Distributed learning over dynamic networks with adversarial agents

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Georgetown University

Outline of the talk

1. Motivation

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2. Problem statement

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3. Our result

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4. Extension to time-varying networks

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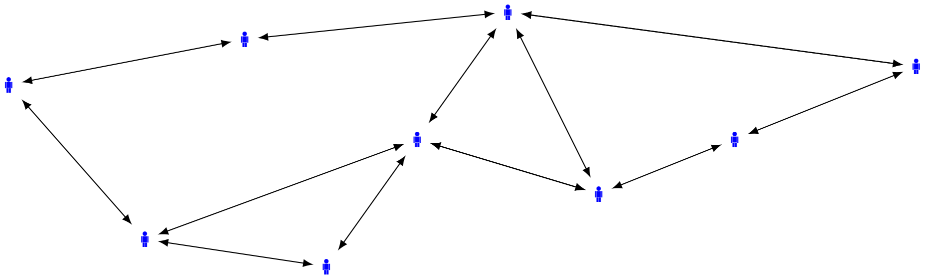
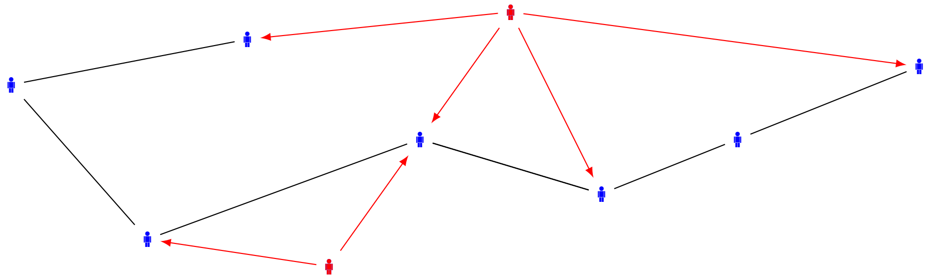
5. Discussion

Distributed (hypothesis testing) learning



- Example: Social networks, sensor networks.
- Each agent wants to know the state of the environment.
- Example: everyone wants to know "best college to attend"

Distributed (hypothesis testing) learning



- There are some adversarial agents in the network.
- We consider Byzantine faults scenario.
- Adversarial agents may send arbitrary information and may not follow the specified algorithm.

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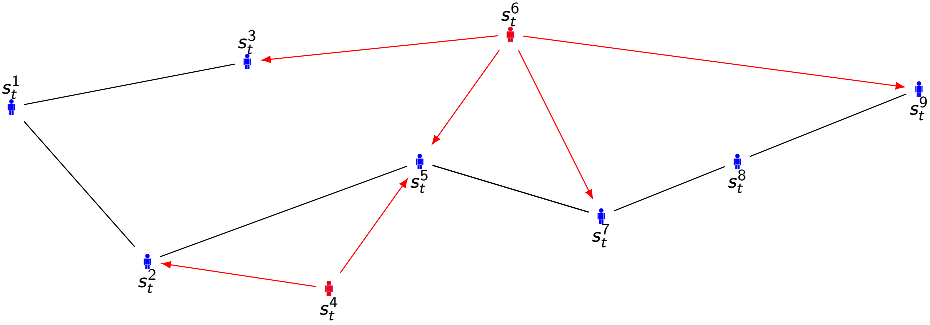

- At every time slot t, each agent collects *partial* information about the state of the environment.
- There are m possible states of the environment but there is only one true state.
- Can agents learn the true state?

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All the good agents in the network should be able to achieve vector consensus via iteratively sharing information with one hop neighbors.

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Condition (Identifiability condition IC)

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- Observe the signal s_t^i and update the marginal distribution
- Update the vector μ_t^i based on the distribution of observed signal and η_t^i

Algorithm analysis

$$\eta_t^i(\theta) = \sum_{j=1}^{n-\phi} \mathbf{A}_{ij}[t] \log \mu_{t-1}^j(\theta) \quad \forall \theta$$

where $\mathbf{A}[t]$ is a random row stochastic matrix which depends on the cumulative observed signals till time t and the behavior of faulty agents.

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$$\phi_t^i(\theta, \theta^*) := \log \frac{\mu_t^i(\theta)}{\mu_t^i(\theta^*)} = \underbrace{\sum_{j=1}^{n-\phi} \mathbf{A}_{ij}[t] \phi_{t-1}^j(\theta, \theta^*)}_{\text{Other agents' influence}} + \underbrace{\sum_{k=1}^{t} \mathcal{L}_k^i(\theta, \theta^*)}_{\text{Agent's private signal influence}}$$

where $\mathcal{L}_k^i(\theta, \theta^*) := \log \frac{l_i(s_k^i|\theta)}{l_i(s_k^i|\theta^*)}$

Analysis (cont...)

$$\phi_t^i(\theta, \theta^*) = \sum_{r=1}^{t} \sum_{j=1}^{n-\phi} \mathbf{\Phi}_{ij}(t, r+1) \sum_{k=1}^{r} \mathcal{L}_k^j(\theta, \theta^*)$$

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where $\mathbf{\Phi}(t, r) = \mathbf{A}[t] \ldots \mathbf{A}[r]$ for $r \in [1, t+1]$

Result [Su and Vaidya, 2016]

- Every good agent i will concentrate its vector on the true state θ^* almost surely, i.e., $\mu_t^i(\theta) \xrightarrow{\text{a.s.}} 0 \; \forall \theta \neq \theta^*$.

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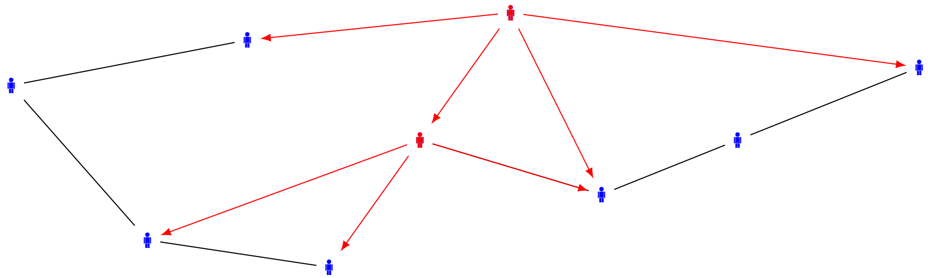
  - In the graph defined by $\mathbf{A}[t]$ at time t, there is only one source component S.
  - S has a path to every good agent in the network.
  - Agents in S can collectively estimate the true state.

Relaxing the network condition

- The network of good agents may not be able to achieve distributed consensus.



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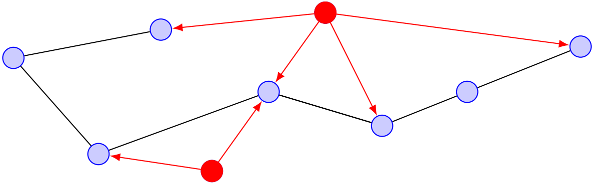
4. $$\phi_t^i(\theta, \theta^*) = \sum_{r=1}^{t} \sum_{j=1}^{n-\phi} \mathbf{\Phi}_{ij}(t, r+1) \sum_{k=1}^{r} \mathcal{L}_k^j(\theta, \theta^*) \xrightarrow{\text{a.s.}} -\infty$$

Our result

Theorem

Under **relaxed identifiability and connectivity conditions for graph of $\mathbf{A}[t]$**, *for the distributed algorithm every agent i will concentrate its vector on the true state θ^* almost surely, i.e.,* $\mu_t^i(\theta) \xrightarrow{\text{a.s.}} 0 \ \forall \theta \neq \theta^*$.

Learning with time-varying networks



Network at time t_1

- A sequence of graphs $\{\mathcal{G}_t | t = 1, 2, \ldots\}$

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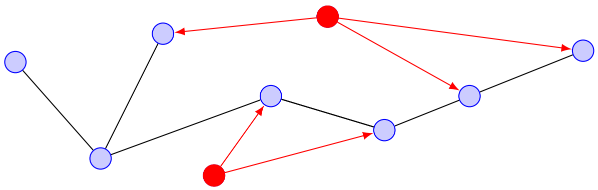

At time t_2

- A sequence of graphs $\{\mathcal{G}_t | t = 1, 2, \ldots\}$
- Set of faulty agents $\mathcal{F}$ is fixed across time.

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Theorem

Under **identifiability of true state by union of a finite sequence of graphs corresponding to $\mathbf{A}[t]$**, *for the distributed algorithm every agent i will concentrate its vector on the true state θ^* almost surely, i.e.,* $\mu_t^i(\theta) \xrightarrow{\text{a.s.}} 0 \ \forall \theta \neq \theta^*$.

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Thank You

Teverberg point

Theorem (Tverberg Theorem)

For any integer $f \geq 1$, for every multiset Y containing at least $(m+1)f+1$ vectors in $\mathbb{R}^m$, there exists a partition $Y_1, \ldots, Y_{f+1}$ of Y into $f+1$ nonempty multisets such that $\cap_{i=1}^{f+1} \mathcal{H}(Y_i) \neq \emptyset$.

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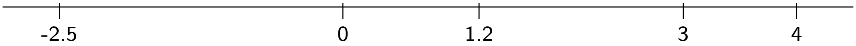

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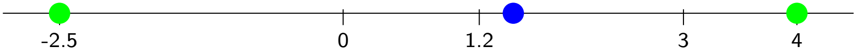

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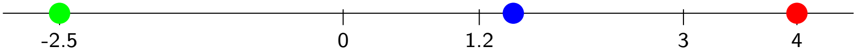

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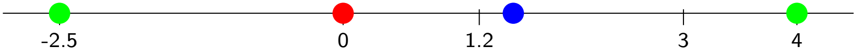
Theorem (Tverberg Theorem)

For any integer $f \geq 1$, for every multiset Y containing at least $(m+1)f+1$ vectors in $\mathbb{R}^m$, there exists a partition $Y_1, \ldots, Y_{f+1}$ of Y into $f+1$ nonempty multisets such that $\cap_{i=1}^{f+1} \mathcal{H}(Y_i) \neq \emptyset$.

- Points in $\cap_{i=1}^{f+1} \mathcal{H}(Y_i)$ are called Tverberg point.
- Example for one dimensional case $m = 1$ and single faulty agent $f = 1$:



▸ Go to algorithm