Almost Budget Balanced Mechanisms for Efficient Allocation of a Divisible Good[1]

Srikrishna Bhashyam

Department of Electrical Engineering
IIT Madras

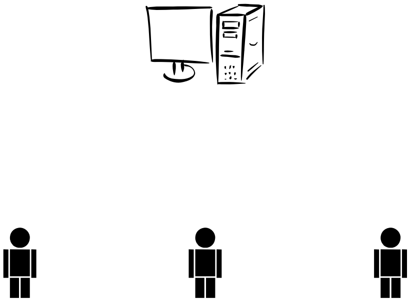
Joint work with
A. K. Chorppath, H. Vinay (IIT Madras)
D. Thirumulanathan, R. Sundaresan (IISc Bangalore)

13 Dec 2013
Northwestern University

[1] Supported by Department of Science and Technology, Govt. of India

Motivation

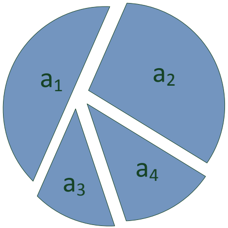
- Fair sharing of internet resources
- Auctioning a public resource

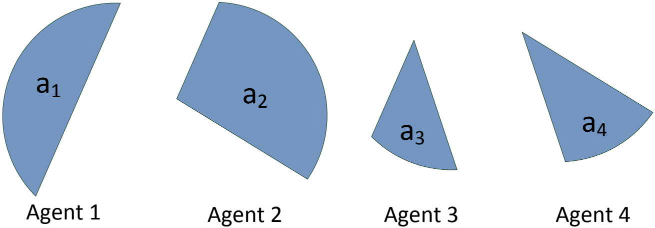


- Efficient allocation depends on privately held information
- How can we counter strategic behavior and be efficient?
- Not interested in maximizing revenue

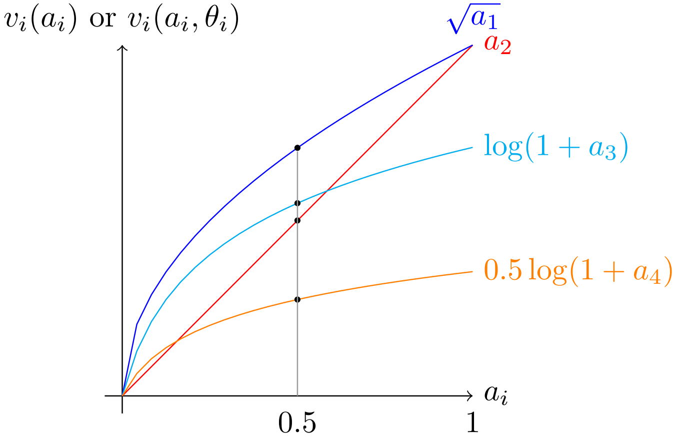
A divisible resource (or good)

Can be split in arbitrary sized parts





What is an efficient allocation?



Allocate resource such that sum valuation is maximized

$$\max_{\{a_i\}} \sum_{i=1}^{n} [\text{Valuation of agent } i]$$

Private Information: Two Scenarios

- **Unknown parameter in valuation functions**: Agent's valuation function known to social planner except for scalar parameter

  e.g. $v_i(a_i, \theta_i) = \theta_i log(1 + a_i), \theta_i$ is private information for agent i

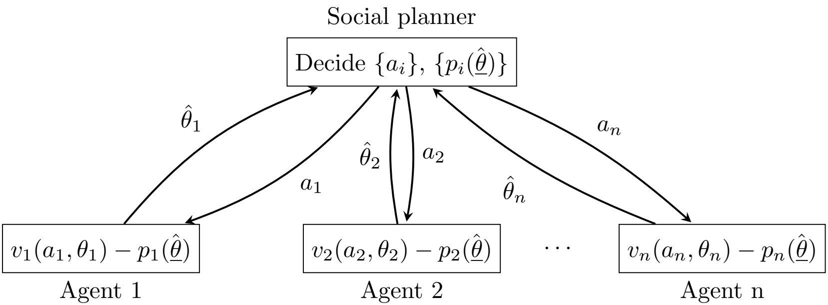
- **Unknown valuation functions**: Agent's full valuation function not known to social planner and other agents

  e.g. $v_i(a_i) = \sqrt{a_i}$ and $v_i(a_i)$ is private information for agent i

Social planner needs to extract private information to achieve efficiency.

Agents can be strategic.

Pricing mechanism to allocate resource



- Agent i reports (bids) $\hat{\theta}_i$
- Social planner allocates a_i to agent i and collects payment $p_i(\underline{\hat{\theta}})$
- Agents know the algorithm used by social planner
- Quasi-linear setting

Budget balance

- Budget surplus: Sum of payments

$$\sum_{i=1}^{n} p_i(\underline{\hat{\theta}})$$

- Strong budget balance: Budget surplus $= 0$
- Weak budget balance: Budget surplus > 0
- Notions of almost budget balance to be defined later

Our work

- Design mechanism (algorithm used by social planner) to achieve:
  - Efficiency (despite strategic behavior)
  - Almost budget balance

- Two scenarios
  - Unknown parameter in valuation functions
  - Unknown valuation functions

- Approach
  - Formulate mechanism design as a convex optimization problem
  - Approximate solutions with closeness guarantee

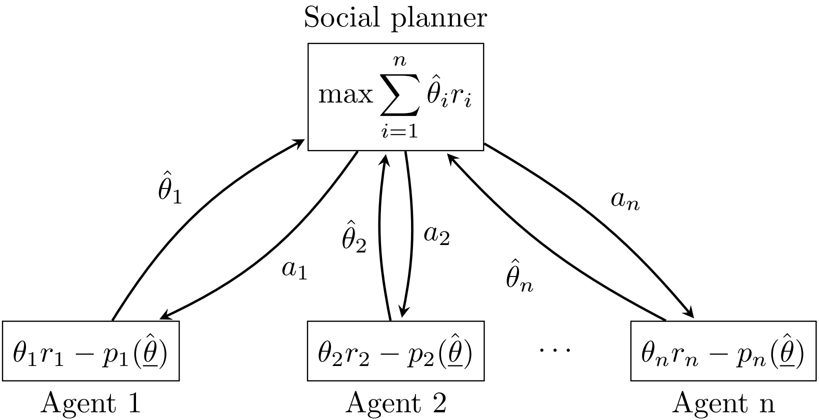
Unknown parameter in valuations

$$v_i(a_i, \theta_i)$$

Setting

- n agents
- $v_i(a_i, \theta_i)$: Valuation function of agent i
- $\theta_i \in [0, 1]$, $\Theta = [0, 1]^n$
- $v_i(\cdot, \theta_i)$ is concave, non-decreasing, $v_i(a_i, 0) = 0$
- Example: $\theta_i \log(1 + a_i)$

Example: Max-weight scheduling



- (Normalized) Queue length θ_i, Instantaneous rate r_i

Vickrey-Clarke-Groves (VCG) Mechanism[2] [3] [4]

- Social planner maximizes $\sum_i v_i(a_i, \hat{\theta}_i)$ to get $\{a_i^*\}$
- Payment for agent i

$$p_i(\underline{\hat{\theta}}) = -\sum_{j \neq i} v_j(a_j^*, \hat{\theta}_j) + h_i(\underline{\hat{\theta}}_{-i}),$$

where

$$h_i(\underline{\hat{\theta}}_{-i}) = \sum_{j \neq i} v_j(a_{-i,j}^*, \hat{\theta}_j)$$

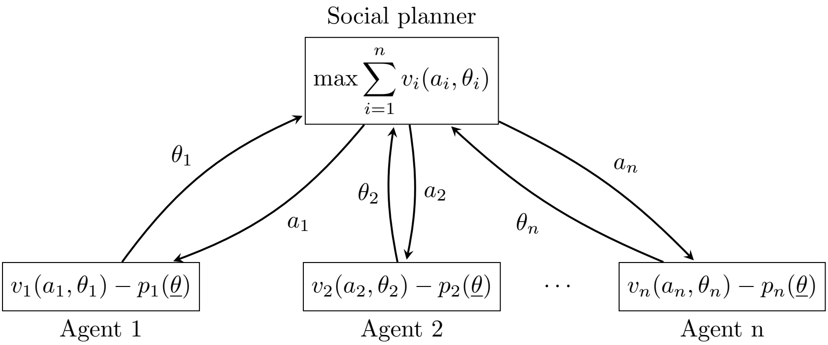
- $\underline{\hat{\theta}}_{-i}$: Vector of bids of all agents except agent i
- $\{a_{-i,j}^*\}$: Allocation when agent i does not participate

[2] W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, The Journal of Finance, vol. 16, no. 1, pp. 8-37, 1961.

[3] E. Clarke, Multipart pricing of public goods, Public Choice, vol. 2, pp. 19-33, 1971.

[4] T. Groves, Incentives in teams, Econometrica, vol. 41, no. 4, pp. 617-631, 1973.

Vickrey-Clarke-Groves (VCG) Mechanism



- Agent i's best strategy $\hat{\theta}_i = \theta_i$ regardless of strategies of other agents
- Mechanism is dominant strategy incentive compatible (DSIC)
- Mechanism is efficient
- Replace $\hat{\theta}_i$ by θ_i henceforth

Budget balance and rebates

- Strong budget balance not possible in general[5]
- To reduce budget surplus: Redistribute payments as rebates

$$\text{Payment for agent } i\text{:} \;\; p_i(\underline{\theta}) = p_{VCG,i}(\underline{\theta}) - r_i(\underline{\theta}_{-i})$$

- Mechanism is still in the VCG class $\Longrightarrow$ Efficient
- Rebates for discrete (indivisible) goods
  - Guo & Conitzer[6] and Moulin[7]

---

[5] J. Green, J.-J Laffont, Characterization of satisfactory mechanisms for the revelation of preferences for public goods, Econometrica, vol. 45, pp. 427-438, 1977.

[6] M. Guo and V. Conitzer, Worst-case optimal redistribution of VCG payments in multi-unit auctions, Games and Economic Behavior, vol. 67, no. 1, pp. 69-98, September 2009.

[7] H. Moulin, Almost budget-balanced VCG mechanisms to assign multiple objects, Journal of Economic Theory, vol. 144, no. 1, pp.96-119, January 2009.

Rebates: Desired properties

- Feasibility (F) or Weak budget balance

$$\sum_i p_i(\underline{\theta}) > 0$$

- Voluntary participation (VP)

$$v_i(a_i^*, \theta_i) - p_i(\underline{\theta}) \geq 0 \;\; \forall i,$$

assuming payoff for not participating in mechanism is 0

Rebates: Desired properties

- Deterministic and anonymous rebates
  - Two agents with identical bids get identical rebates

- Rebates a deterministic function of the ordered bids[8]

$$r_i(\underline{\theta}_{-i}) = g((\underline{\theta}_{-i})_{[1]}, (\underline{\theta}_{-i})_{[2]}, \ldots, (\underline{\theta}_{-i})_{[n-1]})$$

[8] M. Guo, V. Conitzer, Worst-case optimal redistribution of VCG payments in multi-unit auctions, Games and Economic Behavior, vol. 67, pp. 69-98, Sep. 2009.

Notions of almost budget balance

Worst-case design

- Moulin: Minimize the worst-case ratio of the sum of payments to the sum of valuations

$$\min \sup_{\underline{\theta} \in \Theta} \frac{p_V(\underline{\theta}) - \sum_{i=1}^{n} r_i(\underline{\theta}_{-i})}{\sigma_V(\underline{\theta})}$$

- Guo & Conitzer: Maximize the worst-case ratio of sum of rebates to sum of payments

$$\max \inf_{\underline{\theta} \in \Theta} \frac{\sum_{i=1}^{n} r_i(\underline{\theta}_{-i})}{p_V(\underline{\theta})}$$

Sum of valuations: $\sigma_V(\underline{\theta}) = \sum_{i=1}^{n} v_i(a_i^*, \theta_i)$
Sum of VCG payments: $p_V(\underline{\theta}) = \sum_{i=1}^{n} p_{VCG,i}(\underline{\theta})$

Almost budget balance and linear rebates

- Discrete good case[9] [10]
  - Both notions yield same optimal rebates
  - Linear rebates are optimal

$$r_i(\underline{\theta}_{-i}) = c_0 + c_1(\underline{\theta}_{-i})_{[1]} + c_2(\underline{\theta}_{-i})_{[2]} + \cdots + c_{n-1}(\underline{\theta}_{-i})_{[n-1]}$$

- Divisible good case
  - We use Moulin's notion of almost budget balance
  - Restrict ourselves to linear rebates
  - Optimality of linear rebates not yet explored
  - Objective function depends only on ordered bids: Assume agents are ordered according to bids

$$r_i(\underline{\theta}_{-i}) = c_0 + c_1\theta_1 + \cdots + c_{i-1}\theta_{i-1} + c_{i+1}\theta_{i+1} + \cdots + c_{n-1}\theta_n$$

  - $\hat{\Theta} = \{\underline{\theta} \in \Theta | 1 \geq \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0\}$

[9] M. Guo, V. Conitzer, Worst-case optimal redistribution of VCG payments in multi-unit auctions, Games and Economic Behavior, vol. 67, pp. 69-98, Sep. 2009.

[10] S. Gujar, Y. Narahari, Redistribution mechanisms for assignment of heterogeneous objects, Journal of Artificial Intelligence Research, vol. 41, pp. 131-154, 2011.

Optimization problem to design rebates

$$\min_{\{c_i\}} \sup_{\underline{\theta}\in\hat{\Theta}} \frac{p_V(\underline{\theta}) - \sum_{i=1}^{n} r_i(\underline{\theta}_{-i})}{\sigma_V(\underline{\theta})}$$

subject to:

- (F) Feasibility constraints

$$\sum_i r_i(\underline{\theta}_{-i}) \leq p_V(\underline{\theta}) \;\; \forall\theta \in \hat{\Theta}$$

- (VP) Voluntary Participation constraints $\forall i$

$$r_i(\underline{\theta}_{-i}) \geq -v_i(a_i^*, \theta_i) + p_{VCG,i}(\underline{\theta}) \stackrel{\triangle}{=} n_i(\underline{\theta}) \;\; \forall\theta \in \hat{\Theta}$$

With linear rebates

$$\text{Note that} \quad \sum_i r_i(\underline{\theta}_{-i}) = nc_0 + \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i)$$

Problem with linear rebates

$$\min_{\{c_i\}} \sup_{\underline{\theta} \in \hat{\Theta}} \frac{p_V(\underline{\theta}) - \sum_{i=0}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i)}{\sigma_V(\underline{\theta})}$$

subject to:

$$\text{(F)} \quad nc_0 + \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_V(\underline{\theta}) \;\; \forall \theta \in \hat{\Theta}$$

$$\text{(VP)} \quad c_0 + \sum_{j=1}^{i-1} c_j\theta_j + \sum_{j=i+1}^{n} c_j\theta_j \geq -v_i(a_i^*, \theta_i) + p_{VCG,i}(\underline{\theta}) = n_i(\underline{\theta}) \;\; \forall \theta \in \hat{\Theta} \;\; \forall i$$

Simplification 1

- Some good choices of $\underline{\theta}$ are $\underline{e}_k = (1, \cdots, 1, 0, \cdots, 0)$ with k ones, for $k = 0, 1, \cdots, n$
- Using the $\underline{e}_k$'s in (F) and (VP): $c_0 = c_1 = 0$
- (VP) constraints equivalent to:

$$\sum_{i=2}^{k} c_i \geq 0 \;\; \forall k = 2, 3, \cdots, n-1$$

Simplification 2

$$L = \sup_{\underline{\theta} \in \hat{\Theta}} \frac{p_V(\underline{\theta}) - \sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i)}{\sigma_V(\underline{\theta})}$$

is equivalent to saying that:

L is the smallest number such that:

$$\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) + L\sigma_V(\underline{\theta}) \geq p_V(\underline{\theta}) \ \ \forall \underline{\theta} \in \hat{\Theta}$$

Uncertain convex program: WoCLP

$$\min_{\{c_i\},L} L$$

subject to:

- (F) Feasibility constraints

$$\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_V(\underline{\theta}) \;\; \forall \theta \in \hat{\Theta}$$

- (VP) Voluntary Participation constraints $\forall i$

$$\sum_{i=2}^{k} c_i \geq 0 \;\; \forall k = 2, 3, \cdots, n-1$$

- (W)

$$\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) + L\sigma_V(\underline{\theta}) \geq p_V(\underline{\theta}) \;\; \forall \underline{\theta} \in \hat{\Theta}$$

Another notion of almost budget balance: OpELP

Optimal-in-expectation design[11]

- Assume a prior distribution on the private information
- Minimize the ratio of expected budget surplus to expected sum of valuations
- Equivalent to maximizing expected sum of rebates

$$\mathbb{E}\left[\sum_i r_i(\underline{\theta}_{-i})\right]$$

subject to (F) and (VP) constraints

[11] M.Guo, V. Conitzer, Optimal-in-Expectation Redistribution Mechanisms, Proc. of 7th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2008), Padgham, Parkes, Muller and Parsons (eds.), May 12-16, 2008, Estoril, Portugal.

OpELP

$$\max_{\{c_i\}} \sum_{i=2}^{n-1} c_i(i\mathbb{E}[\theta_{i+1}] + (n-i)\mathbb{E}[\theta_i])$$

subject to:

- (F) Feasibility constraints

$$\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_V(\underline{\theta}) \ \ \forall\theta \in \hat{\Theta}$$

- (VP) Voluntary Participation constraints $\forall i$

$$\sum_{i=2}^{k} c_i \geq 0 \ \ \forall k = 2, 3, \cdots, n-1$$

Unknown valuations

$v_i(a_i)$

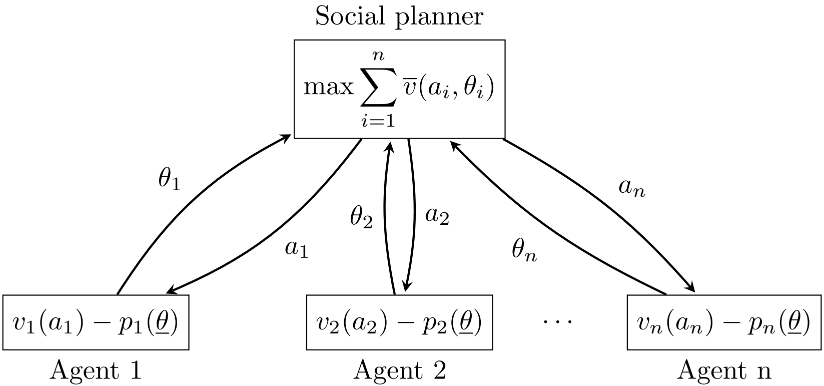
Setting

- n agents
- $v_i(a_i)$: Valuation function of agent i
  - Unknown to social planner
- Social planner announces a scalar-parametrized surrogate valuation function set

$$\{\overline{v}(\cdot, \theta), \theta \in [0, \infty)\}$$

- Concave, strictly increasing, continuous, continuously differentiable
- More assumptions on $v_i(\cdot)$, $\overline{v}(a, \theta)$
  - For existence of efficient Nash equilibrium

Scalar Strategy VCG (SSVCG) mechanism[12] [13]



- Efficient DSIC mechanism not possible
- Efficient Nash equilibria exist

[12] S. Yang, B. Hajek, VCG-Kelly mechanisms for divisible goods: Adapting VCG mechanisms to one-dimensional signals, IEEE JSAC, vol. 25, no. 6, pp. 1237-1243, Aug. 2007.

[13] R. Johari, J. N. Tsitsiklis, Efficiency of scalar-parametrized mechanisms, Operations Research, vol. 57, no. 4, pp. 823-839, 2009.

Scalar Strategy VCG (SSVCG) mechanism

- At least two agents have infinite marginal valuation at 0

$$v_i'(0) = v_j'(0) = \infty$$

  for some agents i, j with $i \neq j$
- For every $\gamma \in (0, \infty)$ and $a \geq 0$, there exists a $\theta > 0$ such that

$$\overline{v}'(a, \theta) = \gamma$$

  (derivative with respect to a)

Need a rich enough surrogate valuation function class
Example: $\overline{v}(a_i, \theta_i) = \theta_i a_i^x, \ x < 1$

Rebates and desired constraints

Payment for agent i

$$p_i(\underline{\theta}) = -\sum_{j \neq i} \overline{v}(a_j^*, \theta_j) + \sum_{j \neq i} \overline{v}(a_{-i,j}^*, \theta_j) - r_i(\underline{\theta})$$

Rebates with following properties:

- **Feasibility (F) or weak budget balance:**

$$\sum_{i=1}^{n} p_i(\underline{\theta}) \geq 0 \implies \sum_{i=1}^{n} r_i(\underline{\theta}) \leq p_V(\underline{\theta})$$

- **Voluntary participation (VP):** Agents better off by participating.

$$v_i(a_i^*) - p_i(\underline{\theta}) \geq 0 \implies r_i(\underline{\theta}) \geq n_i(\underline{\theta})$$

- **Deterministic and anonymous**

Problem formulation - Almost budget balanced SSVCG Mechanism

Worst case problem:

$$\min_{\underline{r}} \sup_{\underline{\theta} \in \Theta_{ne}} f(\underline{r}, \underline{\theta})$$

$$\text{(F)} \sum_{i=1}^{n} r_i(\theta_{-i}) \leq p_{VCG}(\underline{\theta}) \; \forall \underline{\theta} \in \Theta_{ne}$$

$$\text{(VP)} \; r_i(\theta_{-i}) \geq n_i(\underline{\theta}) \; \forall \underline{\theta} \in \Theta_{ne}, i,$$

$p_{VCG}(\underline{\theta}) =$ sum payment without rebates
$n_i(\underline{\theta}) \quad = -v_i(a_i^*) - \sum_{j \neq i} \overline{v}(a_j^*, \theta_j) + \sum_{j \neq i} \overline{v}(a_{-i,j}^*, \theta_j)$

Three difficulties in problem formulation

1) Characterizing the Nash equilibria set Θ_{ne}
2) Dependency of (VP) constraints on true valuations
3) Choosing the appropriate objective function $f(\underline{r}, \underline{\theta})$

1) Characterizing the Nash equilibria set

$\Theta_{ne} = \bigcup_{\{(v_i)\}} \{\text{Nash equilibria for valuations } (v_i)\}$

For a given surrogate function $\overline{v}(a, \theta)$, the set $\Theta_{ne} = [0, \infty)^n$

Outline of argument:

- If $v_i(a_i) \in \{\overline{v}(a, \theta) | \theta > 0\}$ for some $\theta = \alpha_i$, then mechanism reduces to VCG mechanism
- $(\alpha_1, \ldots, \alpha_n)$ is a Nash equilbrium
- Choosing various valuations will give all of $[0, \infty)^n$

2) Simplifying the (VP) constraints

If $v_i(0) = 0\ \forall i$ and $\overline{v}(a, 0) = 0$, following are equivalent

$$(a)\ r_i(\theta_{-i}) \geq n_i(\underline{\theta})\ \forall i, \underline{\theta} \in \Theta_{ne}$$

$$(b)\ \sum_{i=2}^{k} c_i \geq 0\ \forall k = 2, 3, \ldots, n-1$$

- Removes true valuations
- Reduces to a finite number of constraints

3) Choosing the appropriate objective

- $\frac{\text{sum of payments}}{\text{sum of valuations}}$[14] has unknown true valuations
- Sum of payments gives infinity in the worst case
- $\frac{\text{sum of rebates}}{\text{sum VCG payment}}$[15] cannot give closeness guarantee
- $\frac{\sum p_i(\underline{\theta})}{\sum \theta_i}$
  - For $\overline{v}(a, \theta) = \theta f(a)$, it minimizes sum payments for normalized equilibria
  - Provides closeness guarantee

[14] H. Moulin, Almost budget-balanced VCG mechanisms to assign multiple objects, JET, 2009

[15] M. Guo and V. Conitzer, Worst-case optimal redistribution of VCG payments in multi-unit auctions, GEB, 2009

Worst-case problem

$$\min_{\underline{c}} \sup_{\underline{\theta} \in \Theta_{ne}} f(\underline{c}, \underline{\theta})$$

$$(\text{F}) \sum_{i=2}^{n-1} c_i (i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}) \ \forall \underline{\theta} \in \Theta_{ne}$$

$$(\text{VP}) \sum_{i=2}^{k} c_i \geq 0 \ \forall k = 2, 3, \dots, n-1$$

Let $\quad L(n) = \sup_{\theta \in \Theta_{ne}} f(\underline{c}, \underline{\theta}) = \sup_{\theta \in \Theta_{ne}} \dfrac{p_{VCG}(\underline{\theta}) - \sum r_i(\theta_{-i})}{\sum \theta_i}$

Final optimization problem: SSVCG-WoCLP

For $\overline{v}(a, \theta) = \theta f(a)$, we have

$$\min_{\underline{c}, L(n)} L(n)$$

$$1) \sum_{i=2}^{n-1} c_i (i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}) \ \forall \underline{\theta} \in \Theta_s$$

$$2) \sum_{i=2}^{k} c_i \geq 0 \ \forall k = 2, 3, \ldots, n-1$$

$$3) \sum_{i=2}^{n-1} c_i (i\theta_{i+1} + (n-i)\theta_i) + L(n) \sum_{i=1}^{n} \theta_i \geq p_{VCG}(\underline{\theta}), \forall \underline{\theta} \in \Theta_s$$

- $\Theta_s = \{\underline{\theta} \in \Theta | 1 = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0\}$

SSVCG-OpELP

Similar to the VCG case except for Θ_s instead of $\hat{\Theta}$

$$\max_{\{c_i\}} \sum_{i=2}^{n-1} c_i(i\mathbb{E}[\theta_{i+1}] + (n-i)\mathbb{E}[\theta_i])$$

subject to:

- (F) Feasibility constraints

$$\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_V(\underline{\theta}) \ \ \forall \theta \in \Theta_s$$

- (VP) Voluntary Participation constraints $\forall i$

$$\sum_{i=2}^{k} c_i \geq 0 \ \ \forall k = 2, 3, \cdots, n-1$$

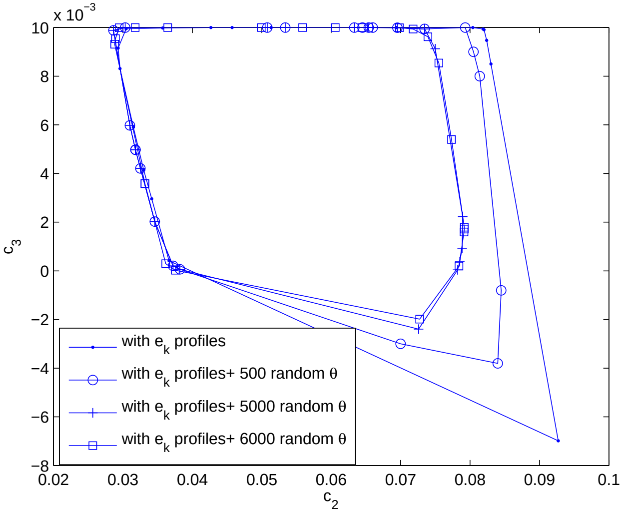
Constraint Sampling

Constraint sampling

- Uncertain convex program (UCP)[16] in all 4 cases
- WoCLP: Need to sample (F) and (W) constraints
- OpELP: Need to sample (F) constraints
- Random sampling approach
  - $\underline{e}_k$'s + Random samples of $\underline{\theta}$
- Two types of results
  - Number of samples required to make sampled constraint set close to actual with high probability
  - Number of samples required for value of sampled problem to be close to value of actual problem

[16] G. Calafiore, M. C. Campi, Uncertain convex programs: randomized solutions and confidence levels, Mathematical Programming – Online first, DOI 10.1007/s10107-003-0499-y, 2004

Constraint sampling: Illustration



8 agents, c_2, c_3 shown

Closeness guarantee

- Closeness of sampled constraint set and actual constraint set[17]
  - For a given ϵ, δ, number of samples $m(\epsilon, \delta)$ such that $\mathbb{P}[\text{Violation probability} \leq \epsilon] \geq 1 - \delta$
- Closeness of value of sampled program and actual UCP
  - For a given τ, number of samples $m(\tau)$ such that $|\text{Value of SCP - Value of UCP}| \leq \tau$
  - Two results: probabilistic guarantee, deterministic guarantee

[17] G. Calafiore, M. C. Campi, Scenario approach on robust control design, IEEE Transactions on Automatic Control, vol. 51, no. 5, pp. 742-752, 2006.

Closeness guarantee: Results

Probabilistic guarantee

- Under a Lipschitz condition on $\sigma_V(\underline{\theta})$, and a restriction on the parameter set:[18]
  - Number of samples $m(\tau, \delta, \nu)$ for the value to be $\tau-$close with probability $\geq 1 - \delta$

Deterministic guarantee

- Under some assumptions on the valuation functions: [19]
  - Example: Valuations of the form $\theta_i f(a_i)$
  - Number of samples $m(\tau)$ for the value to be $\tau-$close

[18] A. K. Chorppath, S. Bhashyam, R. Sundaresan, "A convex optimization framework for almost budget balanced allocation of a divisible good," IEEE Transactions on Automation Science and Engineering, vol.8, no.3, pp.520-531, July 2011.

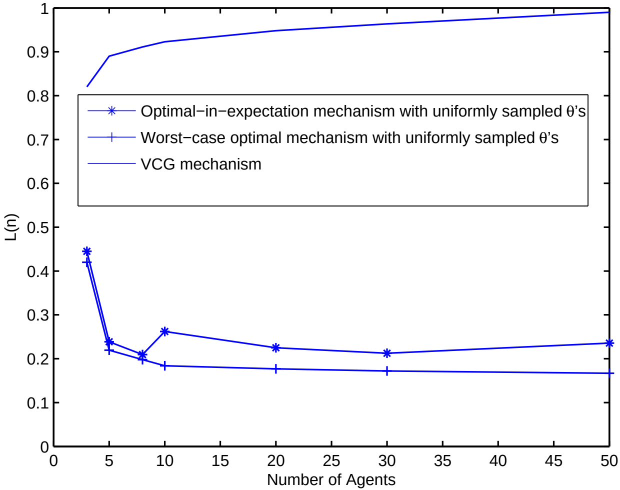
[19] D. Thirumulanathan, Resource allocation for strategic users, Masters' thesis, ECE, IISc Bangalore, 2012. http://www.ece.iisc.ernet.in/ nathan.d/project.pdf

Numerical Results

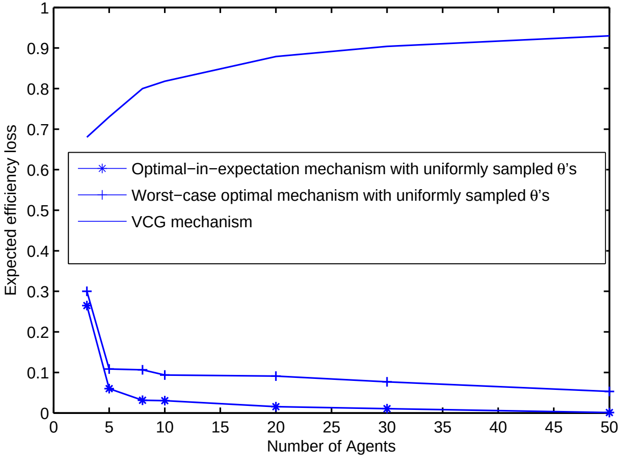
Simulation Setup: ABB-VCG

- $m = 2836n$ random samples from $\hat{\Theta}$ used to generate constraints
- Numerical solution: Determine approximate objective and rebate
- Using numerical solution for rebates, Monte Carlo simulations for a larger set (500,000) of $\underline{\theta}'$s
- Valuation function: $v_i(a_i, \theta_i) = \theta_i \log(1 + a_i)$

Numerical results



Numerical results



Simulation Setup: ABB-SSVCG

- 10,000 random samples of Θ_s are used to generate constraints
- Numerical solution: Determine approximate objective and rebate
- Using numerical solution for rebates, Monte Carlo simulations for a larger set (100,000) of $\underline{\theta}'$s
- Surrogate valuation $\overline{v}(a, \theta) = \theta\sqrt{a}$

Optimal mechanism in worst-case sense

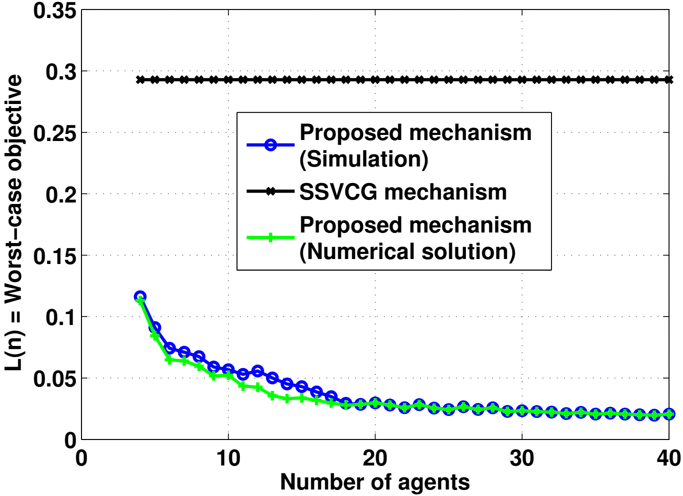


Figure : $\overline{v}(a, \theta) = \theta\sqrt{a}$

Optimal mechanism in expected sense

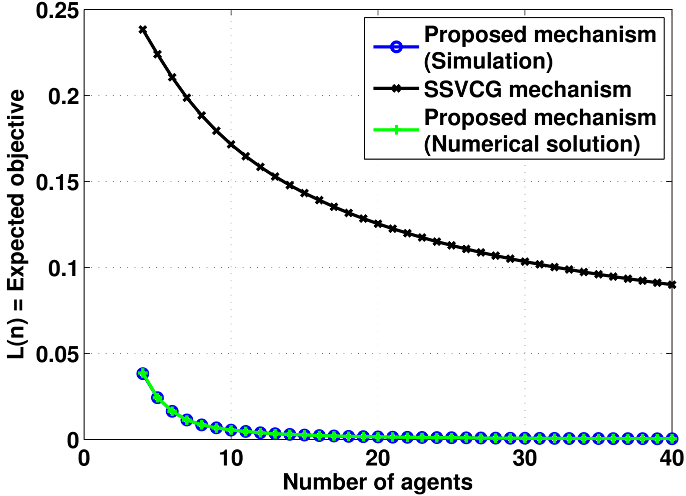


Figure : $\overline{v}(a, \theta) = \theta\sqrt{a}$

Summary: Allocation of a divisible good

- Efficiency and almost budget balance
- Unknown parameter case
  - VCG + rebates
  - Two designs: worst-case, optimal-in-expectation
- Unknown valuations case
  - Existence of efficient Nash equilibrium
  - SSVCG + rebates
  - Two designs: worst-case and optimal-in-expectation
- Formulation as an uncertain convex program
- Constraint sampling

Open questions

- Optimality of linear rebates
- Possibility of closeness guarantee with fewer samples