# Outage Probability and Goodput with ARQ in Multiple Access Channels

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***Abstract*—We consider a block fading multiple access channel with $N$ users transmitting information to a single receiver. We assume channel state is known only at the receiver and ARQ is used to combat fading and improve the transmission reliability. The overall goodput for such a system is a function of user transmission rates and outage probabilities. In this paper, we provide approximations for common and individual outage probability for a general $N$ user symmetric Gaussian MAC channel based on the geometry of the capacity region. Using these approximations, we analyse the optimal goodput as a function of the number of MAC users and the signal-to-noise ratio (SNR). With common outage, we observe a SNR threshold above which the optimal outage probability (for optimal goodput) decreases with the number of users, and below which the optimal outage probability increases with users. When individual outage is considered, the optimal outage probability decreases with increasing users for all SNRs.**

## I. Introduction

Automatic repeat request (ARQ) is a common technique used to improve the link level reliability in a wireless network. ARQ in its simplest form can be viewed as repetitive coding over multiple frames, wherein a data frame is retransmitted until received correctly at the receiver. ARQ is particularly useful in wireless networks where there is high cost of improving the physical layer reliability through coding and signal processing techniques, because of the inherent unreliability of the channel. Most current wireless standards employ ARQ and its variants like HARQ to improve the overall reliability of a link.

Wu et al., analysed ARQ in a point-to-point wireless link with Rayleigh fading in [1]. With ARQ, it was shown that the optimal frame error probability that maximizes the overall goodput is about $30\%$. Hence, simple high rate error control codes coupled with ARQ can be used to achieve the optimal goodput, thus reducing the overall complexity of the system [2]. In this paper, we analyse ARQ in a symmetric multiple access channel where are there are multiple transmitters and a single receiver.

In a MAC channel, there are two different notions of outage probabilities. The first is the common or joint outage probability where all the users are in outage even if one of the user cannot be decoded correctly. The other is the individual outage probability, which equals the probability that a particular user is in outage irrespective of the outage of other users. The capacity region of MAC with $N$ users is bounded by $2^N - 1$ mutual information rate constraints. Obtaining a general expression for outage is a messy combinatorial problem. Hence in this paper we obtain an approximation on common and individual outage probabilities based on the geometry of the capacity region. Using the approximation, the goodput is analysed for the common outage probability. For the individual outage, the mathematical analysis for goodput is tedious even with the approximation. Hence we present numerical results and discuss them.

The use of ARQ as error controlling scheme is long proposed in [2]. For a fading point-to-point channel, throughput maximization with delay constraints is considered in [3]. The diversity-multiplexing trade-off for MAC with ARQ is discussed in [4]. In [5], [6] the throughput capacity region for fading MAC and the optimum power and rate allocation schemes are provided. In [7] common and individual outage capacity regions are obtained for fading MAC and optimal power allocation strategies are discussed. All of the above works on MAC channels assume CSIT. A fading MAC without CSIT was considered in [8] and a comparison of throughput with joint multi-user decoding and single-user decoding was presented. In [9] it was shown that considering individual outages results in a better throughput compared to common outage for a two user case. In this paper, we consider a $N$ user fading MAC without CSIT and discuss outages and goodput involved with it.

The paper is organized as follows: The system model and the two notions of outage probabilities are introduced in Section II, and the outage approximations are provided in Section III for a symmetric $N$ user MAC. The goodput analysis is provided in Section IV. Finally, in Section V, we summarize our results.

## II. System Model

We consider a $N$ user symmetric discrete time MAC with Rayleigh fading [9]. The received signal is given by

$$y = \sum_{i=1}^{N} \sqrt{\gamma} h_i x_i + z \tag{1}$$

where $x_i$ is the information transmitted by the $i$-th user and $z$ is the additive Gaussian noise. We assume that the the input alphabet $x_i$ and $z$ are i.i.d. $\mathcal{CN}(0,1)$. The fading between the $i$-th user and the receiver is denoted by $h_i$ and is assumed to be i.i.d. $\mathcal{CN}(0,1)$ across the users and time. In this paper, we assume a symmetric MAC channel, *i.e.*, the channel gains of all the users are equal and the common gain (SNR) is denoted by $\gamma$. We also assume receiver has the knowledge of instantaneous fading states of the user channels (CSIR).

For a $N$-user MAC, conditioned on the fading states, the capacity region $\mathcal{C}$ is a set of achievable rate vectors $\mathbf{R} = [R_1 \cdots R_N]$ is given by

$$\mathcal{C} = \{\mathbf{R}\colon R_{\mathcal{S}} \leq \log_2(1+\gamma X_{\mathcal{S}}), \forall \mathcal{S} \subseteq \mathcal{N}\}, \quad (2)$$

where $\mathcal{N} = \{1, 2, \cdots, N\}, R_{\mathcal{S}} = \sum_{i \in \mathcal{S}} R_i$, $X_{\mathcal{S}} = \sum_{i \in \mathcal{S}} X_i$ and $X_i = |h_i|^2$. Since $h_i \in \mathcal{CN}(0,1)$, $X_i$ is unit mean exponential random variable.

We now define the common and the individual outage probabilities for a $N$ user MAC. With common outage, packet error for any user is considered to be packet error for all the users and all users are to retransmit the information. With individual outage, packet error for any user is treated independently and only those users are to retransmit the information.

### A. Common outage

In Figure 1(a) the capacity of a two user MAC is illustrated. Observe that the capacity region depends on the fade states. So for a given rate vector, the common outage probability is the probability that the rate vector $\mathbf{R}$ is outside the capacity region. Hence for an $N$-user MAC, the common outage probability is $\mathbb{P}(\mathbf{R} \notin \mathcal{C})$, which equals

$$\varepsilon_c(N) = 1 - \mathbb{P}(R_{\mathcal{S}} \leq \log_2(1+\gamma X_{\mathcal{S}}), \forall \mathcal{S} \subseteq \mathcal{N}). \quad (3)$$

The goodput with common outage probability is given by

$$G_C(N) = \sum_{i=1}^{N} R_i(1-\varepsilon_c(N)) = (1-\varepsilon_c(N)) \sum_{i=1}^{N} R_i. \quad (4)$$

where $R_i$ is the transmission rate of user $i$ in $N$-user MAC.

### B. Individual outage

The individual capacity region for a user $i$ is the set of achievable rate vectors where message decoding of user $i$ is successful irrespective of the decoding of other users. It is the union of the capacity region $\mathcal{C}$, defined in (2), (where decoding of all users are successful) and the region where message decoding of user $i$ is successful with decode failure of any user from set $\mathcal{G} \subseteq \mathcal{N} \setminus i$. Hence the individual outage for user $i$ in $N$ user MAC is

$$\varepsilon_i(N) = 1 - \mathbb{P}\left(\mathcal{C} \cup \left(\bigcup_{\mathcal{G}} \mathcal{C}_{\mathcal{G}}\right)\right), \forall \mathcal{G} \subseteq \mathcal{N} \setminus i \quad (5)$$

where $\mathcal{C}_{\mathcal{G}}$ is

$$\mathcal{C}_{\mathcal{G}} = \left\{R_{\mathcal{G}_c} \leq \log_2\left(1 + \frac{\gamma X_{\mathcal{G}_c}}{1+\gamma X_{\mathcal{G}}}\right), \forall \mathcal{G}_c \subseteq \mathcal{N} \setminus \mathcal{G}\right\}$$

The goodput with individual outages is

$$G_I(N) = \sum_{i=1}^{N} R_i(1-\varepsilon_i(N)). \quad (6)$$

For two users, the common outage region is illustrated in Figure 1(a) and the individual outage region for the first user in Figure 1(b).

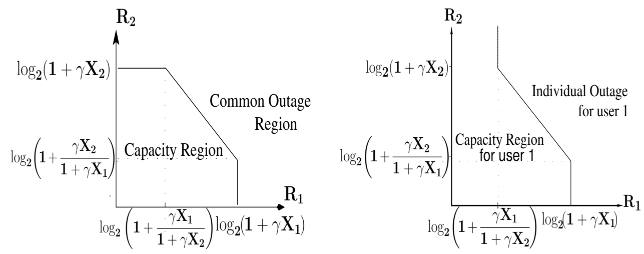


Fig. 1. Common and individual outage regions of a 2-user MAC

## III. Outage probability approximations

Since the capacity region of a $N$ user MAC is bounded by $2^N - 1$ mutual information constraints, computation of outage probabilities becomes extremely messy for $N > 3$. Hence we make approximations by limiting the number of rate constraints. To understand the impact of the rate constraints we first look at the structure of capacity region for low and high SNRs.

The capacity region is the convex hull of its vertices [5]. The vertices on the axes are defined by individual channel SNRs, *i.e.*, $\log_2(1+\gamma X_i), i \in \mathcal{N}$. The coordinates of other vertices are defined by SINRs given by $\log_2(1+\frac{\gamma X_i}{1+\gamma X_{\mathcal{G}}}), \mathcal{G} \subseteq \mathcal{N} \setminus i$. At very low SNR (corresponds to small $\gamma$), it is easy to see that $\frac{\gamma X_i}{1+\gamma X_{\mathcal{G}}} \approx \gamma X_i$ and hence the SINRs equals their corresponding SNRs. Hence the capacity region resembles to that of $N$ parallel individual channels constrained only by individual rate constraints $R_i \leq \log_2(1+\gamma X_i), i \in \mathcal{N}$.

At high SNR, the probability of the events $R_{\mathcal{S}} \leq \log_2(1+\gamma X_{\mathcal{S}})$ will be dominated by the event $R_{\mathcal{N}} \leq \log_2(1+\gamma X_{\mathcal{N}})$. Hence in this case, the dominating rate constraint is $R_{\mathcal{N}} \leq \log_2(1+\gamma X_{\mathcal{N}})$. See Figure 2 for an illustration of these rate constraint for 3 user MAC capacity region.

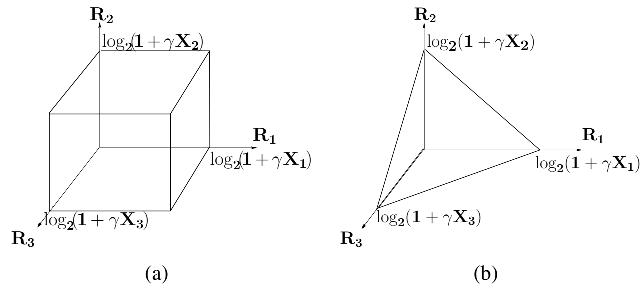


Fig. 2. High and low SNR approximations of a 3 user MAC capacity region.

### A. Common outage probability

Retaining only individual rate constraints and the total sum rate constraint, define

$$\tilde{\varepsilon}_c(N) = 1 - \mathbb{P}\left(\sum_{i=1}^{N} X_i \geq \omega_N, X_1 \geq \alpha_1, ...., X_N \geq \alpha_N\right),$$

where $\omega_N = \frac{2^{\sum_{i=1}^{N} R_i}-1}{\gamma}$ and $\alpha_i = \frac{2^{R_i}-1}{\gamma}$. Observe that the probability $1 - \tilde{\varepsilon}_c(N)$ is obtained by retaining only the individual rate constraints $R_k \leq \log_2(1+\gamma X_k), k \in \mathcal{N}$ and the total rate constraint $R_{\mathcal{N}} \leq \log_2(1+\gamma X_{\mathcal{N}})$ in the probability expression (3). Hence $1-\tilde{\varepsilon}_c(N)$ upper bounds $1-\varepsilon_c(N)$ which implies $\tilde{\varepsilon}_c(N)$ lower bounds the actual outage probability. In the next lemma, we obtain an expression for $\tilde{\varepsilon}_c(N)$.

**Lemma 1.** *The lower bound on the common outage probability in a MAC channel is given by*

$$\tilde{\varepsilon}_c(N) = 1 - e^{-S_n}\frac{\Gamma(N, \omega_N - S_N)}{\Gamma(N)} \tag{7}$$

*where* $S_N = \sum_{i=1}^{N} \alpha_N$, *where* $\Gamma(n,x) = \int_x^\infty t^{n-1}e^{-t}\mathrm{d}t$ *is the standard upper incomplete gamma function.*

*Proof:* The proof proceeds by induction on $N$. For $N = 2$, the probability equals,

$$1 - \mathbb{P}(X_1 + X_2 \geq \omega_2, X_1 \geq \alpha_1, X_2 \geq \alpha_2),$$

which equals $1-e^{-\omega_N}(1+\omega_N - S_2)$ which equals $\tilde{\varepsilon}_c(2)$ using the expansion of incomplete gamma function for integer scale exponent. Let the statement be true for $N = k$. We will now prove the result for $N = k+1$. For $N = k+1$,

$$\tilde{\varepsilon}_c(k+1) = 1- \mathbb{P}\left(\sum_{i=1}^{k} X_i \geq \omega_N - X_{k+1}, X_1 \geq \alpha_1, ...., X_{k+1} \geq \alpha_{k+1}\right).$$

Also observe that $X_1 \geq \alpha_1, ...., X_k \geq \alpha_k$ implies $\sum_{i=1}^{k} X_i \geq S_k$. Conditioning on $X_{k+1} = y$ and using the induction hypothesis for $k$, we obtain

$$\tilde{\varepsilon}_c(k+1) = 1- \int_{\alpha_{k+1}}^{\omega_{k+1}-S_k} \frac{e^{-S_k}}{(k-1)!}\Gamma(k, \omega_{k+1} - S_k - y)e^{-y}\mathrm{d}y + \int_{W_{k+1}}^{\infty} e^{\sum_{i=1}^{k}\alpha_i}e^{-y}\mathrm{d}y.$$

Using the indicator function to represent the incomplete gamma function $\Gamma(s,x) = \int_0^\infty e^{-t}t^{s-1}\mathbf{1}(t \geq x)\mathrm{d}t$ and exchanging integrals, we have

$$\tilde{\varepsilon}_c(k+1) = 1- \frac{e^{-S_k}}{(k-1)!}\int_0^\infty t^{k-1}e^{-t}\Big(e^{-\max(\alpha_{k+1}, \omega_{k+1}-S_k-t)} - e^{-(\omega_{k+1}-S_k)}\Big)\mathrm{d}t + \int_{\omega_{k+1}}^{\infty} e^{\sum_{i=1}^{k}\alpha_i}e^{-y}\mathrm{d}y.$$

Using the fact that $S_k + \alpha_{k+1} = S_{k+1}$, and simplifying the integrals, we obtain

$$\tilde{\varepsilon}_c(k+1) = 1- e^{-\omega_{k+1}}\frac{(\omega_{k+1} - S_{k+1})^k}{k!} + \frac{e^{-S_{k+1}}}{(k-1)!}\Gamma(k, \omega_{k+1} - S_{k+1}).$$

The incomplete gamma function satisfies the property $\Gamma(s+1, x) = s\Gamma(s,x) + x^s e^{-x}$, and hence

$$\tilde{\varepsilon}_c(k+1) = 1 - \frac{e^{-S_{k+1}}}{k!}\Gamma(k+1, \omega_{k+1} - S_{k+1}),$$

proving the induction. ■

In Figure 3, the success probability $1-\tilde{\varepsilon}_c(N)$ is plotted as a function of rate (taken to be equal for all users) for different $N$ and SNR. We observe that the lower bound provides a good approximation.

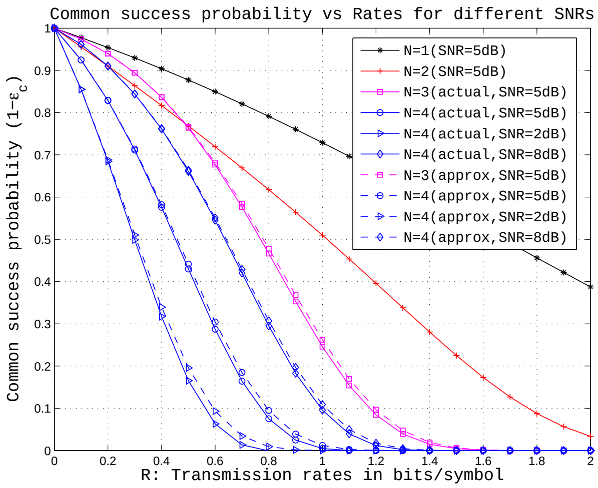


Fig. 3. Success probability $1-\tilde{\varepsilon}_c(N)$ versus the transmission rate $R$ for different number of users and SNRs. We assume that all the users transmit at the same rate.

### *B. Individual outage probability*

The union in (5) makes it very difficult to obtain a lower bound for the individual outage probabilities. Instead we focus on approximating the individual outage probability by discarding some constraints in (5). More precisely, we retain the individual rate constraints and total rate constraint as done in common outage approximations. In addition, for the user $i$, we consider the constraint $R_i \leq \log_2(1 + \frac{\gamma X_i}{1+\gamma\sum_{k=1,k\neq i}^{N} X_k})$, *i.e.*, the event that message decoding succeeds even with interference from all other users. This approximation is denoted by $\tilde{\varepsilon}_i(N)$ and equals

$$\tilde{\varepsilon}_i(N) = 1 - \mathbb{P}\left[\left\{X_i \geq \alpha_i, \sum_{k=1}^{N} X_k \geq \omega_N\right\} \bigcup \left\{X_i \geq \alpha_i + \alpha_i\gamma\sum_{\substack{k=1,\\k\neq i}}^{N} X_k\right\}\right], \tag{8}$$

where $\omega_N = \frac{2^{\sum_{i=1}^{N} R_i}-1}{\gamma}$ and $\alpha_i = \frac{2^{R_i}-1}{\gamma}$. Figure 4, shows the closeness of this approximation for $N = 3$ and $N = 4$.

In the next lemma we evaluate the probability in (8).

**Lemma 2.** *The approximated individual outage probability* $\tilde{\varepsilon}_i(N)$ *is given by*

$$\tilde{\varepsilon}_i(N) = 1 - \frac{e^{-\omega_N}}{\Gamma(N)}\left((\omega_N - \alpha_i)^{N-1} - (\omega_{N-1})^{N-1}\right) - \frac{e^{-\alpha_i}}{\Gamma(N-1)}\left(\Gamma(N-1, \omega_N - \alpha_i) + \frac{\gamma(N-1, \omega_N - \alpha_i)}{(1+\gamma\alpha_i)^{N-1}}\right),$$

where $\omega_{N-1} = \frac{2^{\sum_{k=1,k\neq i}^{N} R_k}-1}{\gamma}$.

*Proof:* Since $X_k$ are i.i.d exponential, $X_{\mathcal{G}} = \sum_{k=1,k\neq i}^{N} X_k$ is gamma distributed with shape parameter $N-1$ and is independent of $X_i$. Using basic set axioms, the RHS of (8) can be simplified to

$$\tilde{\varepsilon}_i(N) = \mathbb{P}(X_i \leq \alpha_i) + \mathbb{P}\big(\underbrace{X_{\mathcal{G}} \leq \omega_N - X_i, X_i \leq \alpha_i + \gamma\alpha_i X_{\mathcal{G}}, X_i \geq \alpha_i}_{T1}\big).$$

We have $\mathbb{P}(X_i \leq \alpha_i) = 1 - e^{-\alpha_i}$. We evaluate the probability of the event $T1$ by first conditioning on $X_{\mathcal{G}} = y$. So we have

$$\mathbb{P}(T_1|X_{\mathcal{G}} = y) = \int_{\alpha_i}^{\min(\omega_N - y, \alpha_i + \gamma\alpha_i y)} e^{-x}\mathrm{d}x = \left\{e^{-\alpha_i} - e^{\min(\omega_N - y, \alpha_i + \gamma\alpha_i y)}\right\}.$$

Now averaging over $X_{\mathcal{G}}$ we obtain

$$\begin{aligned}
&\mathbb{P}(T_2) = \\
&\int_0^{\omega_N - \alpha_i} \left(e^{-\alpha_i} - e^{\min(\omega_N - y, \alpha_i + \gamma\alpha_i y)}\right) \frac{y^{(N-2)}e^{-y}}{(N-2)!}\mathrm{d}y, \\
&= \frac{e^{-\alpha_i}}{N-2!}\int_0^{\omega_N - \alpha_i} y^{N-2}e^{-y}\mathrm{d}y \\
&- \frac{1}{N-2!}\int_0^{\frac{\omega_N - \alpha_i}{1+\gamma\alpha_i}} e^{-\alpha_i}e^{-(1+\gamma\alpha_i)y}y^{N-2}\mathrm{d}y \\
&- \frac{e^{-\alpha_i}}{N-2!}\int_{\omega_{N-i}}^{\omega_N - \alpha_i} e^{-\omega_N + y}y^{N-2}e^{-y}\mathrm{d}y.
\end{aligned}$$

Evaluating the above integrals we obtain the result. ■

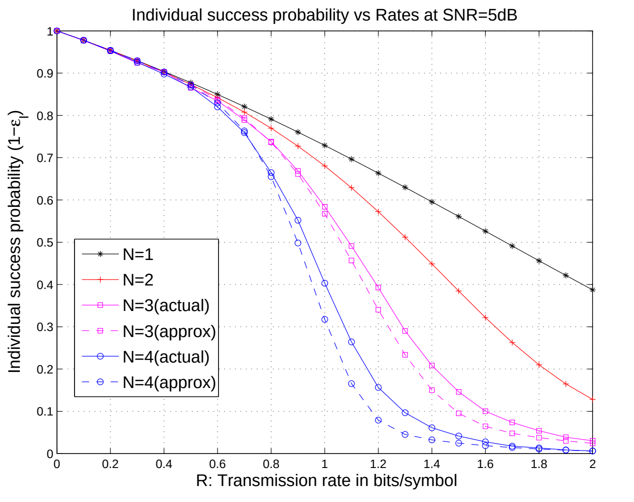


Fig. 4. Success probability $1 - \tilde{\varepsilon}_i(N)$ versus the transmission rate $R$ for different number of users. Here $\gamma = \sqrt{10}$ (5dB) and we assume that all the users transmit at the same rate.

In Figure 4 the actual and approximated individual outage probabilities are plotted as a function of $R$ for different $N$.

For $N = 1, 2$ the above approximations for both the individual and common outage corresponds to the exact outage probability. In the next section, we use these approximations to analyse the optimal goodput in $N$-user MAC.

## IV. Goodput of MAC channels

In a single user case, a frame is retransmitted till it is successful decoded. Hence the goodput (long term average of rate at which information is successfully received) denoted by $G$ is function of both transmission rate and packet error probability and is given by

$$G = R(1-\varepsilon), \tag{9}$$

where $\epsilon$ is the link outage probability when the transmission rate is $R$. Since $\epsilon$ is a function of the rate, the goodput can be considered as a function that depends entirely on $\epsilon$ or $R$. It is intuitive that a small transmission rate would lead to a lower outage. However the goodput which is a product of rate and success probability will be low. On the other hand, for a large rate, the outage probability will be high leading to a smaller goodput. So there is an optimal rate of transmission that would maximize the goodput. Also observe that this optimal rate would translate to an optimal outage probability for the channel.

We now briefly describe the results from [1] that deal with a point to point channel. The packet or frame error probability corresponding to transmission rate $R$ over a fading channel, denoted by $\varepsilon$

$$\varepsilon = \mathbb{P}\left(\log_2\left(1 + \gamma|h|^2\right) \leq R\right) = 1 - \exp\left(-\frac{2^R - 1}{\gamma}\right).$$

Hence the the optimal packet error probability (PEP) for a single user system ($N = 1$) that maximizes the goodput [1] is

$$\varepsilon^* = 1 - \exp\left(\frac{1}{\gamma} - \frac{1}{W(\gamma)}\right). \tag{10}$$

where $W(.)$ is the LambertW function [10]. Hence the optimal outage probability depends only on $\mathtt{SNR}$. Next we see that in MAC channels, optimal outage will also depend on $N$.

### A. Goodput with common outage

In the $N$ user case with common outage, the goodput is $G_C(N, \bar{R}) = (1 - \varepsilon_c(N, \bar{R}))\sum_{i=1}^{N} R_i$, where $\bar{R} = [R_1, R_2, ..., R_N]$ represents the vector of rates. In a symmetric MAC setting, the common outage and goodput are symmetric function of rates. Hence, intuitively the rates of all the users are equal at the optimal goodput, which is proved in the following lemma.

**Lemma 3.** *Let* $R_{av} = N^{-1}\sum_{i=1}^{N} R_i$. *Also let* $\bar{R}_{av} = [R_{av}, ..., R_{av}]$, i.e., *a vector with all the* $N$ *entries equal to*$R_{av}$. *Then*

$$G_C(N, \bar{R}) \leq G_C(N, \bar{R}_{av})$$

*Proof:* Since $\sum R_i$ is same in both the cases, it suffices to prove that the success probability is higher when all the users use a common $R_{av}$ rate rather than $\bar{R}$ rates. By Jensen's inequality we have (with a slight abuse of notation) $\omega_N$'s equal and $S_N(\bar{R}) \geq S_N(\bar{R}_{av})$. Also from Lemma 1, and expansion of the incomplete gamma function, the success probability equals $e^{-\omega_N}\sum_{k=0}^{N-1}\frac{(\omega_N - S_N)^k}{k!}$, which is a decreasing function of $S_N$. Hence the success probability with $\bar{R}_{av}$ is greater than the success probability with $\bar{R}$, thus proving the lemma. ■

Since we are interested in maximum goodput, by previous lemma without loss of generality we assume that the rates of all the users are equal. Hence the goodput for the common outage is

$$G_C(N,R) = (1-\varepsilon_c(N,R))NR. \tag{11}$$

where $R$ is the common rate of transmission. It is easy to observe that there exists, at a particular SNR, an optimal transmission rate $R^*$ as a function of $N$ that maximizes goodput. This optimal rate corresponds to an optimal outage probability which we denote by $\varepsilon_c^*(N)$.

For $N = 1$, it was shown in [1] that the optimal outage probability is about $50\%$ at 2 dB SNR. However, it is not clear if such high outage is optimal when there are multiple users. In Figure 5 the optimal outage probability is plotted as a function of SNR for different users. We make the following observations[1]:

1) In the low SNR regime, the optimal outage probability increases with the number of users.
2) In the high SNR regime, the optimal outage probability decreases with the number of users.

At 2 dB SNR, we observe that the optimal outage probability is larger than $50\%$ in the multi-user scenario and increases with the number of users. When SNR $= 16$ dB the optimal outage probability for $N = 1$ is $30\%$ while it is about $20\%$ for $N = 4$. To show the existence of a crossover point we first show that $\varepsilon_c^*(N) < \varepsilon_c^*(N+1)$ when $\gamma \to 0$, and then prove that $\varepsilon_c^*(N) > \varepsilon_c^*(N+1)$ when $\gamma \to \infty$. See Appendix A for the main arguments in the proof.

*An intuitive explanation for the cross over:* At high and low SNRs, we can "view" the MAC channel as a point-to-point link and use the fact that for a single user channel, the optimal outage increases with decreasing SNR (from Figure 5 or (10))

- In the low-SNR regime, as explained in Section III, the capacity region looks *almost* like a cube in $N$ dimension (see Figure 2(a)). Hence by treating the signals of other users as residual noise, the *effective* SNR of each user is $\frac{\gamma}{\gamma(N-1)+1}$ which is smaller than $\gamma$ and decreases with increasing $N$. So each user can be viewed as a point-to-point link with an effective SNR that decreases with increasing $N$. Hence it follows that the optimal outage probability increases with increasing $N$.
- At high-SNR, the capacity region almost looks like a simplex (see Figure 2(b)). Any point in the simplex can be obtained by time sharing. Assume that each user transmits for $1/N$ time slots. Hence the goodput equals $\frac{NR}{N}(1-\epsilon)$, where $R$ is the rate and $\epsilon$ is the outage probability of the single transmitting user. Since the user transmits only $1/N$ fraction of the time, the power of the user can be boosted by a factor $N$, thereby increasing the *effective* SNR to $\gamma N$. So in the high-SNR regime with $N$ users, each user can be viewed as a point-to-point link (with time sharing) with an effective SNR that increases with the number of users. Hence it follows that the optimal outage probability decreases with increasing $N$.

[1]These results hold when the number of users is not scaling to $\infty$.

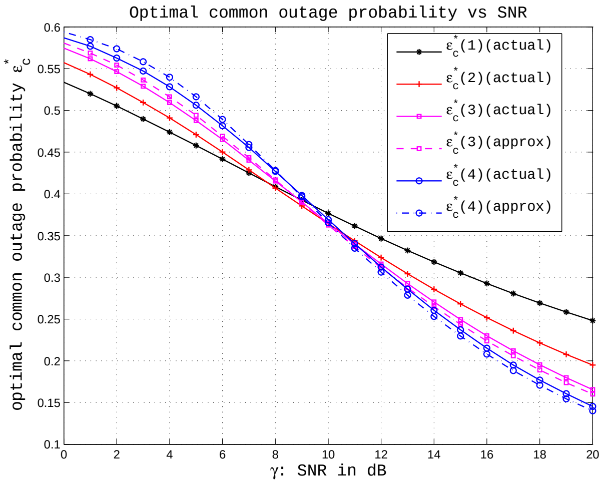


Fig. 5. Optimal common outage probability that maximizes the goodput versus the SNR for different $N$.

## B. Goodput with individual outage

We now analyse the goodput behaviour considering the individual outage probability. The goodput with individual outages is a symmetrical function of user rates. Hence the optimal goodput is achieved when all the users have equal rate, something similar to the case of common outage. Hence with equal rates the goodput (6) is

$$G_I(N) = NR(1-\varepsilon_I(N)).$$

The goodput analysis (even with the approximation) is very intractable. Instead we look at the numerical results to obtain some intuition. Unlike in the common outage case, from Figure 6 we observe that the optimal outage probability decreases with increasing $N$ for any SNR.

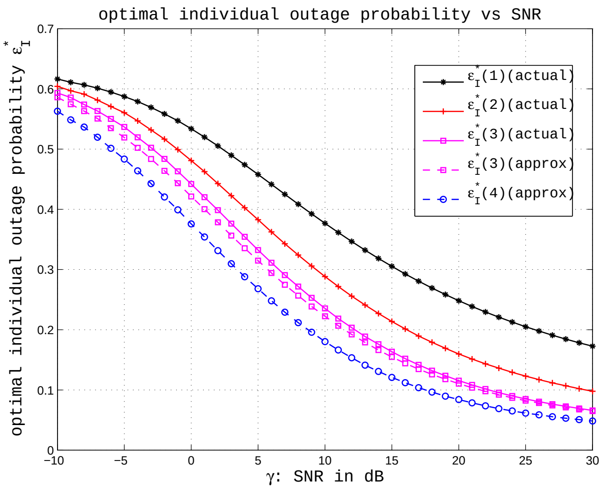


Fig. 6. Optimal individual outage probability that maximizes the goodput versus the SNR for different $N$.

## V. Conclusion

In this paper, a symmetric Gaussian fading MAC channel with $N$ users with channel state information available only at the receiver is considered. A tight lower bound to the common outage probability and a good approximation to the individual outage probability are provided.

It is shown that the optimal common outage probability that maximizes goodput (using ARQ) increases with the number of users at low SNR and decreases with number of users at high SNR. Hence in a network with large number of users operating at high SNR, it is not optimal to rely entirely on ARQ to achieve the best goodput. Instead the reliability has to be increased (especially for large $N$) by using appropriate codes. However, at low SNR, relying on ARQ would be sufficient to achieve the optimal goodput when joint decoding is employed. When individual outage is considered, the optimal outage probability decreases with increasing number of users for all SNR.

## Appendix A
## Existence of a crossover point for common outage probability

Finding the optimal outage is equivalent to finding the optimal rate for which throughput maximizes.

Differentiating (11) with respect to $R$ and setting it to zero we see that the optimal $R$ satisfies

$$\frac{\varepsilon_c'(N,R)}{1-\varepsilon_c(N,R)} = \frac{1}{R}.$$

Using the approximation from Lemma 1,

$$\frac{\tilde{\varepsilon_c}'(N)}{1-\tilde{\varepsilon_c}(N)} = \frac{N\ln(2)2^R}{\gamma} + \frac{N\ln(2)(2^{NR}-2^R)}{\gamma}\left\{\frac{(\omega-N\alpha)^{N-1}e^{-(\omega-N\alpha)}}{\Gamma(N,\omega-N\alpha)}\right\},$$

which is an increasing function of $N$.

This proves that $R^*(N+1) \leq R^*(N)$ and clearly indicates that each user transmission rate should decrease as the number of users in MAC increases.

Hence the optimal rate $R$ must be chosen to satisfy

$$NR2^R + (NR2^{NR} - NR2^R). \left\{\frac{(\omega-N\alpha)^{N-1}e^{-(\omega-N\alpha)}}{\Gamma(N,\omega-N\alpha)}\right\} = \frac{\gamma}{\ln(2)}. \quad (12)$$

The above equation (12) cannot be solved in general to obtain a closed form expression for optimal $R$. Instead we will look at the asymptotes as $\gamma \to 0$ and $\gamma \to \infty$ to prove the crossover.

In the above equation $(\omega - N\alpha)$ is a function of $R$ and $\gamma$. And it is obvious that as $\gamma \to 0$, $R \to 0$ and as $\gamma \to \infty$, $R \to \infty$. However the optimal rate $R$ is always upper bounded by $NR \leq \log_2(1+N\gamma)$, maximum sum rate in symmetric MAC with CSIT. Hence the asymptotic behaviour is determined by $R$ when $\gamma \to 0$ and $\gamma$ when $\gamma \to \infty$.

For $\gamma \to 0$, $R \approx 0$ and $\omega - N\alpha \approx 0$. The approximation in (7) reduces to $\tilde{\varepsilon_c}(N) \approx 1 - e^{-N\alpha}$ (which is the common outage probability for low SNR approximation shown in Figure 2(a)). With this (12) reduces to

$$NR2^R \approx \frac{\gamma}{\ln(2)}$$

and we get optimal rate, for low SNR, as

$$R^* = \log_2\left(\frac{\frac{\gamma}{N\ln(2)}}{W\left(\frac{\gamma}{N\ln(2)}\right)}\right). \quad (13)$$

By substituting the above equation (13) into $\tilde{\varepsilon_c}(N)$ we obtain $\tilde{\varepsilon_c}^*(N) \leq \tilde{\varepsilon_c}^*(N+1)$.

For $\gamma \to \infty$, $R$ is large and however $\omega - N\alpha \approx 0$. The approximation in (7) reduces to $\tilde{\varepsilon_c}(N) \approx 1 - \frac{\Gamma(N,\omega)}{\Gamma(N)}$ (common outage probability for high SNR approximation shown in Figure 2(b)). Also, by approximating $\Gamma(N,\omega-N\alpha) \approx (N-1)!$ and $\omega \approx \frac{2^{NR}}{\gamma}$, (12) reduces to

$$NR\ 2^{NR}\frac{2^{(NR)^{N-1}}}{(N-1)!\gamma^{N-1}} = \frac{\gamma}{\ln(2)},$$

and we get optimal rate, for high SNR, as

$$R^* = \frac{1}{N^2}\log_2\left(\frac{\frac{N!\gamma^N}{\ln(2)}}{W\left(\frac{N!\gamma^N}{\ln(2)}\right)}\right). \quad (14)$$

With this optimal rate, $\omega = \left(\frac{N!}{\ln(2)W\left(\frac{N!\gamma^N}{\ln(2)}\right)}\right)^{1/N}$. Substituting this into the above expression for asymptotic $\tilde{\varepsilon_c}(N)$, we see $\tilde{\varepsilon_c}^*(N) \geq \tilde{\varepsilon_c}^*(N+1)$ for high SNR. This proves the existence of crossover in the optimal common outage curves.

## References


[1] P. Wu and N. Jindal, "Coding versus ARQ in fading channels: How reliable should the PHY be?" *IEEE Trans. on Info. Theory*, vol. 59, no. 12, pp. 3363 –3374, Dec. 2011.

[2] S. Lin, D. Costello, and M. Miller, "Automatic-repeat-request error-control schemes," *IEEE Communications Magazine*, vol. 22, no. 12, pp. 5 –17, Dec. 1984.

[3] N. Ahmed and R. Baraniuk, "Throughput maximization for ARQ-like systems in fading channels with coding and queuing delay constraints," in *Proc., IEEE Asilomar*, vol. 2, Nov. 2004, pp. 1463 – 1467 Vol.2.

[4] Y.-H. Nam, K. Azarian, H. El Gamal, and P. Schniter, "Cooperation through ARQ," in *Proc., IEEE Signal Proc. Adv. in Wireless Comm. (SPAWC)*, Jun. 2005, pp. 1023 – 1027.

[5] D. Tse and S. Hanly, "Multiaccess fading channels. I. Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. on Info. Theory*, vol. 44, no. 7, pp. 2796 –2815, Nov. 1998.

[6] M. Mohseni, R. Zhang, and J. Cioffi, "Optimized transmission for fading multiple-access and broadcast channels with multiple antennas," *IEEE Journal on Sel. Areas in Communications*, vol. 24, no. 8, pp. 1627 – 1639, Aug. 2006.

[7] L. Li, N. Jindal, and A. Goldsmith, "Outage capacities and optimal power allocation for fading multiple-access channels," *IEEE Trans. on Info. Theory*, vol. 51, no. 4, pp. 1326 – 1347, Apr. 2005.

[8] D. Tuninetti and G. Caire, "The throughput of some wireless multiaccess systems," *IEEE Trans. on Info. Theory*, vol. 48, no. 10, pp. 2773 – 2785, Oct. 2002.

[9] R. Narasimhan, "Individual outage rate regions for fading multiple access channels," in *Proc., IEEE Intl. Symposium on Information Theory*, Jun. 2007, pp. 1571 –1575.

[10] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W Function," in *Advances in Computational Mathematics*, 1996, pp. 329–359.