PSD Estimation Using the DFT

1 Introduction

The power spectral density (or simply power spectrum) of a random signal can be estimated using the DFT. There are two basic approaches: one is called *periodogram analysis* and the other an indirect approach based on the autocorrelation sequence.

2 Estimating the Power Spectrum of a Random Signal Using the Periodogram

Let $x[n]$ be a stationary random signal and $v[n] = x[n] \cdot w[n]$ where be $w[n]$ is a window that selects L samples. An estimate of the power spectrum is given by

$$I(\omega) = \frac{1}{LU} \left| V(e^{j\omega}) \right|^2$$

where $V(e^{j\omega})$ is the Fourier transform of $v[n]$. The constant U anticipates the need for normalization to remove the bias in the spectral estimate. When the $w[n]$ is the rectangular window, this estimate is called the *periodogram* (otherwise, it is called the *modified periodogram*). Explicit computation of the periodogram can be carried out only at discrete frequencies. If $\omega_k = 2\pi k/N$, the DFT is a natural choice:

$$I(\omega_k) = \frac{1}{LU} \left| V[k] \right|^2 .$$

The scale factor U is normally chosen to be the mean-square value of $w[n]$, which for the rectangular window is 1.

An estimator is said to be *consistent* if both bias and variance tend to zero as the data length tends to infinity. The periodogram is not a consistent estimator because the variance of the estimate does not reduce to zero with increasing window length. This problem is dealt with by averaging many estimates.[1] If the data record is sufficiently long, the individual estimates are obtained from short non-overlapping blocks. If overlapping blocks are considered, an overlap of up to half gives further reduction in the variance of the estimate. If the data length increases to infinity, periodogram averaging produces an asymptotically unbiased, consistent PSD estimate.

Let $x[n] = A\cos(\omega_0 n + \theta) + e[n]$ where θ is a uniform random variable between 0 and 2π and $e[n]$ is a zero-mean white noise sequence that has constant power spectrum. Let $A = 0.5$, $\omega_0 = 2\pi/21$, and random phase $0 \leq \theta < 2\pi$. Let $e[n]$ be uniformly distributed such that $-\sqrt{3} < e[n] \leq \sqrt{3}$. Generate 1024 samples of $x[n]$.

1. That the periodogram is not a consistent estimator can be see by computing the PSD of $e[n]$ for various lengths. Take 16, 64, 256 and 1024 samples of $e[n]$ and zero-pad

[1] When $w[n]$ is the rectangular window the method of averaging periodograms is called *Bartlett's procedure.*

them appropriately to get a 1024-length sequence. Plot the PSD estimate for each case. Do the fluctuations diminish with increasing data length? What is the shape of the true PSD?

2. Compute the periodogram of the entire data $x[n]$ (no averaging).

3. Now let the length of each block be 64. There will be 16 non-overlapping blocks. Compute the averaged periodogram PSD estimate.

4. Repeat by increasing the noise variance. Also try overlapping blocks.

   For this $x[n]$, the expected value of the averaged periodogram at the frequency ω_0 is $A^2L/4 + \sigma_e^2$. For this noise distribution, $\sigma_e^2 = 1$.

3 PSD From Estimated Autocorrelation Sequence

Our second approach is based on the Wiener-Kinchine theorem, which says that the autocorrelation sequence and PSD are Fourier transform pairs. Therefore we first estimate the autocorrelation sequence $\phi_{xx}[m]$ and then compute its Fourier transform to obtain the PSD estimate.

Let us first understand why the periodogram is not a consistent estimator, i.e., why the variance does not decrease with increasing data length. It can be show that

$$I(\omega) = \frac{1}{LU} \sum_{m=-(L-1)}^{L-1} c_{vv}[m] e^{-j\omega m}$$

where $c_{vv}[m] = \sum_{n=0}^{L-1} v[n]v[n{+}m]$ (which is nothing but the aperiodic autocovariance sequence estimate for $v[n] = w[n]\, x[n]$). Observe that as m gets close to $L-1$ only a few samples of $v[n]$ enter into the computation. This results in poor estimates (large variance) for these lag values. This variability manifests itself in the Fourier transform as fluctuations at all frequencies, which is why the periodogram is not a consistent estimator.

As explained above, in the periodogram *all* lag values are implicitly involved in the PSD estimate. On the other hand, if we explicitly compute the lag values and discard the poor estimates for large m, we would have better control over the PSD estimate. Therefore, in this second approach, we will compute the lag values only up to M $(< L)$ and then compute their DFT to get the PSD estimate.

Note that the DFT can also be used for computing the lag values. If only a few lags have to be computed, the time-domain approach is more efficient; otherwise, the DFT approach is better (because of the FFT algorithm), the break-even point occurring for $M < 100$.

An estimate of the autocorrelation sequence can be obtained using the `corr` command in Scilab. Let us estimate the PSD for the signal given in the previous section using the autocorrelation approach.

1. Write down the expression for the correlation sequence of $x[n]$ given in the previous section. Let $M = 64$, and estimate these correlation lags using the command `corr(x,64)` and plot the result. How does it compare with the theoretical value?

2. Once the autocorrelation sequence has been obtained, the PSD is obtained by first forming the finite length sequence

$$s[m] = \begin{cases} \hat{\phi}_{xx}[m]\, w_c[m] & 0 \leq m \leq M-1 \\ 0 & M \leq m \leq N-M \\ \hat{\phi}_{xx}[N-m]\, w_c[N-m] & N-M+1 \leq m \leq N-1 \end{cases}$$

   where $w_c[n]$ is a symmetric window applied to the correlation sequence. If it is triangular, the resulting estimate is non-negative (because $W_c(e^{j\omega}) \geq 0$ for $-\pi < 0 \leq \pi$). If $w_c[n]$ is one of the other commonly used windows, e.g., the rectangular window, non-negativity is not guaranteed. For simplicity, we will use the rectangular window. The DFT $S[k]$ of $s[m]$ is the PSD estimate using the autocorrelation method. Choose $N = 1024$ and compare this estimate with that obtained in the previous section.