EE5040: Adaptive Signal Processing
Problem Set 6: Least-squares

1. (Sayed VII.10, Cholesky method) Consider the normal equations $H^H H\hat{\mathbf{w}} = H^H\mathbf{x}$, where H is $N \times M$. Assume H has full column rank (i.e., the rank of H is M) so that $H^H H$ is positive definite. The normal equations can be solved by the standard method of Gaussian elimination. They can also be solved by appealing to the Cholesky factorization of $H^H H$. Every positive definite matrix admits a unique triangular factorization of the form $H^H H = LL^H$, where L is lower-triangular with positive entries on its diagonal. The matrix L is called the Cholesky factor of $H^H H$. Show that the normal equations can be solved by means of the following two steps.

   (a) Solve the lower triangular system of equations $L\hat{\mathbf{y}} = H^H\mathbf{x}$ for $\hat{\mathbf{y}}$.

   (b) Solve the upper triangular system of equations $L^H\hat{\mathbf{w}} = \hat{\mathbf{y}}$ for $\hat{\mathbf{w}}$.

2. (Sayed VII.19, Stochastic properties of least-squares solutions) Let $\mathbf{x} = H\mathbf{w}_0 + \mathbf{v}$, where $\mathbf{w}_0$ is an unknown vector that we wish to estimate and $\mathbf{v}$ is a zero-mean random vector with covariance matrix $\sigma_v^2\mathbf{I}$. Moreover, H is $N \times M$, $N \geq M$, and has full column rank. The LS estimate of $\mathbf{w}_0$ is $\hat{\mathbf{w}} = (H^H H)^{-1}H^H\mathbf{x}$. In order to study the stochastic proerties of this least-squares solution, we need to treat it $\mathbf{x}$ as a random quantity.

   (a) Verify that $\hat{\mathbf{w}} = \mathbf{w}_0 + (H^H H)^{-1}H^H\mathbf{v}$, and conclude that $\hat{\mathbf{w}}$ is an unbiased estimator.

   (b) Show that $E[(\mathbf{w}_0 - \hat{\mathbf{w}})(\mathbf{w}_0 - \hat{\mathbf{w}})^H] = \sigma_v^2(H^H H)^{-1}$.

   (c) Assume $N > M$ and let $\hat{\boldsymbol{\sigma}}_v^2 = ||\tilde{\mathbf{x}}||^2/(N - M)$, where $\tilde{\mathbf{x}} = \mathbf{x} - H\hat{\mathbf{w}}$, denote an estimator of the noise variance. Verify that $E[||\tilde{\mathbf{x}}||^2] = \sigma_v^2\mathrm{Tr}(I - P_H)$, where P_H is the projection matrix onto $R(H)$. Show further that $\mathrm{Tr}(I - P_H) = N - M$ and conclude that $\hat{\boldsymbol{\sigma}}_v^2$ is an unbiased estimate of σ_v^2.