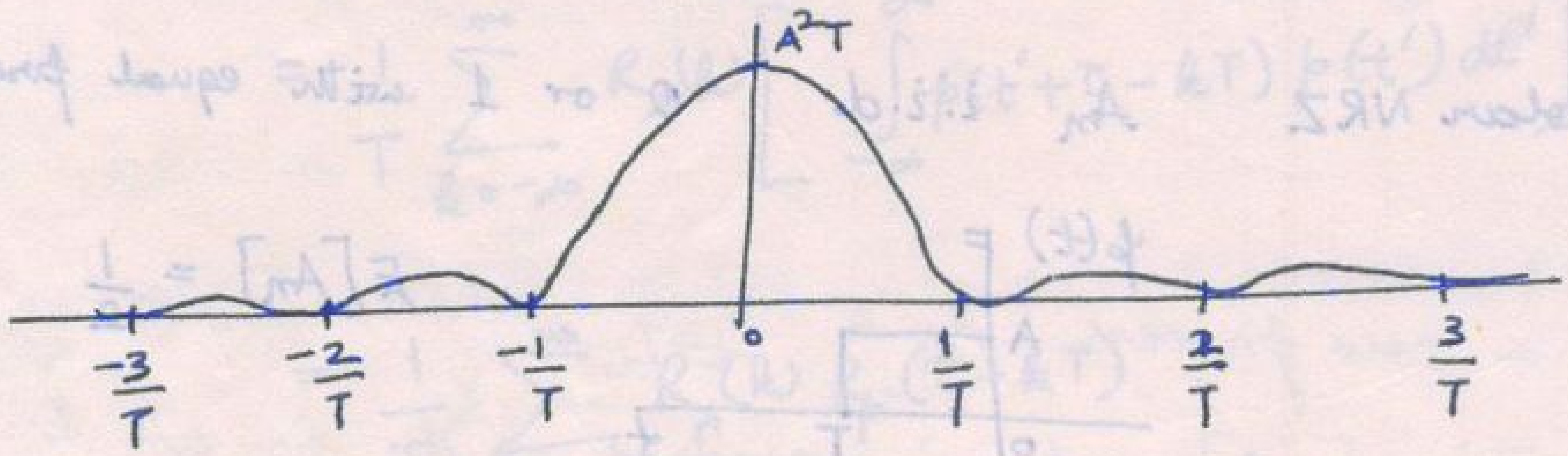
Similar to (1) except

(2) A_n is -1 or 1 with equal probability

$$E[A_n] = 0, \quad R_A(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0. \end{cases}$$



$$S_X(f) = A^2 T \, \mathrm{sinc}^2(fT)$$



(3) $\{A_n\}$ uncorrelated $\quad E[A_n] = m_A$.

$$\Rightarrow \quad R_A(k) = \begin{cases} R_A(0) & k=0 \\ m_A^2 & k \neq 0. \end{cases}$$

$$\sum_{k=-\infty}^{\infty} R_A(k) e^{-j2\pi fkT} = (R_A(k) - m_A^2) + \sum_{k=-\infty}^{\infty} m_A^2 e^{-j2\pi fkT}$$

$$= (R_A(k) - m_A^2) + \frac{m_A^2}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right).$$

$$S_X(f) = \frac{|P(f)|^2}{T} \left[(R_A(k) - m_A^2) + \frac{m_A^2}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \right]$$

$$= \underbrace{\frac{|P(f)|^2}{T} (R_A(k) - m_A^2)}_{\text{continuous component}} + \underbrace{\frac{m_A^2}{T^2} \sum_{n=-\infty}^{\infty} \left|P\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)}_{\text{impulsive component.}}$$

$\boxed{\text{Lecture 33:}}$

Sampling:

- For a deterministic signal $x(t)$ that is bandlimited to the frequency range $[-W_0, W_0]$, the following sampling expansion can be written

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

provided that $\frac{1}{T} = W$ (sampling frequency) is greater than $2W_0$.

- Suppose a continuous random process X_t is sampled at a sampling frequency $\frac{1}{T}$ samples per second to produce a discrete-time random process $Y_n = X_{nT}$.

If X_t is W.S.S. with zero-mean and power spectral density $S_X(f)$ with the property that $S_X(f) = 0$ for $|f| \geq W$, then

$$X_t \overset{m.s}{=} \sum_{n=-\infty}^{\infty} X_{nT} \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)},$$

for any $T \leq \frac{1}{2W}$.

Equivalently, we can say that $\lim_{N\to\infty} E\left[|X_t - X_t^N|^2\right] = 0$

where $X_t^N = \sum_{n=-N}^{N} X_{nT} \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$.

$$E\left[|X_t - X_t^N|^2\right] = E\left[\left| X_t - \sum_{n=-N}^{N} X_{nT} \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}\right|^2\right]$$

$$= E\left[|X_t|^2\right]$$

$$-\sum_{n=-N}^{N} E\left[X_t X_{nT}^*\right] \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

$$-\sum_{n=-N}^{N} E\left[X_t^* X_{nT}\right] \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

$$+\sum_{n=-N}^{N} \sum_{m=-N}^{N} E\left[X_{nT} X_{mT}^*\right] \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)} \frac{\sin\left(\pi\left(\frac{t}{T}-m\right)\right)}{\pi\left(\frac{t}{T}-m\right)}$$

First term: $E\left[|X_t|^2\right] = R_X(0)$.

Second term: $\displaystyle\sum_{n=-N}^{N} R_X(t-nT) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$

$R_X(\tau)$ is a bandlimited signal with bandwidth W.

$$\cancel{\Rightarrow R_X(0) = \sum_{n=-N}^{N} R_X(nT) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}}$$

$$R_X(t-\tau) = \sum_{n=-\infty}^{\infty} R_X(nT-\tau) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)} \qquad \left(T < \frac{1}{2W}\right)$$

$$\Rightarrow R_X(0) = \sum_{n=-\infty}^{\infty} R_X(nT-t) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

Third term: $\displaystyle\sum_{n=-N}^{N} R_X(nT-t) \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)} = R_X(0)$

Second & third terms are equal to $R_X(0)$.
(since $R_X(0)$ is real).

Fourth term: $$\sum_{n=-N}^{N}\sum_{m=-N}^{N} R_X((n-m)T)\,\frac{\sin\left(\pi\left(\frac{t}{T}-m\right)\right)}{\pi\left(\frac{t}{T}-m\right)}\,\frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

$$= \sum_{k=-2N}^{2N} R_X(kT) \sum_{\substack{(n,m) \\ \text{such that} \\ n-m=k \\ \& \ -N\le n,m\le N}} \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}\,\frac{\sin\left(\pi\left(\frac{t}{T}-m\right)\right)}{\pi\left(\frac{t}{T}-m\right)}$$

As $N\to\infty$, we get

$$\sum_{k=-\infty}^{\infty} R_X(kT)\sum_{n=-\infty}^{\infty} \frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}\,\frac{\sin\left(\pi\left(\frac{t}{T}-(n-k)\right)\right)}{\pi\left(\frac{t}{T}-(n-k)\right)} \quad \text{—}(*)$$

Using the sampling expansion for $\dfrac{\sin\left(\pi\left(\frac{t-s}{T}\right)\right)}{\pi\left(\frac{t-s}{T}\right)}$,

we get

$$\frac{\sin\left(\pi\left(\frac{t-s}{T}\right)\right)}{\pi\left(\frac{t-s}{T}\right)} = \sum_{n=-\infty}^{\infty} \frac{\sin\left(\pi\left(n-\frac{s}{T}\right)\right)}{\pi\left(n-\frac{s}{T}\right)}\,\frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}.$$

$$\Rightarrow \quad (*) = \sum_{k=-\infty}^{\infty} R_X(kT)\,\frac{\sin\left(\pi\left(\frac{t}{T}-k\right)\right)}{\pi\left(\frac{t}{T}-k\right)} = R_X(0).$$

Therefore, we have

$$\lim_{N\to\infty} E\left[|X_t - X_t^N|^2\right] = R_X(0) - R_X(0) - R_X(0) + R_X(0) = 0.$$

Another approach to show that the fourth term is $R_X(0)$.

We know $$R_X(0) = \sum_{n=-\infty}^{\infty} R_X(nT-t)\,\frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

$$= \sum_{n=-\infty}^{\infty}\left[\underbrace{\sum_{m=-\infty}^{\infty} R_X((n-m)T)\,\frac{\sin\left(\pi\left(\frac{t}{T}-m\right)\right)}{\pi\left(\frac{t}{T}-m\right)}}_{\text{sampling expansion for } R_X(nT-t)}\right]\frac{\sin\left(\pi\left(\frac{t}{T}-n\right)\right)}{\pi\left(\frac{t}{T}-n\right)}$$

$$= \text{Fourth term.}$$

Lecture 34:

Mean Square Estimation:

① Approximation of a random variable Y by a constant c such that $E[(Y-c)^2]$ is minimum.

$$MSE = E[(Y-c)^2] = \int_{-\infty}^{\infty}(y-c)^2 f_Y(y)\,dy$$

$$\frac{d\,MSE}{dc} = -2\int_{-\infty}^{\infty}(y-c)f_Y(y)\,dy = 0 \Rightarrow c = E[Y].$$

$$\frac{d^2 MSE}{dc^2} = 2.$$

$\therefore$ The best estimate of Y is $E[Y]$.

$$E[(Y-c)^2] = E\left[(Y-E[Y]+E[Y]-c)^2\right]$$

$$= E\left[(Y-E[Y])^2\right] + E\left[(E[Y]-c)^2\right] + 2E\left[(Y-E[Y])(E[Y]-c)\right]$$

$$= E\left[(Y-E[Y])^2\right] + (E[Y]-c)^2 + 2(E[Y]-c)\left[E[Y]-E[Y]\right]$$

$$= E\left[(Y-E[Y])^2\right] + \left[E[Y]-c\right]^2$$

$$\geq E\left[(Y-E[Y])^2\right]$$

$\left[E[Y]-c\right]^2 \geq 0$ and is equal to 0 if $c = E[Y]$

(2) MS estimation of a random variable Y by a function $g(X)$ of a random variable X.

$$MSE = E\left[(Y-g(X))^2\right] = \int_{-\infty}^{\infty} E\left[(Y-g(x))^2/X=x\right] f_X(x)\,dx$$

Minimizing MSE is equivalent to minimizing

$$MSE(x) = E\left[(Y-g(X))^2/X=x\right] \text{ for each } x.$$

$E\left[(Y-g(X))^2/X=x\right]$ is minimized if

$$g(x) = E\left[Y/X=x\right].$$

$$E\left[(Y-g(X))^2/X=x\right] = E\left[\left(Y-E[Y/X=x]+E[Y/X=x]-g(x)\right)^2/X=x\right]$$

$$= E\left[(Y-E[Y/X=x])^2\right] + E\left[(E[Y/X=x]-g(x))^2\right]$$

$$+2E\left[(Y-E[Y/X=x])(E[Y/X=x]-g(x))\right]$$

$$= E\left[(Y-E[Y/X=x])^2\right] + (E[Y/X=x]-g(x))^2$$

$$\geq E\left[(Y-E[Y/X=x])^2\right]$$

$(E[Y/X=x]-g(x))^2 \geq 0$ and is equal to 0 if

$$g(x) = E\left[Y/X=x\right]$$

$\Rightarrow E[Y/X=x]$ is the MMSE estimate & the MMSE is $E\left[(Y-E[Y/X=x])^2\right]$.

(3) Homogeneous linear MS estimation

$$\hat{Y} = g(X) = aX. \quad (g(x) \text{ is restricted to be } ax).$$

$$MSE = E\left[(Y-aX)^2\right]$$

$$\frac{d\,MSE}{da} = 2E\left[(Y-aX)(-X)\right] = 0$$

$$\Rightarrow E\left[(Y-aX)X\right] = 0 \Rightarrow a = \frac{E[XY]}{E[X^2]}.$$

$$\frac{d^2 MSE}{da^2} = 2E[X^2].$$

$$E\left[(Y-aX)^2\right] = E\left[\left(Y - \frac{E[XY]}{E[X^2]}X + \frac{E[XY]}{E[X^2]}X - aX\right)^2\right]$$

$$= E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)^2\right] + E\left[\left(\frac{E[XY]}{E[X^2]}X - aX\right)^2\right]$$

$$+ 2E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)\left(\frac{E[XY]}{E[X^2]}X - aX\right)\right]$$

$$= E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)^2\right] + \left(\frac{E[XY]}{E[X^2]} - a\right)^2 E[X^2]$$

$$+ 2\left[\cancel{\frac{E^2[XY]}{E[X^2]}} - \cancel{aE[XY]} - \cancel{\frac{E^2[XY]}{E[X^2]}} + \cancel{\frac{aE[XY]}{E[X^2]}E[X^2]}\right]$$

$$= E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)^2\right] + \left(\frac{E[XY]}{E[X^2]} - a\right)^2 E[X^2]$$

$$\geq E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)^2\right]$$

The MMSE estimate of Y is $\dfrac{E[XY]}{E[X^2]}X$ and the MMSE is

$$E\left[\left(Y - \frac{E[XY]}{E[X^2]}X\right)^2\right].$$

TIME		I	II	III	IV	V	VI	VII	VIII
MONDAY	SUBJECT		B						
TUESDAY	SUBJECT	B							
WEDNESDAY	SUBJECT						B		
THURSDAY	SUBJECT								
FRIDAY	SUBJECT	B							
SATURDAY	SUBJECT								

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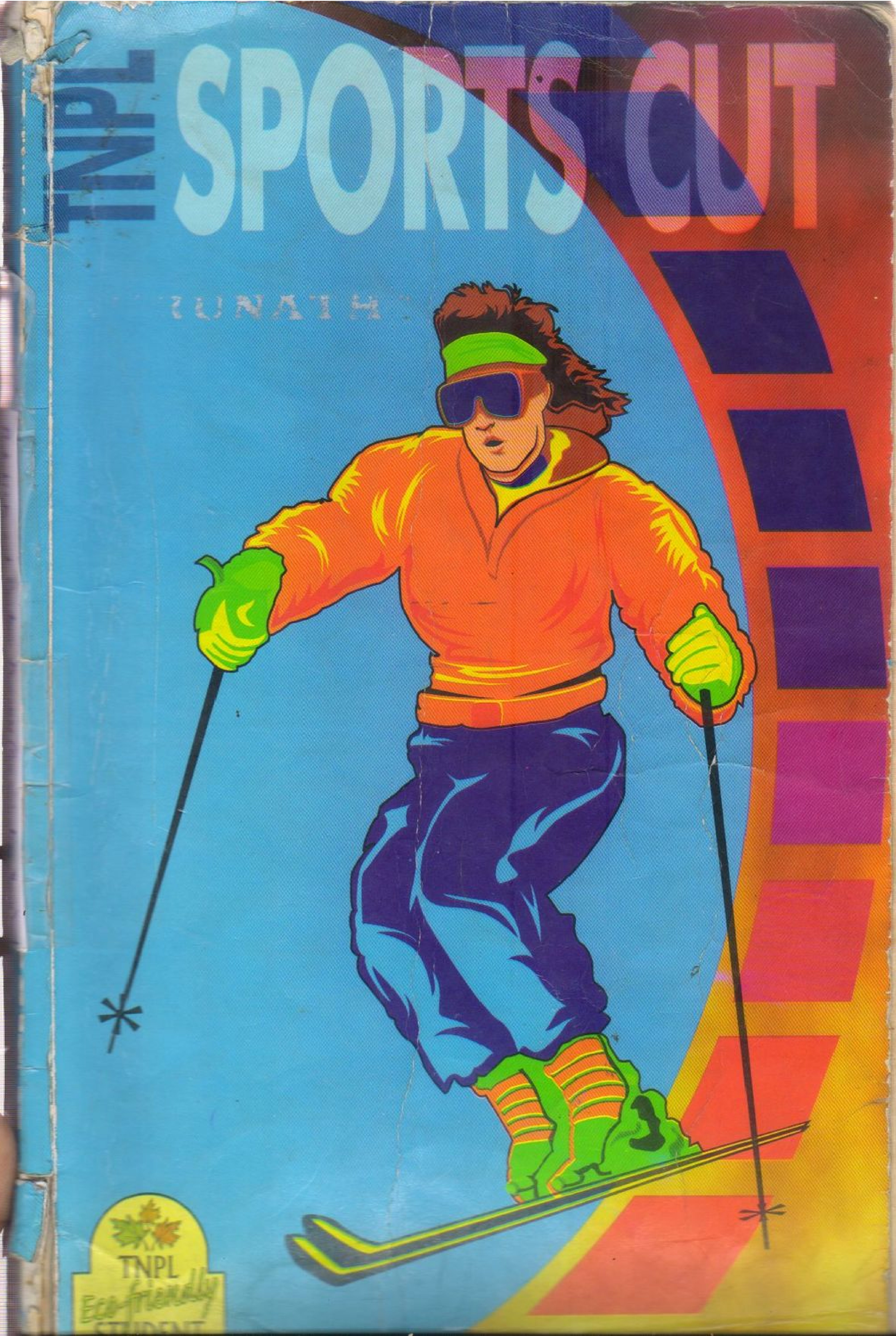
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