$Y_1 + 2Y_2 = 5X + N_1 + 2N_2$
$Y_1 + 4Y_2 = 9X + N_1 + 4N_2$

(1) Let $Y_1 = X + N_1$ where $X = \pm 1$ w.p $\frac{1}{2}$

$Y_2 = 2X + N_2$ N_1, N_2 indep. of X

& N_1, N_2 i.i.d $\sim N(0, 1)$. $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\hat{X}$ = MMSE estimate of X from Y_1 & Y_2.

$$\hat{X} = E[X/\underline{Y}] = E[X/Y_1, Y_2] = \Pr[X=1/Y_1, Y_2] - \Pr[X=-1/Y_1, Y_2].$$

$$\Pr[X=1/Y_1 = y_1, Y_2 = y_2] = \frac{f(y_1, y_2/x=1)\,P[x=1]}{f(y_1, y_2)}$$

$$= \frac{\frac{1}{(2\pi)(1)} e^{-\frac{1}{2}\left[\left((y_1-1) \;\; (y_2-2)\right)\begin{bmatrix} y_1-1 \\ y_2-2 \end{bmatrix}\right]} \cdot \frac{1}{2}}{\frac{1}{2}\left(\frac{1}{2\pi} e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]}\right) + \frac{1}{2}\left(\frac{1}{2\pi} e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}\right)}$$

$$= \frac{e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]}}{e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]} + e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}}$$

$$\Pr[X=-1/Y_1=y_1, Y_2=y_2] = \frac{e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}}{e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]} + e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}}.$$

$$\hat{X} = \frac{e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]} - e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}}{e^{-\frac{1}{2}[(y_1-1)^2+(y_2-2)^2]} + e^{-\frac{1}{2}[(y_1+1)^2+(y_2+2)^2]}}.$$

$$= \frac{e^{-\frac{1}{2}(y_1^2 - 2y_1 + 1 + y_2^2 - 4y_2 + 4)} - e^{-\frac{1}{2}(y_1^2 + 2y_1 + 1 + y_2^2 + 4y_2 + 4)}}{\left(\quad '' \quad \right) + \left(\quad '' \quad \right)}$$

$$= \frac{e^{+y_1+2y_2} - e^{-y_1-2y_2}}{e^{y_1+2y_2} + e^{-y_1-2y_2}} = \tanh(y_1 + 2y_2)$$

LMMSE estimate of X from Y_1 and Y_2.

a) $E[X]=0$

(a) $E[Y_1]=0$ $\qquad \hat{X} = K_0 \underline{Y}$ where $K_0 R_Y = R_{XY}$. $\qquad Y_1+2Y_2 = \underbrace{5X+N_1+2N_2}_{5}$

$E[Y_2]=0$. $\qquad$ 30 $\quad$ 25

$$R_Y = \begin{bmatrix} 1+1 & 2 \\ 2 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$R_{XY} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$K_0 \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \Rightarrow K_0 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \frac{1}{6}$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{6}$$

$$\hat{X} = \frac{1}{6}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{Y_1 + 2Y_2}{6}.$$

(b) MSE achieved by LMMSE estimate.

$$E[|\tilde{X}|^2] = E[(X-\hat{X})(X-\hat{X})] = E[X(X-\hat{X})] = E[X^2] - E[X\hat{X}]$$

$$E[X^2] = 1$$

$$E[X\hat{X}] = E[X K_0 \underline{Y}] = K_0 R_{XY}^H = K_0 R_Y K_0^H = \frac{1}{6}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{5}{6}.$$

$$MSE = 1 - \frac{5}{6} = \frac{1}{6}.$$

(3) $Y_1 = X_1 + N_1$ $\qquad$ – X_1 zero mean, variance 1.

$Y_2 = X_2 + N_2$ $\qquad$ – N_1, N_2 i.i.d zero mean, variance σ^2 indep. of X_1

$X_2 = \rho X_1 + \sqrt{1-\rho^2}\, Z$ $\qquad$ – Z zero-mean, variance 1 indep of X_1, X_2, N_1, N_2

$E[X_2^2] = \rho^2 + 1 - \rho^2 = 1$.

(a) Estimate X_1 from Y_1 $\qquad \hat{X}_1 = k_0 Y_1 \qquad k_0(1+\sigma^2) = 1$

$$\Rightarrow \hat{X}_1 = \frac{Y_1}{1+\sigma^2}.$$

(b) Estimate Y_2 from Y_1 $\qquad \hat{Y}_2 = k Y_1 \qquad k(1+\sigma^2) = R_{Y_1Y_2} = E[X_1X_2] = \rho.$

$$\hat{Y}_2 = \hat{X}_2 + \hat{N}_2 = \rho\hat{X}_1 = \frac{\rho Y_1}{1+\sigma^2} \qquad \hat{Y}_2 = \frac{\rho Y_1}{1+\sigma^2}.$$

(c) let $E_2 = Y_2 - \hat{Y}_2 = Y_2 - \frac{\rho Y_1}{1+\sigma^2}$

Estimate X_2 from E_2.

$$\hat{X}_{2e} = k_e E_2 \qquad k_e R_{E_2} = R_{E_2 X_2}$$

$$R_{E_2X_2} = E\left[\left(Y_2 - \frac{\rho Y_1}{1+\sigma^2}\right)X_2\right] = 1 - \frac{\rho \cdot \rho}{1+\sigma^2} = 1 - \frac{\rho^2}{1+\sigma^2}.$$

$$R_{E_2} = E\left[(Y_2-\hat{Y}_2)(Y_2-\hat{Y}_2)\right] = E\left[Y_2(Y_2-\hat{Y}_2)\right]$$
$$= E[Y_2^2] - E[Y_2\hat{Y}_2]$$
$$= 1+\sigma^2 - \frac{\rho}{1+\sigma^2}E[Y_2Y_1] = 1+\sigma^2 - \frac{\rho}{1+\sigma^2}E[X_1X_2]$$
$$= 1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}$$

$$k_e = \frac{1 - \frac{\rho^2}{1+\sigma^2}}{1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}}.$$

$$\hat{X}_{2e} = \frac{1 - \frac{\rho^2}{1+\sigma^2}}{1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}}\left(Y_2 - \frac{\rho Y_1}{1+\sigma^2}\right).$$

(d) Estimate X_2 from Y_1

$$\hat{X}_{21} = \rho \hat{X} = \frac{\rho Y_1}{1+\sigma^2}.$$

(e) $$\hat{X}_2 = \hat{X}_{21} + \hat{X}_{2e} = \frac{\rho Y_1}{1+\sigma^2} + \frac{1-\frac{\rho^2}{1+\sigma^2}}{1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}}\left(Y_2 - \frac{\rho Y_1}{1+\sigma^2}\right)$$

$$= \frac{\sigma^2}{1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}}\left(\frac{\rho Y_1}{1+\sigma^2}\right) + \frac{1-\frac{\rho^2}{1+\sigma^2}}{1+\sigma^2 - \frac{\rho^2}{1+\sigma^2}} Y_2.$$

$$= \frac{\rho\sigma^2}{(1+\sigma^2)^2 - \rho^2} Y_1 + \frac{1+\sigma^2 - \rho^2}{(1+\sigma^2)^2 - \rho^2} Y_2.$$

$\hat{X}_2$ from Y_1, Y_2.

$$\hat{X}_2 = [\; K_0 \;] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$[\; K_0 \;] = R_{X_2Y} R_Y^{-1}$$

$$= [\rho \quad 1] \begin{bmatrix} 1+\sigma^2 & \rho \\ \rho & 1+\sigma^2 \end{bmatrix}^{-1}$$

Direct answer for 3(e)

$$= [\rho \quad 1] \begin{bmatrix} 1+\sigma^2 & -\rho \\ -\rho & 1+\sigma^2 \end{bmatrix} \frac{1}{(1+\sigma^2)^2 - \rho^2}$$

$$\hat{X}_2 = \frac{\rho\sigma^2}{(1+\sigma^2)^2 - \rho^2} Y_1 + \frac{1+\sigma^2-\rho^2}{(1+\sigma^2)^2 - \rho^2} Y_2 \quad = [\rho\sigma^2 \quad -\rho^2+1+\sigma^2] \frac{1}{(1+\sigma^2)^2 - \rho^2}$$