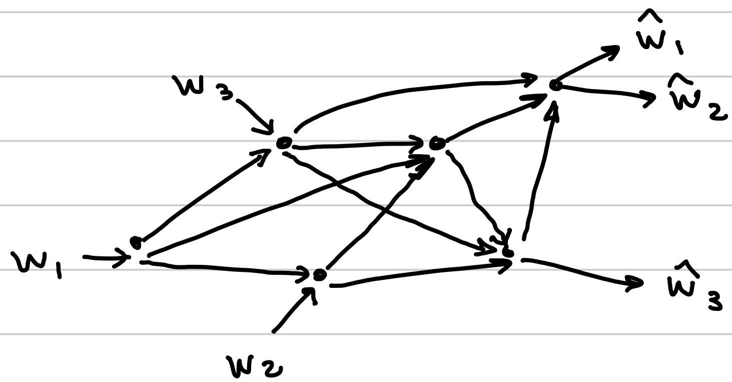
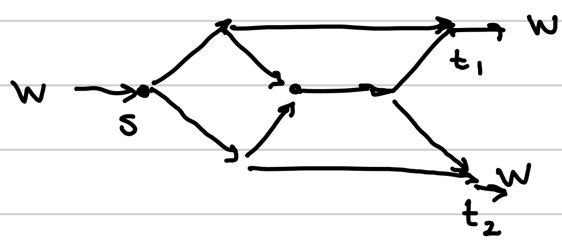
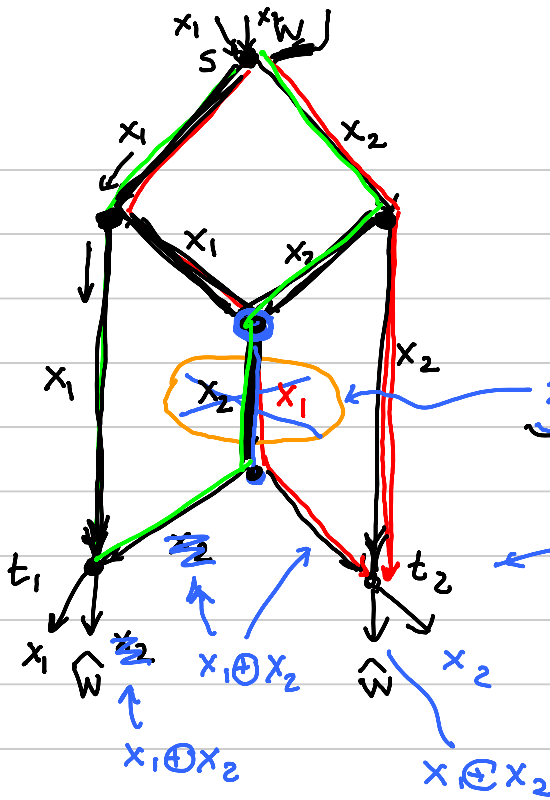
Lecture 5 - Network Coding

Note Title 27-Jun-19



Multicast Network Coding





$\mathcal{N}$: $G = (V, E)$

each edge $e \in E$ has a fixed capacity C_e

$\Rightarrow$ edge e can carry $n C_e$ bits over n channel uses

$x_1 \oplus x_2$

$(x_2, x_1 \oplus x_2) \to (x_1, x_2)$

$$x_1,\ x_1 \oplus x_2 \Rightarrow \begin{cases} x_1 \oplus (x_1 \oplus x_2) = x_2 \\ x_1 \end{cases}$$

A $(2^{nR}, n)$ multicast network code for an acyclic network is defined by:

A collection of encoders:

For each edge $e = (\underline{u}, v)$

the encoder $f_{n,e} : \prod_{(w,\underline{u}) \in E} \{1, \dots, \underbrace{2^{n C_{(w,u)}}}\} \longrightarrow \{1, \dots, 2^{n C_e}\}$

For each terminal node $t \in T$, where T = set of terminal nodes $\subseteq V$

$$g_{n,t} : \prod_{(w,t) \in E} \{1, \dots, 2^{n C_{(w,t)}}\} \longrightarrow \{1, \dots, 2^{nR}\}$$

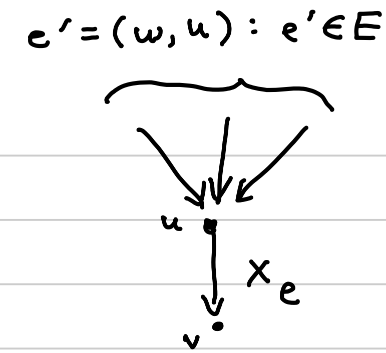
The probability of error for this code (averaged over all possible messages $W \in \{1, \dots, 2^{nR}\}$, $W \sim \text{Uniform}\{1, \dots, 2^{nR}\}$)

$$\boxed{P_e^{(n)} = \Pr\Big(\bigcup_{t \in T} \{ g_{n,t}(Y_t^n) \neq W \} \Big)}$$

For $e = (u,v) \in E$:

Where X_e^n = information carried by edge

$= f_{n,e}(X_{e'}^n : e' = (w,u), e' \in E)$



For any $v \in V$:

$Y_v^n = (X_{(u,v)}^n : (u,v) \in E)$.

Notice that X_e^n is some deterministic function of W that is being calculated in a step-by-step process across this network.

The step-by-step functions are sometimes called "local encoding functions".

Each of those functions can also be represented by a single deterministic function of the source information W.

For example, for $e = (u,v)$

$$X_e^n = f_{n,e}(X_{e'}^n : e' = (w,u), e' \in E) = f_{n,e}(Y_u^n)$$

can also be represented by some function $X_e^n = F_{n,e}(W)$.

Achievability:

Random encoder design:

For each edge $e=(u,v)\in E$ and $y_u^n \in \underbrace{\mathcal{Y}_u^n}_{} = \prod_{(w,u)\in E} \{1,\dots,2^{nC(w,u)}\}$,

choose $f_{n,e}(y_u^n) \sim$ iid Uniform $\{1,\dots,2^{nC_e}\}$.

Fix this code. Let $F_{n,e}(w)$ represent the end-to-end operation that chooses X_e^n.

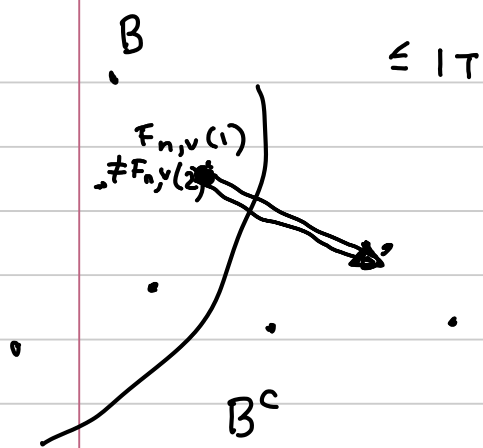
Design the decoder:

For each $t\in T$ and each output $y_t^n \in \underbrace{\mathcal{Y}_t^n}_{}$,

$$g_{n,t}(\underbrace{y_t^n}_{}) = \begin{cases} w & \text{if } \overbrace{(F_{n,(u,t)}(w) : (u,t)\in E)}^{} = \underbrace{y_t^n}_{} \\ & \text{and } \nexists\ \hat{w}\neq w \ (F_{n,(u,t)}(\hat{w}) : (u,t)) = y_t^n \\ \text{"error"} & \text{otherwise} \end{cases}$$

$$F_{n,t}(w) \triangleq (F_{n,(u,t)}(w) : (u,t)\in E)$$

$$
\begin{aligned}
E[P_e^{(n)}] &= E\left[\frac{1}{2^{nR}} \sum_{\omega=1}^{2^{nR}} Pr\left(\bigcup_{t\in T} g_{n,t}(F_{n,t}(\omega)) \neq \omega\right)\right] \\
&= E\left[Pr\left(\bigcup_{t\in T} g_{n,t}(F_{n,t}(1)) \neq 1\right)\right] \\
&\leq \sum_{t\in T} E\left[Pr\left(g_{n,t}(F_{n,t}(1)) \neq 1\right)\right] \\
&\leq |T| \max_{t\in T} E\left[Pr\left(g_{n,t}(F_{n,t}(1)) \neq 1\right)\right] \\
&\leq |T| \max_{t\in T} \sum_{\hat{\omega}\neq 1} Pr\left(F_{n,t}(\hat{\omega}) = F_{n,t}(1)\right) \\
&\lesssim |T| \max_{t\in T} 2^{nR} Pr\left(F_{n,t}(2) = F_{n,t}(1)\right) \\
&= |T| \max_{t\in T} 2^{nR} Pr\left(\bigcup_{\substack{B\subset V:\, s\in B \\ t\notin B}} \{F_{n,v}(2) \neq F_{n,v}(1)\ \forall v\in B,\ F_{n,v}(2) = F_{n,v}(1)\ \forall v\notin B\}\right)
\end{aligned}
$$



$$\leq |T| \max_{t \in T} 2^{nR} \sum_{\substack{B \subset V:\, s \in B \\ t \in B^c}} \underbrace{\underbrace{\Pr\left(F_{n,v}(2) \neq F_{n,v}(1) \ \forall v \in B, \ F_{n,v}(2) = F_{n,v}(1) \ \forall v \notin B \right)}_{\Pr\left(F_{n,v}(2) \neq F_{n,v}(1) \ \forall v \in B \right) \Pr\left(F_{n,v}(2) = F_{n,v}(1) \ \forall v \in B^c \mid F_{n,v}(2) \neq F_{n,v}(1) \ \forall v \in B \right)}}_{\leq \Pr\left(F_{n,v}(2) = F_{n,v}(1) \ \forall v \in B^c \mid F_{n,v}(2) \neq F_{n,v}(1) \ \forall v \in B \right)}$$

$$\leq |T| \max_{t \in T} 2^{nR} \, 2^{|V|} \max_{\substack{B \subset V:\, s \in B \\ t \in B^c}} \prod_{\substack{e \in E:\, e = (i,j) \\ i \in B \\ j \in B^c}} 2^{-nC_e}$$

$$= \underbrace{|T|\, 2^{|V|}} \overbrace{\underbrace{\max_{t\in T} \max_{\substack{B\subset V:\, s\in B\\ t\in B^c}}} 2^{-n\left(\sum_{\substack{e\in E:\, e=(i,j)\\ i\in B\\ j\notin B}} C_e - R\right)}}$$

$\longrightarrow 0$ as $n\to\infty$ provided that

$$\boxed{R < \min_{t\in T} \min_{\substack{B\subset V:\, s\in B\\ t\in B^c}} \sum_{\substack{(i,j)\in E:\, i\in B\\ j\in B^c}} C_e}$$