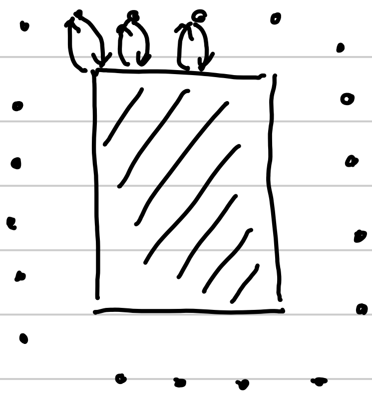
Lecture # 6 : General Multi-terminal Networks



$$\underbrace{p(y^{(1)}, y^{(2)}, \ldots, y^{(K)} \mid x^{(1)}, \ldots, x^{(K)})}$$

Let $W^{(ij)}$ = msg. from node i intended for node j

$\in \{1, \ldots, 2^{nR^{(ij)}}\}$

Note Title 27-Jun-19

A $((2^{nR_{ij}})_{i,j \in \{1,\dots,K\}}, n)$ code for our network is defined by a family of encoders, where for each node v, at each time $t \in \{1,\dots,n\}$, the encoder for node v at time i

$$f_{n,t}^{(v)} : \left(\prod_{l=1}^{K} \{1,\dots,2^{nR_{vl}}\} \right) \times (\mathcal{Y}^{(v)})^{t-1} \longrightarrow \underline{\mathcal{X}^{(v)}}.$$

and a family of decoders:

$$\underbrace{g_n^{(v)}} : \left(\prod_{l=1}^{K} \{1,\dots,2^{nR_{vl}}\} \right) \times (\mathcal{Y}^{(v)})^n \longrightarrow \prod_{l=1}^{K} \{1,\dots,2^{nR_{lv}}\}$$

we will use the notation

$$\underbrace{g_n^{(j,v)} : \left(\prod_{l=1}^{K} \{1,\dots,2^{nR_{vl}}\} \right) \times (\mathcal{Y}^{(v)})^n \longrightarrow \{1,\dots,2^{nR_{jv}}\}}$$

to represent the component (j,v) from the decoder

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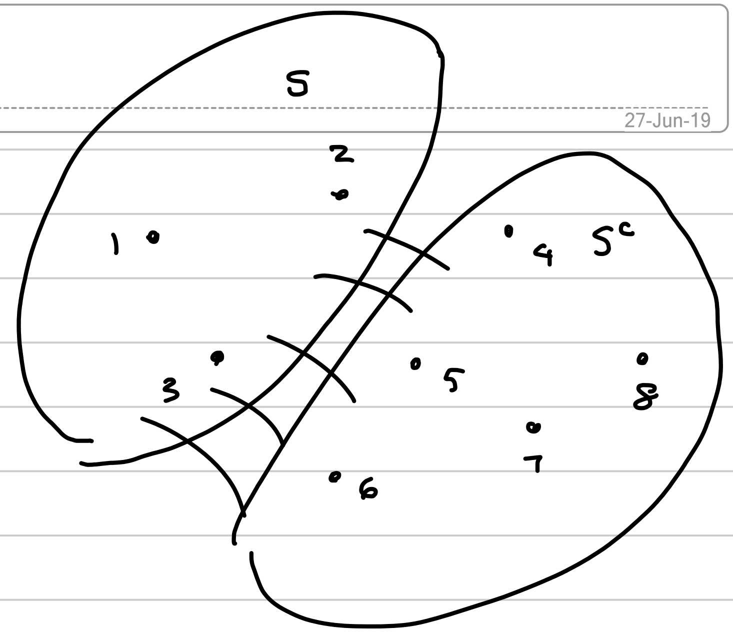
The error probability for this code is

$$\Pr\left(\bigcup_{i=1}^{K} \bigcup_{j=1}^{K} \left\{ g_n^{(i,j)}\left(W^{(j,1)}, W^{(j,2)}, \dots, W^{(j,K)}, (Y^{(j)})^n \right) \neq W^{(i,j)} \right\} \right).$$

We will say that a rate vector $(R_{ij} : i \in \{1, \dots, K\}, j \in \{1, \dots, K\})$ is achievable if $\exists$ a seq. of $(2^{nR_{ij}} : i,j \in \{1, \dots, K\})$ with $P_e^{(n)} \to 0$ as $n \to \infty$.

We will try to derive the following converse:
For any subset $S \subset \{1, \dots, K\}$

$$\boxed{\sum_{i \in S, j \in S^c} R_{ij} \leq I\left(\underline{X^{(S)}} ; Y^{(S^c)} \mid \underline{X^{(S^c)}} \right)}$$



where $X^{(S)} = (X^{(v)} : v \in S)$

$X^{(S^c)} = (X^{(v)} : v \in S^c = \{1, \dots, K\} \setminus S)$

$Y^{(S)} = (Y^{(v)} : v \in S)$

$Y^{(S^c)} = (Y^{(v)} : v \in S^c)$

Let $T = \{(i,j) : i \in S,\ j \in S^c\}$.

Let $T^c = \{(i,j) : i,j \in \{1, \dots, K\}\} \setminus T$.

Let $W^{(T)} = (W^{(ij)} : (ij) \in T)$

$W^{(T^c)} = (W^{(ij)} : (ij) \in T^c)$

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$$\sum_{(ij)\in T} nR_{ij} = \sum_{(ij)\in T} H(W^{(ij)}) \quad \text{since } W^{(ij)} \sim \text{Unif}(\{1,\dots,2^{nR_{ij}}\})$$

$$= H(W^{(T)}) \quad \text{since all } W^{(ij)} \text{ are indep.}$$

$$= H(W^{(T)} \mid \underbrace{W^{(T^c)}}) \quad \text{since } W^{(T)} \perp\!\!\!\perp W^{(T^c)}$$

$$= I(W^{(T)}; (Y^{(S^c)})^n \mid W^{(T^c)}) + H(W^{(T)} \mid (Y^{(S^c)})^n, W^{(T^c)})$$

$$\leq I(W^{(T)}; (Y^{(S^c)})^n \mid \underbrace{W^{(T^c)}}) + \underbrace{n\varepsilon_n}$$

for some $\varepsilon_n \to 0$ by Fano's inequality since $((Y^{(S^c)})^n, W^{(T^c)})$ contains all the decoder inputs for all nodes in S^c used to reconstruct the msg.s $W^{(T)}$.

$$= H((Y^{(S^c)})^n \mid W^{(T^c)}) - H((Y^{(S^c)})^n \mid W^{(T)}, W^{(T^c)}) + n\varepsilon_n$$

$$= \sum_{t=1}^{n} \Big[H\big(Y_t^{(S^c)} \mid \underbrace{W^{(T^c)}, (Y^{(S^c)})^{t-1}} \big) - H\big(Y_t^{(S^c)} \mid W^{(T)}, W^{(T^c)}, (Y^{(S^c)})^{t-1} \big) \Big] + n\varepsilon_n$$

chain rule for entropy

$$= \sum_{t=1}^{n} H\big(\underbrace{Y_t^{(S^c)}} \mid \underbrace{X_t^{(S^c)}}, \underbrace{W^{(T^c)}, (Y^{(S^c)})^{t-1}} \big) - H\big(Y_t^{(S^c)} \mid X_t^{(S^c)}, W^{(T)}, W^{(T^c)}, (Y^{(S^c)})^{t-1} \big) \Big] + n\varepsilon_n$$

since $X_t^{(S^c)}$ is a det. func. of thevalues $W^{(T^c)}$ and $(Y^{(S^c)})^{t-1}$!

$$\leq \sum_{t=1}^{n} \Big[H\big(Y_t^{(S^c)} \mid X_t^{(S^c)} \big) - H\big(Y_t^{(S^c)} \mid \underbrace{W^{(T)}}, \underbrace{W^{(T^c)}} (Y^{(S^c)})^{t-1}, X_t^{(S^c)}, \underbrace{X_t^{(S)}} \big) \Big] + n\varepsilon_n$$

by conditioning reduces entropy

$$= \sum_{t=1}^{n} \Big[H\big(Y_t^{(S^c)} \mid X_t^{(S^c)} \big) - H\big(Y_t^{(S^c)} \mid X_t^{(S)}, X_t^{(S^c)} \big) \Big] + n\varepsilon_n$$

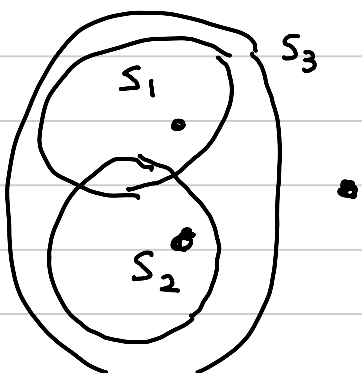
since $\underbrace{(W^{(T)}, W^{(T^c)}, (Y^{(S^c)})^{t-1})} \to \underbrace{(X_t^{(S)}, X_t^{(S^c)})} \to \underbrace{Y_t^{(S^c)}}$

form a Markov chain

$$= \sum_{t=1}^{n} I\left(X_t^{(S)}; Y_t^{(S^c)} \mid X_t^{(S^c)}\right) + n\varepsilon_n$$

With a bit more work, this gives

$$\boxed{\sum R_{ij} \le I\left(X^{(S)}; Y^{(S^c)} \mid X^{(S^c)}\right)}$$



$$\boxed{\begin{aligned} S_1 &: \quad R_1 \le I(X_1; Y \mid X_2) \\ S_2 &: \quad R_2 \le I(X_2; Y \mid X_1) \\ S_3 &: \quad R_1 + R_2 \le I(X_1, X_2; Y) \end{aligned}}$$

$$(W^{(T)}, W^{(T^c)}, (Y^{(S^c)})^{t-1}) \to (X_t^{(S)}, X_t^{(S^c)}) \to Y_t^{(S^c)}$$

$$p(y_t^{(S^c)} \mid (x_t^{(S)}, x_t^{(S^c)}), (w^{(T)}, w^{(T^c)}, (Y^{(S^c)})^{t-1}))$$

$$= p(y_t^{(S^c)} \mid (x_t^{(S)}, x_t^{(S^c)}))$$