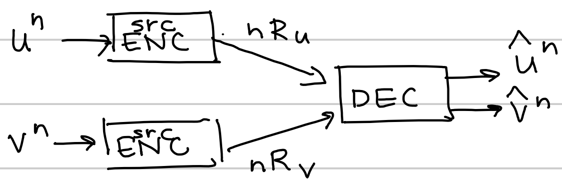
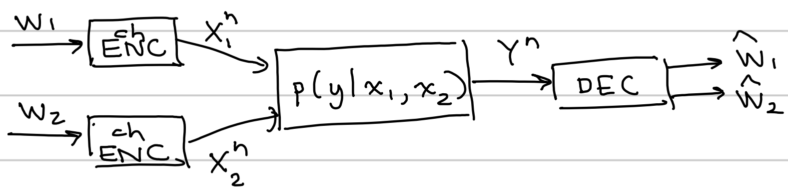
Network Information Theory : Lecture 3

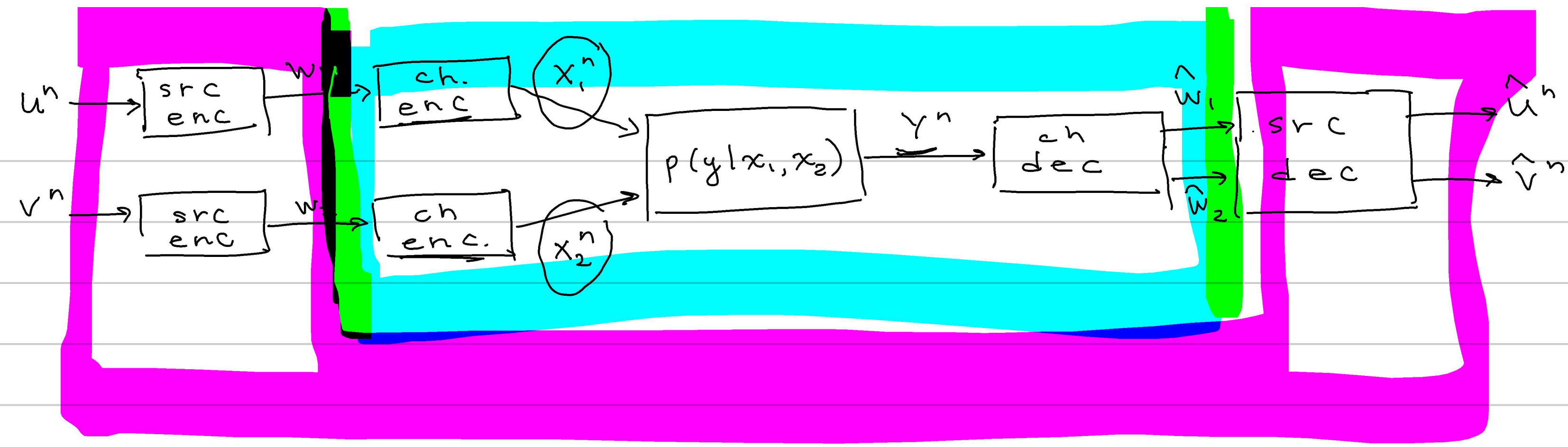
Note Title 26-Jun-19

Last time : Multiple Access Source Coding

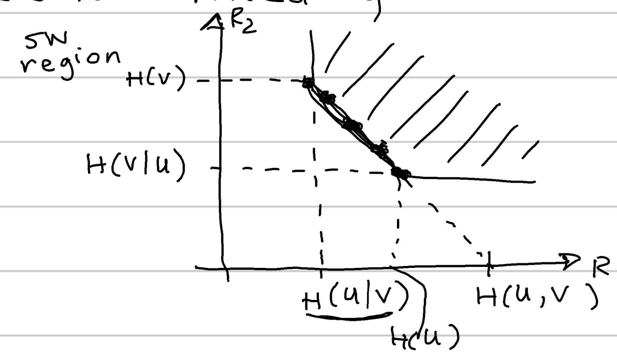


Today Multiple Access Channel Coding





A $((2^{nR_1}, 2^{nR_2}), n)$ multiple access channel code is defined by
(MAC)



a pair of encoders

$$f_{1,n} : \{1, \ldots, 2^{nR_1}\} \longrightarrow \mathcal{X}_1^n$$

$$f_{2,n} : \{1, \ldots, 2^{nR_2}\} \longrightarrow \mathcal{X}_2^n$$

and a single decoder

$$g_n : \mathcal{Y}^n \longrightarrow \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{nR_2}\}$$

We assume that the messages W_1 and W_2 are independent & uniformly distr

The probability of error for this code is

$$P_e^{(n)} = \Pr\left(g_n(Y^n) \neq (W_1, W_2) \mid X_1^n = f_{1,n}(W_1), X_2^n = f_{2,n}(W_2) \right).$$

A rate vector (R_1, R_2) is achievable if $\exists$ a seq. of $((2^{nR_1}, 2^{nR_2}), n)$ MAC codes $\{((f_{1,n}, f_{2,n}), g_n)\}_{n=1}^{\infty}$ with $P_e^{(n)} \to 0$ as $n \to \infty$.

The capacity region for the MAC is the closure of the set of all achievable rates.

To derive the capacity, we again use an argument in 2 parts:

Achievability

Converse

Achievability

Random code Design: Fix $p_1(x_1)$ on alphabet $\mathcal{X}_1$, $p_2(x_2)$ on alphabet $\mathcal{X}_2$.

Draw the codewords $f_{1,n}(1), f_{1,n}(2), \ldots, f_{1,n}(2^{nR_1}) \sim \text{iid} \prod_{i=1}^{n} p_1(x_{1i})$

$f_{2,n}(1), f_{2,n}(2), \ldots, f_{2,n}(2^{nR_2}) \sim \text{iid} \prod_{i=1}^{n} p_2(x_{2i})$.

Design the decoder : For each $y^n \in \mathcal{Y}^n$

$$g_n(y^n) = \begin{cases} (w_1, w_2) & \text{if } (f_{1,n}(w_1), f_{2,n}(w_2), y^n) \in A_\varepsilon^{(n)} \\ & \text{and } \nexists\ (\hat{w}_1, \hat{w}_2) \neq (w_1, w_2) \text{ s.t. } \\ & (f_{1,n}(\hat{w}_1), f_{2,n}(\hat{w}_2), y^n) \in A_\varepsilon^{(n)} \\ \text{"error"} & \text{otherwise} \end{cases}$$

$$E[P_e^{(n)}] = E\left[\frac{1}{2^{nR_1}} \cdot \frac{1}{2^{nR_2}} \sum_{(w_1,w_2) \in \{1,\dots,2^{nR_1}\} \times \{1,\dots,2^{nR_2}\}} Pr\left(g_n(Y^n) \neq (w_1,w_2) \mid X_1^n = f_{1,n}(w_1), X_2^n = f_{2,n}(w_2)\right)\right]$$

$$= \frac{1}{2^{nR_1}} \frac{1}{2^{nR_2}} \sum_{(w_1,w_2)} \underbrace{E\left[Pr\left(g_n(Y^n) \neq (w_1,w_2) \mid X_1^n = f_{1,n}(w_1), X_2^n = f_{2,n}(w_2)\right)\right]}_{\text{the same } \forall\ w_1, w_2 \text{ by symmetry of code design}}$$

$$= \underbrace{E\left[Pr\left(g_n(Y^n) \neq (1,1) \mid X_1^n = f_{1,n}(1), X_2^n = f_{2,n}(1)\right)\right]}$$

Events of interest: $E_{ij} = \{(\underline{f_{1,n}(i)}, \underline{f_{2,n}(j)}, Y^n) \in A_\varepsilon^{(n)}\}$

Error events: $\underbrace{E_{11}^c}$: codewords transmitted are not jointly typical with Y^n received

$\underbrace{(i,j) \neq (1,1)}$ $\underbrace{E_{ij}}$: codewords (i,j) are jointly typical with Y^n but (i,j) was not the message pair sent.

$$E[P_e^{(n)}] = E[\Pr(g_n(Y^n) \neq (1,1) \mid (X_1^n, X_2^n) = (f_{1,n}(1), f_{2,n}(1)))]$$

$$= \Pr\Big(E_{11}^c \cup \bigcup_{(i,j)\neq(1,1)} E_{ij} \,\Big|\, \underbrace{(X_1^n, X_2^n) = (f_{1,n}(1), f_{2,n}(1))}_{\text{"(1,1) sent"}}\Big)$$

$$\leq \Pr(\underline{E_{11}^c} \mid (1,1)\text{ sent}) + \underline{\sum_{j\neq 1} \Pr(\underline{E_{1j}} \mid (1,1)\text{ sent})} + \sum_{i\neq 1} \Pr(\underline{E_{i1}} \mid (1,1)\text{ sent})$$

$$+ \sum_{i\neq 1}\sum_{j\neq 1} \Pr(\underline{E_{ij}} \mid (1,1)\text{ sent}) \qquad \text{by union bound}$$

$\Pr(E_{11}^c \mid (1,1)\text{ sent}) \longrightarrow 0$ as $n \to \infty$ by AEP (WLLN)

$$\sum_{j\neq 1} \Pr(E_{1j} \mid (1,1)\text{ sent}) = (2^{nR_2}-1)\Pr(E_{12} \mid (1,1)\text{ sent}) \quad \text{by symmetry of code design}$$

$$\le 2^{nR_2} \sum_{(x_1^n, \hat{x}_2^n, y^n)\in A_\varepsilon^{(n)}} \underbrace{p_1(x_1^n)\, p_2(\hat{x}_2^n)\, p(y^n \mid x_1^n)}$$

$$\le 2^{nR_2} \sum_{(x_1^n, \hat{x}_2^n, y^n)\in A_\varepsilon^{(n)}} p_2(\hat{x}_2^n)\, p(x_1^n, y^n)$$

$$\le 2^{nR_2}\, |A_\varepsilon^{(n)}|\, 2^{-n(H(X_2)-\varepsilon)}\, 2^{-n(H(X_1,Y)-\varepsilon)}$$

$$\le 2^{nR_2}\, 2^{n(H(X_1,X_2,Y)+\varepsilon)}\, 2^{-n(H(X_2)-\varepsilon)}\, 2^{-n(H(X_1,Y)-\varepsilon)}$$

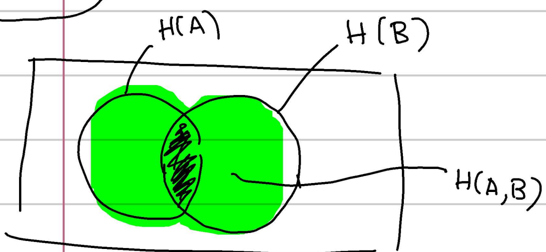
$$= 2^{-n(I(X_1,Y;X_2)-3\varepsilon-R_2)}$$

$$\xrightarrow[n\to\infty]{} 0 \quad \text{provided} \quad R_2 < I(X_1,Y;X_2) - 3\varepsilon$$

$$= \underbrace{\cancel{I(X_1;X_2)}}_{=0 \text{ since } X_1 \perp\!\!\!\perp X_2} + \underbrace{I(Y;X_2 \mid X_1)}_{= I(X_2;Y\mid X_1)}$$

$$\boxed{R_2 < I(X_2;Y\mid X_1)}$$

$$I(A;B) = H(A) + H(B) - H(A,B)$$



Similarly $\sum_{i\neq 1} \Pr(E_{i1} \mid (1,1)\text{ sent}) \longrightarrow 0$ as $n\to\infty$ provided

$$\boxed{R_1 < I(X_1;Y|X_2)}$$

Finally $\sum_{i\neq 1}\sum_{j\neq 1} \Pr(E_{ij} \mid (1,1)\text{ sent}) \le 2^{nR_1} 2^{nR_2} \sum_{(\hat{x}_1^n,\hat{x}_2^n,y^n)\in A_\varepsilon^{(n)}} p_1(\hat{x}_1^n)\, p_2(\hat{x}_2^n)\, p(y^n)$

$$\le 2^{n(R_1+R_2)} |A_\varepsilon^{(n)}| \, 2^{-n(H(X_1)-\varepsilon)} 2^{-n(H(X_2)-\varepsilon)} 2^{-n(H(Y)-\varepsilon)}$$

$$\le 2^{n(R_1+R_2)} 2^{n(H(X_1,X_2,Y)+\varepsilon)} 2^{-n(\underbrace{H(X_1)+H(X_2)}_{H(X_1,X_2)}+H(Y)-3\varepsilon)} \text{ since } X_1 \perp\!\!\!\perp X_2$$

$$= 2^{-n(H(X_1,X_2)+H(Y)-H(X_1,X_2,Y)-4\varepsilon-(R_1+R_2))}$$

$$= 2^{-n(I(X_1,X_2;Y)-4\varepsilon-(R_1+R_2))}$$

$\longrightarrow 0$ as $n\to\infty$ provided $\boxed{R_1+R_2 < I(X_1,X_2;Y)}$

$E[P_e^{(n)}] \to 0$ as $n \to \infty$ provided

$$\boxed{\begin{aligned} R_1 &< I(X_1;Y|X_2) \\ R_2 &< I(X_2;Y|X_1) \\ R_1+R_2 &< I(X_1,X_2;Y) \end{aligned}}$$

taking the union over all $p(x_1)p(x_2)$ and then the convex hull and the closure turns out to give the complete capacity region for MAC.