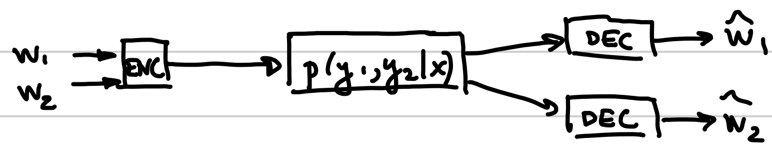
Network Information Theory : Lecture #4 June 26, 2019

Broadcast Channel



$W_1 \perp\!\!\!\perp W_2$

We look at a special case of broadcast channels:

Physically degraded broadcast channels

$$p(y_1, y_2 | x) = \underbrace{p(y_1 | x) p(y_2 | y_1)}$$

A $((2^{nR_1}, 2^{nR_2}), n)$ broadcast channel (BC) code is defined by

a single encoder

$$f_n : \{1, \dots, 2^{nR_1}\} \times \{1, \dots, 2^{nR_2}\} \longrightarrow \mathcal{X}^n$$

and a pair of decoders

$$g_{1,n} : \mathcal{Y}_1^n \longrightarrow \{1, \dots, 2^{nR_1}\}$$

$$g_{2,n} : \mathcal{Y}_2^n \longrightarrow \{1, \dots, 2^{nR_2}\}.$$

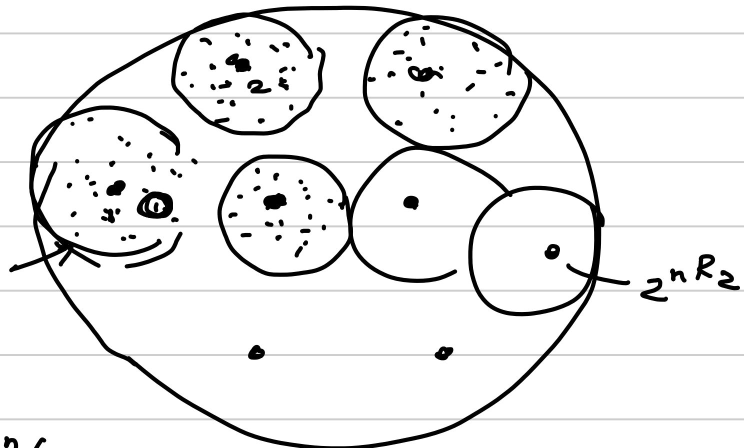
The average probability of error for the code is

$$P_e^{(n)} = \Pr\left((g_{1,n}(Y_1^n), g_{2,n}(Y_2^n)) \neq (W_1, W_2) \mid X^n = f_n(W_1, W_2)\right)$$

$$(W_1, W_2) \sim \text{Unif}\left(\{1, \dots, 2^{nR_1}\} \times \{1, \dots, 2^{nR_2}\}\right)$$

A rate pair (R_1, R_2) is <u>achievable</u> if $\exists$ a seq. of $((2^{nR_1}, 2^{nR_2}), n)$ BC codes $\{(f_n, (g_{1,n}, g_{2,n}))\}_{n=1}^{\infty}$ with $P_e^{(n)} \to 0$ as $n \to \infty$.

The <u>capacity region</u> is the closure of the set of achievable rates.



<u>Achievability</u> :

Random code design: Fix $p(u)$. Fix $p(x|u)$

Draw $\underbrace{U^n(1), U^n(2), \ldots, U^n(2^{nR_2})}$ ~ iid $\prod_{i=1}^{n} p(u_i)$.

For each w_2, draw

$$\underbrace{f_n(1, w_2), f_n(2, w_2), \ldots, f_n(2^{nR_1}, w_2)} \sim \text{iid} \prod_{i=1}^{n} p(x_i | u_i)$$

where $u^n = U^n(w_2)$.

Decoder design: For each $y_2^n \in \mathcal{Y}_2^n$

$$g_{2,n}(y_2^n) = \begin{cases} w_2 & \text{if } (U^n(w_2), Y_2^n) \in A_\varepsilon^{(n)} \\ & \text{and } \nexists\ \hat{w}_2 \neq w_2 \text{ s.t } (U^n(\hat{w}_2), Y_2^n) \in A_\varepsilon^{(n)} \\ \text{"error"} & \text{otherwise.} \end{cases}$$

For each $y_1^n \in \mathcal{Y}_1^n$,

$$g_{1,n}(y_1^n) = \begin{cases} w_1 & \text{if } \exists \text{ a unique } w_2 \text{ s.t. } (u^n(w_2), f_n(w_1,w_2), Y_1^n) \in A_\varepsilon^{(n)} \\ & \text{and } \nexists\ (\hat{w}_1, \hat{w}_2) \neq (w_1,w_2) \text{ s.t. } (u^n(\hat{w}_2), f_n(\hat{w}_1,\hat{w}_2), Y_1^n) \in A_\varepsilon^{(n)} \\ \text{"error"} & \text{otherwise} \end{cases}$$

$$E[P_e^{(n)}] = E\left[\frac{1}{2^{nR_1}} \cdot \frac{1}{2^{nR_2}} \sum_{(w_1,w_2)\in\{1,\dots,2^{nR_1}\}\times\{1,\dots,2^{nR_2}\}} \Pr\left((g_{1,n}(Y_1^n), g_{2,n}(Y_2^n)) \neq (w_1,w_2) \,\middle|\, X^n = f_n(w_1,w_2)\right)\right]$$

$$= E\left[\Pr\left((g_{1,n}(Y_1^n), g_{2,n}(Y_2^n)) \neq (1,1) \,\middle|\, X^n = f_n\underbrace{(1,1)}\right)\right]$$

Events of interest at receiver 2:

$$E_j^{(2)} = \{(\underbrace{u^n(j)}, \underbrace{Y_2^n}) \in A_\varepsilon^{(n)}\}$$

Events of interest at receiver 1:

$$E_j^{(1)} = \{(u^n(j), Y_1^n) \in A_\varepsilon^{(n)}\}, \quad E_{ij}^{(1)} = \{(u^n(j), f_n(i,j), Y_1^n) \in A_\varepsilon^{(n)}\}$$

$$E[P_e^{(n)}] = \Pr\Big((E_1^{(2)})^c \cup \bigcup_{j\neq 1} E_j^{(2)} \cup \underbrace{(E_{11}^{(1)})^c}_{} \cup \underbrace{\bigcup_{j\neq 1} E_j^{(1)}}_{} \bigcup_{i\neq 1} E_{i1}^{(1)} \,\Big|\, \underbrace{x^n = f_n(1,1)}_{\text{"(1,1) sent"}} \Big)$$

$$\leq \Pr((E_1^{(2)})^c \mid (1,1)\text{ sent}) + \Pr((E_{11}^{(1)})^c \mid (1,1)\text{ sent})$$

$$+ \underbrace{\sum_{j\neq 1} \Pr(E_j^{(2)} \mid (1,1)\text{ sent})}_{} + \sum_{j\neq 1} \Pr(E_j^{(1)} \mid (1,1)\text{ sent})$$

$$+ \underbrace{\sum_{i\neq 1} \Pr(E_{i1}^{(1)} \mid (1,1)\text{ sent})}_{}$$

$\Pr((E_1^{(2)})^c \mid (1,1)\text{ sent}) + \Pr((E_1^{(1)})^c \mid (1,1)\text{ sent}) \longrightarrow 0$ as $n \to \infty$ AEP

$$\sum_{j\neq 1} \Pr(E_j^{(2)} \mid (1,1)\text{ sent}) \leq 2^{nR_2} \sum_{(\hat{u}^n, y_2^n) \in A_\varepsilon^{(n)}} p(\hat{u}^n)\, p(y_2^n)$$

$$\leq \underbrace{2^{nR_2}}_{} \, 2^{n(H(\underbrace{U, Y_2}_{}) + \varepsilon)} \, 2^{-n(H(\underbrace{U}_{}) - \varepsilon)} \, 2^{-n(H(\underbrace{Y_2}_{}) - \varepsilon)}$$

$$= 2^{-n(I(U;Y_2) - 3\varepsilon - R_2)} \xrightarrow[n\to\infty]{} 0 \quad \text{provided } R_2 < I(U;Y_2)$$

Similarly

$$\sum_{j\neq 1} \Pr\left(E_j^{(1)} \mid (1,1)\text{ sent}\right) \le 2^{-n(I(U;Y_1)-3\varepsilon-R_2)} \xrightarrow[n\to\infty]{} 0 \quad \text{provided } R_2 < I(U;Y_1)$$

Finally:

$$\sum_{i\neq 1} \Pr\left(E_{i1}^{(1)} \mid (1,1)\text{ sent}\right) = (2^{nR_1}-1)\Pr\left(E_{21}^{(1)} \mid (1,1)\text{ sent}\right)$$

$$\le 2^{nR_1} \sum_{(u^n,\hat{x}^n,y_1^n)\in A_\varepsilon^{(n)}} p(u^n)\,p(\hat{x}^n|u^n)\,p(y_1^n|u^n)$$

$$\le 2^{nR_1}\, 2^{n(H(U,X,Y_1)+\varepsilon)}\, 2^{-n(H(U)-\varepsilon)}\, \frac{2^{-n(H(X,U)-\varepsilon)}}{2^{-n(H(U)+\varepsilon)}} \cdot \frac{2^{-n(H(U,Y_1)-\varepsilon)}}{2^{-n(H(U)+\varepsilon)}}$$

$$= 2^{-n(-H(U,X,Y_1)+H(U)+H(X|U)+H(Y_1|U)-5\varepsilon-R_1)}$$

$$= 2^{-n(-H(X,Y_1|U)+H(X|U)+H(Y_1|U)-5\varepsilon-R_1)}$$

$$= 2^{-n(I(X;Y_1|U) - 5\varepsilon - R_1)}$$

$\longrightarrow 0$ as $n \to \infty$ provide $R_1 < I(X;Y_1|U)$

$$\left.\begin{aligned} R_1 &< I(X;Y_1|U) \\ R_2 &< I(U;Y_1) \\ R_2 &< I(U;Y_2) \end{aligned}\right\} \rightarrow R_2 < \underbrace{\min\{I(U;Y_1), I(U;Y_2)\}}_{\text{by data processing inequality}} = I(U;Y_2)$$

$$\boxed{\begin{aligned} R_1 &< I(X;Y_1|U) \\ R_2 &< I(U;Y_2) \end{aligned}}$$

This turns out to describe the capacity region of the broadcast channel in the physically degraded case

It turns out that for any pair of channels

$$\underbrace{p(y_1, y_2 | x)}$$

$$\underbrace{\hat{p}(y_1, y_2 | x)}$$

for which
$$p(y_1|x) = \hat{p}(y_1|x)$$
$$p(y_2|x) = \hat{p}(y_2|x)$$

the capacity regions for BCs $p(y_1, y_2|x)$ & $\hat{p}(y_1, y_2|x)$ are the same.

We say that channel $\hat{p}(y_1, y_2|x)$ is <u>stochastically</u> <u>degraded</u> if $\exists$ a physically degraded channel $p(y_1, y_2|x)$ different from $\hat{p}$

s.t. $\hat{p}(y_1|x) = p(y_1|x) \quad \forall\, x, y_1$

$$\hat{p}(y_2|x) = p(y_2|x) \quad \forall\, x, y_2 .$$

We have solved the capacity region for all stochastically degraded channels.