EE5040: Adaptive Signal Processing
Problem Set 8: Least-squares and recursive least-squares

1. (Sayed, Chapter 34, Lemma 34.4, Complex Householder reflection) Consider a $1 \times n$ vector $\mathbf{z}$ with possibly complex entries. Choose $\mathbf{g} = \mathbf{z} \pm ||\mathbf{z}||e^{j\phi_a}\mathbf{e}_0$ and
$$\Theta = \mathbf{I}_n - 2\frac{\mathbf{g}^H\mathbf{g}}{\mathbf{g}\mathbf{g}^H}$$
to get
$$\mathbf{z}\Theta = \mp||\mathbf{z}||e^{j\phi_a}\mathbf{e}_0.$$
Here $\mathbf{e}_0 = [1\ 0\ 0 \cdots 0]$ and ϕ_a denotes the phase of the leading entry of $\mathbf{z}$.

2. (Sayed, Example 34.1, Using Householder transformations) Assume we are given a 2×3 matrix $A = \begin{bmatrix} 1 & 0.75 & 0.75 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$ and that we wish to reduce it to the form $A\Theta = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \end{bmatrix}$ via Householder tranformations. (Hint: First make the (1,2) and (1, 3) entries 0, and then the (2, 3) entry 0.)

3. (Sayed VII.11, Danger of Squaring) Solving the normal equations $H^H H\hat{\mathbf{w}} = H^H\mathbf{x}$ by forming the matrix $H^H H$ (i.e., by squaring the data) is a bad idea in general. This is because for ill-conditioned matrices H, numerical precision is lost when the matrix product $H^H H$ is formed. Recall that ill-conditioned matrices are those that have a very large ratio of largest to smallest singular values, i.e., they are close to being rank-deficient. Consider that full-rank matrix
$$H = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \\ 1 & 1 \end{bmatrix},$$
where ϵ is a very small positive number that is of the same order of magnitude as the machine precision. Assuming $2 + \epsilon^2 = 2$ in finite precision, what is the rank of $H^H H$?

4. (Sayed VIII.6, Rotation matrix for QR algorithm) In the QR RLS algorithm, we wanted a unitary transformation Θ such that (assume $M = 3$):
$$\begin{bmatrix} \lambda^{1/2}\Phi_{i-1}^{1/2} & \mathbf{h}_i \\ \lambda^{1/2}\mathbf{q}_{i-1}^H & X_i^* \end{bmatrix}\Theta \equiv \left[\begin{array}{ccc|c} \times & 0 & 0 & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \\ \hline \times & \times & \times & \times \end{array}\right]\Theta = \left[\begin{array}{ccc|c} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \hline \times & \times & \times & \times \end{array}\right].$$
One way to implement Θ is via a sequence of three elementary Givens rotations in order to annihilate the three entries of $\mathbf{h}_i$, one at a time. Now since, by assumption, the diagonal entries of $\Phi_{i-1}^{1/2}$ are positive, we have that the diagonal entries of the triangular matrix in the pre-array (left hand side of the equation) are positive. Recall further that we desire a post-array (right hand side of the equation) having a $\Phi_i^{1/2}$ with positive diagonal entries as well. Using the fact that the diagonal entries of the individual Givens rotations will be positive, conclude that the rightmost diagonal entry of Θ is positive.