Quantum Secret Sharing with CSS Codes

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Joint work with Andreas Klappenecker and Robert Raussendorf

Quantum Information Seminar
Department of Physics and Astronomy
University of British Columbia, Vancouver

Outline

Introduction to Secret Sharing

Motivated by the need to secure sensitive information.

i) passwords for secure locations such as bank vaults
ii) strategic military information
iii) secure distributed computing
iv) privacy (anonymous voting)

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Components of Secret Sharing

Encoded secret

Trusted dealer encodes the secret and distributes it among the parties $P = \{P_1, \ldots, P_n\}$

Reconstruction

Authorized subsets of P can recover the secret

Secrecy

Unauthorized subsets cannot learn anything about the secret

Access structure

The collection of all authorized sets

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Quantum Secret Sharing (QSS)

Classical secret to be secured

- Secret is an element of a finite alphabet (usually a finite field $\mathbb{F}_q$)
- Encoded into q orthonormal quantum states

Quantum secret to be secured (quantum state sharing)

- Secret is chosen from a set of q pure states
- Encoded into a linear combination of q orthonormal states

Why quantum secret sharing?

- Enhanced security
- Increased efficiency for classical secrets
- We might require to share a quantum state

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Secret Sharing and Error Correction

Assume that the shares are distributed to n players as s_j, $1 \leq j \leq n$

An authorized set: $\{2, 3, \ldots, 6\}$

Implicitly every subset that can reconstruct the secret is correcting erasure errors on the (qu)bits it does not have access

It suggests that codewords of an error correcting code can be used for secret sharing.

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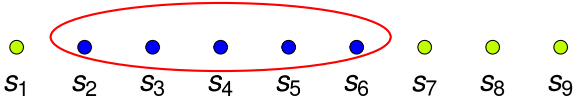

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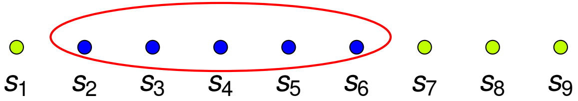

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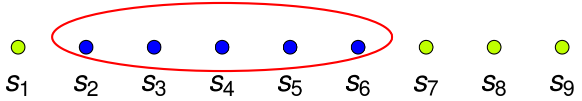

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Classical Codes

An $[n, k, d]_q$ classical code C is a k-dimensional subspace of $\mathbb{F}_q^n$ and it is capable of correcting up to $d - 1$ erasures.

C can be compactly described by a $k \times n$ generator matrix G.

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k1} & g_{k2} & \cdots & g_{kn} \end{bmatrix}.$$

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Dual Codes

Associated to an $[n, k, d]_q$ classical code C is a $[n, n-k, d^\perp]_q$ code called the dual code $C^\perp$.

$$C^\perp = \{x \in \mathbb{F}_q^n \mid x \cdot c = 0 \text{ for all } c \in C\}$$

The generator matrix H of $C^\perp$ is called the parity check matrix of C.

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Secret Sharing Schemes from Codes

- Every $[n, k, d]_q$ code C can be converted to a secret sharing scheme Σ
- The access structure of Σ is defined by the dual code, $C^\perp$

Consider a code C and its dual $C^\perp$

$$C = \left\{ \begin{array}{c} 0000 \\ 0111 \\ 1011 \\ 1100 \end{array} \right\} \quad C^\perp = \left\{ \begin{array}{c} 0000 \\ 0011 \\ 1110 \\ 1101 \end{array} \right\}$$

The encoded secret is any of the codewords in C with the first coordinate dropped.

The authorized sets correspond to codewords in $C^\perp$ that have nonzero first coordinate

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Stabilizer Codes

Pauli group

$$\mathcal{P}_n = \{i^a g_1 \otimes g_2 \otimes \cdots \otimes g_n \mid g_i \in \{I, X, Z, Y = iXZ\}\}$$

A $[[n,k,d]]_q$ stabilizer code Q is the joint eigenspace of an abelian subgroup $S \leq \mathcal{P}_n$.

i) Q is a q^k-dimensional subspace in q^n-dimensional system Hilbert space.

ii) Q can correct for $d-1$ erasures.

$$\varphi : \begin{array}{ccc} I & \mapsto & (0,0) \\ X & \mapsto & (1,0) \\ Z & \mapsto & (0,1) \\ Y & \mapsto & (1,1) \end{array} \qquad X \otimes Z \otimes I \otimes Y \mapsto (1001|0101)$$

The stabilizer can be identified with a classical code by φ.

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CSS Quantum Codes

$$S = \begin{bmatrix} XXXX \\ ZZZZ \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 1111 & 0 \\ 0 & 1111 \end{bmatrix}$$

CSS codes are stabilizer codes with the stabilizer generators consisting of purely X or purely Z operators.

CSS codes are quantum stabilizer codes which are derived from a classical code whose parity check matrix H satisfies $HH^t = 0$. In other words $C \supseteq C^\perp$. The stabilizer (matrix) of the CSS code is

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ex: If $C^\perp = [1111]$, the stabilizer of the quantum code is

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Quantum Secret Sharing and Quantum Codes

What precisely is the correspondence between quantum codes and secret sharing

- Can we take an $[[n, k, d]]_q$ quantum code and convert it into a secret sharing scheme?

A correspondence between QECC and QSS exists but it seems to be limited!

- $[[2k-1, 1, k]]_q$ quantum MDS codes can lead to threshold secret sharing schemes and vice versa, (Cleve et al 1999; Rietjens et al 2005)
- Every QECC does not appear to be a secret sharing scheme

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Some more terminology

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Minimal Codewords

The support of $x = (x_1, x_2, \ldots, x_n)$, is the location of its nonzero components.

ex: $\mathrm{supp}([1,0,1,0]) = \{1,3\}$

We say that x covers y if $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$

ex: (1,1,0,1) covers (1,1,0,0) but not $(1,0,1,1)$

A codeword of $C \subseteq \mathbb{F}_q^n$ is said to be **minimal** if it does not cover any other codeword of C except its scalar multiples

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Sharing Classical Secrets with CSS States

Let Q be a pure $[[n, 1, d]]_2$ CSS code derived from a classical code $C^\perp \subseteq C \subseteq \mathbb{F}_2^n$. Let $\mathcal{E}$ be the encoding given by the CSS code

$$\mathcal{E} : |i\rangle \mapsto \sum_{x \in C^\perp} |x + ig\rangle \quad i \in \mathbb{F}_2, \tag{1}$$

where $g \in C \setminus C^\perp$. Distribute the n qubits as the n shares for a secret sharing scheme, Σ. The minimal access structure Γ is given by

$$\Gamma = \left\{ \text{supp}(c) \mid \; c \text{ is a minimal codeword in } C \setminus C^\perp \; \right\} \tag{2}$$

The reconstruction for an authorized set is to simply take the parity of the set (into an ancilla).

Secret sharing using $[[7,1,3]]_2$ code

$[[7,1,3]]_2$ is derived from a code $C \supseteq C^{\perp}$ with generator matrices

$$G = \left[\begin{array}{c} 1110000 \\ \hline 1010101 \\ 0110011 \\ 0001111 \end{array}\right] \qquad H = \left[\begin{array}{c} 1010101 \\ 0110011 \\ 0001111 \end{array}\right]$$

Encoding for the secret sharing scheme

$$\begin{aligned} |\bar{0}\rangle &= |0000000\rangle + |1010101\rangle + |0110011\rangle + |0001111\rangle \\ &+ |1100110\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \\ |\bar{1}\rangle &= |1111111\rangle + |0101010\rangle + |1001100\rangle + |1110000\rangle \\ &+ |0011001\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle \end{aligned}$$

$$\begin{aligned} C \setminus C^{\perp} = \{&(0100101), (0101010), (1001100), (1110000), \\ &(0011001), (0100101), (0010110), (1111111)\} \end{aligned}$$

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Encoding for the secret sharing scheme

$$\begin{aligned} |\overline{0}\rangle &= |0000000\rangle + |1010101\rangle + |0110011\rangle + |0001111\rangle \\ &+ |1100110\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \\ |\overline{1}\rangle &= |1111111\rangle + |0101010\rangle + |1001100\rangle + |1110000\rangle \\ &+ |0011001\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle \end{aligned}$$

$$\begin{aligned} C \setminus C^\perp &= \{(0100101), (0101010), (1001100), (1110000), \\ &\quad (0011001), (0100101), (0010110), (1111111)\} \end{aligned}$$

Secret sharing using $[[7,1,3]]_2$ code

Take the minimal codeword (1110000), the authorized from this is $\{1,2,3\}$.

To reconstruct the secret compute the parity of these qubits.

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\end{aligned}
$$

Minimal Access Structure

$$
\begin{aligned}
C \setminus C^{\perp} \quad = \quad & \{(0100101), (0101010), (1001100), (1110000), \\
& (0011001), (0100101), (0010110), (1111111)\}
\end{aligned}
$$

$$
\Gamma = \left\{ \begin{array}{c} \{1,2,7\}; \{1,3,5\}; \{1,4,6\}; \{2,3,4\}; \\ \{2,5,6\}; \{3,6,7\}; \{4,5,7\} \end{array} \right\}.
$$

The minimal authorized set has d parties and all the codewords of minimum distance in $C \setminus C^{\perp}$ give rise to minimal authorized sets.

CSS Code Based Secret Sharing

Let Q be a pure $[[n, 1, d]]_q$ CSS code derived from a classical code $C^\perp \subseteq C \subseteq \mathbb{F}_q^n$. Let $\mathcal{E}$ be the encoding given by the CSS code

$$\mathcal{E} : |i\rangle \mapsto \sum_{x \in C^\perp} |x + ig\rangle \quad i \in \mathbb{F}_q, \quad g \in C \setminus C^\perp \text{ and } g \cdot g = \beta \neq 0 \tag{3}$$

Distribute the n qudits as the n shares. The minimal access structure Γ

$$\Gamma = \left\{ \text{supp}(c) \,|\, \ c \text{ is a minimal codeword in } C \setminus C^\perp \ \right\} \tag{4}$$

The reconstruction for an authorized set derived from a minimal codeword $c = \alpha g + s$ for some $s \in C^\perp$ is to compute

$$(\alpha\beta)^{-1} \sum_{j \in \text{supp}(c)} c_j S_j, \text{ where } S_j \text{ is the } j\text{th share} \tag{5}$$

Quantum Secret Sharing and Error Correction

Lemma (Gottesman, 2000)

Suppose we have a set of orthonormal states $|\psi_i\rangle$ encoding a classical secret. Then a set T is an unauthorized set iff

$$\langle\psi_i| F |\psi_i\rangle = c(F) \tag{6}$$

independent of i for all operators F on T. The set T is authorized iff

$$\langle\psi_i| E \left|\psi_j\right\rangle = 0 \quad (i \neq j) \tag{7}$$

for all operators E on the complement of T.

Informally,

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Quantum Secret Sharing and No Cloning

No Cloning Theorem (Wootters, Zurek, Dieks 1982)

We cannot make copies of an unknown quantum state.

No cloning theorem puts restrictions on the permissible authorized sets equivalently, access structures.

- No two authorized sets are disjoint
- The access structure Γ is self-orthogonal

$$\Gamma \subseteq \Gamma^* \text{ where } \Gamma^* = \{A \mid \overline{A} \notin \Gamma\}.$$

Secret Sharing Schemes from Classical Codes

Extended Hamming code given by the following generator matrix.

$$G_C = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & & & & 1 & 1 & 1 \\ & & 1 & & 1 & & 1 & 1 \\ & & & 1 & 1 & 1 & 1 & \end{array}\right]$$

We can check that C self-dual. The punctured code $\rho_1(C)$ and the shortened code $\sigma_1(C)$ are given by the following generator matrices.

$$G_{\rho_1(C)} = \left[\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & & & & 1 & 1 & 1 \\ & 1 & & 1 & & 1 & 1 \\ & & 1 & 1 & 1 & 1 & \end{array}\right] \qquad G_{\sigma_1(C)} = \left[\begin{array}{ccccccc} 1 & & & & 1 & 1 & 1 \\ & 1 & & 1 & & 1 & 1 \\ & & 1 & 1 & 1 & 1 & \end{array}\right]$$

Secret Sharing Schemes from Classical Codes

Now let us form a (CSS) stabilizer code with stabilizer matrix as follows.

$$
\begin{aligned}
S &= \left[\begin{array}{c|c} G_{\sigma_1(C)} & \mathbf{0} \\ \hline \mathbf{0} & \rho_{1(C)^\perp} \end{array}\right] \\
&= \left[\begin{array}{ccccccc|ccccccc}
1 & & & & 1 & 1 & 1 & & & & & & & \\
 & 1 & & 1 & & 1 & 1 & & & & \mathbf{0} & & & \\
 & & 1 & 1 & 1 & 1 & & & & & & & & \\
\hline
 & & & & & & & 1 & & & & 1 & 1 & 1 \\
 & & & \mathbf{0} & & & & & 1 & & 1 & & 1 & 1 \\
 & & & & & & & & & 1 & 1 & 1 & 1 &
\end{array}\right]
\end{aligned}
$$

Encoding the secret

The secret is encoded into the encoded states of the quantum code and each qubit is given as a share.

For this stabilizer code the encoding for $|0\rangle$ and $|1\rangle$ is given as

$$\begin{aligned}
|0\rangle &\mapsto |0000000\rangle + |1000111\rangle + |0101011\rangle + |0011110\rangle \\
&+ |1101100\rangle + |1011001\rangle + |0110101\rangle + |1110010\rangle \\
|1\rangle &\mapsto |1111111\rangle + |0111000\rangle + |1010100\rangle + |1100001\rangle \\
&+ |0010011\rangle + |0100110\rangle + |1001010\rangle + |0001101\rangle
\end{aligned}$$

$$|s\rangle \mapsto \sum_{c\in\sigma_c(C)} \left|s\cdot\overline{X}+c\right\rangle \text{ where } \overline{X}=(1,1,1,1,1,1,1)$$

Recovering the secret

Goal is to recover the secret accessing only the qubits in the authorized set.

The authorized sets are determined by the minimal codewords in $C^\perp$.

Algorithm 1 Recovering the secret

Input: $c \in C^\perp$, a minimal codeword with $c_0 = 1$
for $i \in \mathrm{supp}(c) \setminus 1$ **do**
    Add the ith qubit to the first qubit
end for
for $i \in \mathrm{supp}(c) \setminus 1$ **do**
    Add the first column to the $i$the column
end for

Recovering the secret

Now consider a minimal codeword in $C^\perp$ such that $c_0 = 1$. One such codeword is $(1,1,1,0,0,0,0,1)$. $\mathrm{supp}(c) = \{0,1,2,7\}$, Claim $\{1,2,7\}$ is an authorized set.

$$
\begin{aligned}
|0\rangle &\mapsto |0000000\rangle + |1000111\rangle + |0101011\rangle + |0011110\rangle \\
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Recovering the secret

The key observation is that $|\psi'\rangle = \left|\psi + \overline{X'}\right\rangle$, where $\overline{X'} = (1, 1, 0, 0, 0, 0, 1) = (c_1, c_2, \ldots, c_n)$.

So we need to transform $|\psi'\rangle$ to $|\psi\rangle$.

$$\begin{aligned} &|s\rangle\,(|000000\rangle + |000111\rangle + |101011\rangle + |011110\rangle \\ &\quad + |101100\rangle + |011001\rangle + |110101\rangle + |110010\rangle) \end{aligned}$$

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Correctness of Recovery

$$
S = \left[\begin{array}{ccccccc|ccccccc}
{\color{blue}1} & & & & 1 & 1 & {\color{blue}1} & & & & & & & \\
 & {\color{blue}1} & & 1 & & 1 & {\color{blue}1} & & & & \mathbf{0} & & & \\
 & & 1 & 1 & 1 & 1 & & & & & & & & \\
\hline
 & & & & & & & 1 & & & & 1 & 1 & 1 \\
 & & & \mathbf{0} & & & & & 1 & & 1 & & 1 & 1 \\
 & & & & & & & & & 1 & 1 & 1 & 1 &
\end{array}\right]
$$

$$
\mapsto \left[\begin{array}{ccccccc|ccccccc}
{\color{blue}0} & & & & 1 & 1 & 1 & & & & & & & \\
{\color{blue}0} & 1 & & 1 & & 1 & 1 & & & & \mathbf{0} & & & \\
{\color{blue}0} & & 1 & 1 & 1 & 1 & & & & & & & & \\
\hline
 & & & & & & & 1 & {\color{blue}1} & & & 1 & 1 & {\color{blue}0} \\
 & & & \mathbf{0} & & & & & 1 & & 1 & & 1 & 1 \\
 & & & & & & & & & 1 & 1 & 1 & 1 &
\end{array}\right]
$$

Correctness of Recovery

- Minimal codewords correspond to the undetectable errors of the quantum code
- They also act as the encoded operators of the code

The first operation transforms the stabilizer so that the secret is in the first qubit. The second set of operations transform the encoded operator so that the the encoded states are disentangled from the first qubit.

Quantum Secret Sharing Schemes from Classical Codes

Let $C \subseteq \mathbb{F}_q^n$ be an $[n+1, k, d]_q$ code such that $C^\perp = C$ with generator matrix G_C given as

$$G_C = \left[\begin{array}{c|c} 1 & g \\ \mathbf{0} & \sigma_0(C) \end{array}\right] = \left[\begin{array}{c|c} 1 & \rho_0(C) \\ \mathbf{0} & \end{array}\right]. \tag{8}$$

Then there exists a quantum secret sharing scheme Σ on n parties whose access structure is determined by the minimal cdoewords of C and the dealer is associated to the 1st, coordinate; Σ is encoded using the stabilizer code with the stabilizer matrix given by

$$S = \left[\begin{array}{c|c} \sigma_0(C) & \mathbf{0} \\ \mathbf{0} & \rho_0(C)^\perp \end{array}\right] \tag{9}$$

The secret is recovered using Algorithm 1.

Quantum Secret Sharing and Error Correction

Lemma (Cleve et al, 1999)

Suppose we have any set of orthonormal states $|\psi_i\rangle$ of subspace Q encoding a quantum secret. Then a set T is an unauthorized set iff

$$\langle\psi_i| F |\psi_i\rangle = c(F) \tag{10}$$

independent of i for all operators F on T. The set T is authorized iff

$$\langle\psi_i| E \left|\psi_j\right\rangle = 0 \quad (i \neq j) \tag{11}$$

for all operators E on the complement of T.

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Matroids

A set V and $\mathcal{C} \subseteq 2^V$ form a matroid $\mathcal{M}(V, \mathcal{C})$ if and only if the following conditions hold.
M1) $A, B \in \mathcal{C}$ if and only if $A \not\subseteq B$.
M2) If $x \in A \cap B$, then there exists a $C \in \mathcal{C}$ such that $C \subseteq (A \cup B) \setminus \{x\}$.

We say that V is the ground set and $\mathcal{C}$ the set of minimal circuits of the matroid.

Matroids and secret sharing schemes are related by a correspondence between the minimal circuits and the access structure.

Vector Matroids

To every matrix G, we can associate a matroid.

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k1} & g_{k2} & \cdots & g_{kn} \end{bmatrix}.$$

The ground set is the set of columns of G and the minimal circuits of the matroid $\mathcal{G}$ are the minimally independent columns of G.

Matroids and Secret Sharing Schemes

Given an access structure Γ and a secret sharing scheme Σ that realizes Γ we can associate it to a matroid.

$$\Gamma_e = \{A \cup D \mid \text{ for all } A \in \Gamma_0\}$$

$$\mathcal{C}(A, B) = A \cup B \setminus \left(\bigcap_{C \in \Gamma_e : C \subseteq A \cup B} C\right) \tag{12}$$

$$\mathcal{C}_\Gamma = \{ \text{ minimal sets of } \mathcal{C}(A, B) \text{ for all } A, B \in \Gamma_0 \text{ and } A \neq B\} \tag{13}$$

If $\mathcal{C}_\Gamma$ satisfies the axioms M1 and M2, then we say associate the matroid $\mathcal{M}_\Gamma$ to Γ with the ground set $P \cup D$ and the set of minimal circuits given by $\mathcal{C}_\Gamma$ i.e.

$$\mathcal{M}_\Gamma = \mathcal{M}(P \cup D, \mathcal{C}_\Gamma). \tag{14}$$

Matoids and Quantum Secret Sharing Schemes

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$$S = \left[\begin{array}{c|c} \sigma_0(C) & \mathbf{0} \\ \mathbf{0} & \rho_0(C)^\perp \end{array} \right] \tag{16}$$

Summary

- ◇ Derived new secret sharing schemes based on CSS codes
  - Strengthened the connection between quantum codes and secret sharing schemes
  - Provided a new characterization of the access structure in terms of minimal codewords
- ◇ Sketched some links between quantum secret sharing schemes and matroids

Thanks!

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