Problem Set 6

(1) $H^H H \underline{\hat{w}} = H^H \underline{x}$ and $H^H H = L L^H$ (cholesky factorization)

L is > 0 (has inverse)

$\Rightarrow L \underbrace{L^H \underline{\hat{w}}} = H^H \underline{x}$

$\Rightarrow$ We can solve for $\underline{\hat{y}}$ in $L \underline{\hat{y}} = H^H \underline{x}$ (has a unique soln. $L > 0$)

where $\underline{\hat{y}} = L^H \underline{\hat{w}}$.

Once again we can solve for $\underline{\hat{w}}$ in $L^H \underline{\hat{w}} = \underline{\hat{y}}$ $(L > 0)$

(2) $\underline{X} = H \underline{w}_0 + \underline{V}$

$\underline{V} \sim$ zero mean, $\sigma_V^2 I$

$H : N \times M$, $N \geq M$, full rank (M)

(a) $\underline{\hat{w}} = (H^H H)^{-1} H^H \underline{X} = (H^H H)^{-1} H^H (H \underline{w}_0 + \underline{V})$

$= \underline{w}_0 + (H^H H)^{-1} H^H \underline{V}$.

$\Rightarrow E[\underline{\hat{w}}] = \underline{w}_0 + E\left[(H^H H)^{-1} H^H \underline{V}\right] = \underline{w}_0 + \underline{0} = \underline{w}_0$.

$\Rightarrow \underline{\hat{w}}$ is an unbiased estimator

(b) $E\left[(\underline{w}_0 - \underline{\hat{w}})(\underline{w}_0 - \underline{\hat{w}})^H\right] = ?$

$\underline{w}_0 - \underline{\hat{w}} = -(H^H H)^{-1} H^H \underline{V}$

$\Rightarrow E\left[(\underline{w}_0 - \underline{\hat{w}})(\underline{w}_0 - \underline{\hat{w}})^H\right] = \left[(H^H H)^{-1} H^H H (H^H H)^{-1}\right] \sigma_V^2$

$= \sigma^2 (H^H H)^{-1}$

(c) $\underline{\tilde{X}} = \underline{X} - H \underline{\hat{w}}$

$= \underline{X} - H (H^H H)^{-1} H^H \underline{X} = (I - P_H) \underline{X}$

$E[\|\underline{\tilde{X}}\|^2] = E[\underline{\tilde{X}}^H \underline{\tilde{X}}] = E\left[\underline{X}^H (I - P_H)(I - P_H) \underline{X}\right]$

$= E\left[\underline{X}^H (I - P_H) \underline{X}\right]$

$$= E\left[\mathrm{Tr}\left(\underline{X}^H (I - P_H)\underline{X}\right)\right]$$

$$= E\left[\mathrm{Tr}\left((I - P_H)\,\underline{X}\,\underline{X}^H\right)\right]$$

$$= \mathrm{Tr}\left(E\left[(I - P_H)\underline{X}\,\underline{X}^H\right]\right)$$

$$= \mathrm{Tr}\left[(I - P_H)\, E\left[\underline{X}\,\underline{X}^H\right]\right]$$

$$= \mathrm{Tr}\left[(I - P_H)\left(H\,\underline{w}_0\,\underline{w}_0^H H^H + \sigma_v^2 I\right)\right]$$

$$= \mathrm{Tr}\left[(I - P_H)\sigma_v^2 I\right] \qquad \Big(\text{since } \underbrace{(I - P_H)}_{\perp R(H)}\underbrace{H\underline{w}_0\underline{w}_0^H H^H}_{\in R(H)} = 0\Big)$$

$$= \sigma_v^2\,\mathrm{Tr}(I - P_H)$$

$$\mathrm{Tr}(I - P_H) = \mathrm{Tr}\left(I - H(H^H H)^{-1}H^H\right)$$

$$= \mathrm{Tr}(I) - \mathrm{Tr}\left(H(H^H H)^{-1}H^H\right)$$

$$= N - \mathrm{Tr}\left(H(H^H H)^{-1}H^H\right)$$

$$= N - \mathrm{Tr}\left((H^H H)^{-1}H^H H\right)$$

$$= N - \mathrm{Tr}\left(I_{M\times M}\right)$$

$$= N - M$$

Let $\hat{\sigma}_v^2 = \dfrac{\|\tilde{X}\|^2}{N-M}$

$$E\left[\hat{\sigma}_v^2\right] = \frac{E\left[\|\tilde{X}\|^2\right]}{N-M} = \frac{\sigma_v^2\,\mathrm{Tr}(I - P_H)}{N-M} = \sigma_v^2$$

$\Rightarrow \hat{\sigma}_v^2$ is an unbiased estimate of σ_v^2