EE5040 Problem Set 2

(1) $\hat{\underline{X}} = R_{xy} R_y^{-1} \underline{y}$

$$R_{xy} = E\left[\begin{pmatrix} S_0 \\ S_1 \end{pmatrix} \begin{pmatrix} y_0^* & y_1^* \end{pmatrix}\right] = \begin{bmatrix} E[S_0(S_0+V_0)^*] & E[S_0(S_1+0.5S_0+V_1)^*] \\ E[S_1(S_0+V_0)^*] & E[S_1(S_1+0.5S_0+V_1)^*] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

$$R_y = E\left[\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \begin{pmatrix} Y_0^* & Y_1^* \end{pmatrix}\right] = \begin{bmatrix} E[(S_0+V_0)(S_0+V_0)^*] & E[(S_0+V_0)(S_1+0.5S_0+V_1)^*] \\ E[(S_1+0.5S_0+V_1)(S_0+V_0)^*] & E[(S_1+0.5S_0+V_1)(S_1+0.5S_0+V_1)^*] \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}$$

$$\hat{\underline{X}} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2.25 \end{bmatrix}^{-1} \underline{Y} = \frac{4}{17} \begin{bmatrix} 2 & 1/2 \\ -1/2 & 2 \end{bmatrix} \underline{Y}$$

$$\hat{S}_0 = \frac{8}{17} Y_0 + \frac{2}{17} Y_1 \quad \& \quad \hat{S}_1 = -\frac{2}{17} Y_0 + \frac{8}{17} Y_1 .$$

(2) $\underline{\Delta = 0}$

$$\hat{S}_i = \alpha_0 Y_i + \alpha_1 Y_{i-1} + \alpha_2 Y_{i-2} \triangleq \underline{k}_0^H \underline{Y} \quad \text{where } \underline{k}_0^H = [\alpha_0 \ \alpha_1 \ \alpha_2]$$

$$\underline{Y} = \begin{bmatrix} Y_i \\ Y_{i-1} \\ Y_{i-2} \end{bmatrix}$$

$$Y_i = S_i + 0.5 S_{i-1} + V_i$$

$$R_y = E[\underline{Y}\,\underline{Y}^H] = \begin{bmatrix} R_y(0) & R_y(1) & R_y(2) \\ R_y^*(1) & R_y(0) & R_y(1) \\ R_y^*(2) & R_y^*(1) & R_y(0) \end{bmatrix} \quad \text{where } R_y(k) = E[Y_i Y_{i-k}^*]$$

$$R_{S_i \underline{Y}} = E[S_i \underline{Y}^H] = \begin{bmatrix} E[S_i Y_i^*] & E[S_i Y_{i-1}^*] & E[S_i Y_{i-2}^*] \end{bmatrix}.$$

$$E[S_i Y_i^*] = E[S_i(S_i + 0.5 S_{i-1} + V_i)^*] = 1$$

$$E[S_i Y_{i-1}^*] = E[S_i(S_{i-1} + 0.5 S_{i-2} + V_{i-1})^*] = 0$$

$$E[S_i Y_{i-2}^*] = E[S_i(S_{i-2} + 0.5 S_{i-3} + V_{i-2})^*] = 0.$$

$$R_y(0) = E[Y_i Y_i^*] = E[(S_i + 0.5 S_{i-1} + V_i) Y_i^*]$$

$$= E[S_i Y_i^*] + 0.5 E[S_{i-1} Y_i^*] + E[V_i Y_i^*]$$

$$= 1 + 0.5 E[S_i(S_i + 0.5 S_{i-1} + V_i)^*] + E[V_i(S_i + 0.5 S_{i-1} + V_i)^*]$$

$$= 1 + 0.25 + 1 = 2.25$$

$$R_y(1) = E[Y_i Y_{i-1}^*] = E[(S_i + 0.5 S_{i-1} + V_i) Y_{i-1}^*]$$

$$= E[S_i Y_{i-1}^*] + 0.5 E[S_{i-1} Y_{i-1}^*] + E[V_i Y_{i-1}^*]$$

$$= 0 + 0.5 + 0 = 0.5$$

$$R_y(2) = E[Y_i Y_{i-2}^*] = E[(S_i + 0.5 S_{i-1} + V_i) Y_{i-2}^*]$$

$$= E[S_i Y_{i-2}^*] + 0.5 E[S_{i-1} Y_{i-2}^*] + E[V_i Y_{i-2}^*]$$

$$= 0$$

$$k_o^H = R_{xy} R_y^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2.25 & 0.5 & 0 \\ 0.5 & 2.25 & 0.5 \\ 0 & 0.5 & 2.25 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0.4688 & -0.1096 & 0.0244 \end{bmatrix}.$$

$\underline{\Delta = 1}$

$$\hat{S}_{i-1} = \alpha_0 Y_i + \alpha_1 Y_{i-1} + \alpha_2 Y_{i-2}$$

$$R_{S_{i-1}Y} = \begin{bmatrix} E[S_{i-1} Y_i^*] & E[S_{i-1} Y_{i-1}^*] & E[S_{i-1} Y_{i-2}^*] \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix}$$

$$k_o^H = [\alpha_0 \ \alpha_1 \ \alpha_2] = R_{S_{i-1}Y} R_y^{-1} = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix} R_y^{-1}$$

The model and equations for general L and Δ is in Section 5.4 of [Sayed].