EE6110 Aug.-Nov., 2021

Class #27 (Nov. 09, 2021)

Note Title 27-03-2020

LQM

$n-1$ n

Kalman Filter Algorithm

1 Haykin

Recursive Predictor Form (of the KF)

$\bar{x}(n|Y^{n-1}) \longrightarrow \bar{x}(n+1|Y^{n})$

2 J.M. Mendel

Predictor - Corrector form

$\bar{x}(n|Y^{n}) \longrightarrow \bar{x}(n+1|Y^{n})$

$\bar{x}(n+1|Y^{n}) \longrightarrow \bar{x}(n+1|Y^{n+1})$

Recursive filtering form

$\bar{x}(n-1|Y^{n-1}) \longrightarrow \bar{x}(n|Y^{n})$

$$\bar{x}(n) = F(n,n-1)\,\bar{x}(n-1) + \bar{v}_1(n-1)$$

$$\bar{y}(n) = C(n)\,\bar{x}(n) + \bar{v}_2(n)$$

$y(n) = c^{*}x + v_2$

Summary of the Recursive Predictive form of the KF

acf

step 1: $R(n) = C(n)\,K(n,n-1)\,C^H(n) + Q_2(n)$

step 2: $G(n) = \Big[\,\overbrace{F(n+1,n)\,\underbrace{K(n,n-1)\,C^H(n)}_{\to P(n)}}\,\Big]\,R^{-1}(n)$

Kalman Gain

step 3: $\bar{\alpha}(n) = \bar{y}(n) - C(n)\,\hat{x}(n|\mathcal{Y}^{n-1})$

innovation residual

step 4: $\hat{x}(n+1|\mathcal{Y}^{n}) = F(n+1,n)\,\hat{x}(n|\mathcal{Y}^{n-1}) + G(n)\cdot\bar{\alpha}(n)$

Prediction update

$M \times 1$

Step 5 : $\quad J(n) = \left(\underset{M\times M}{I} - F^{-1}(n+1,n)\, G(n)\, C(n) \right) \underset{M\times M}{K}(n,n-1)$

Step 6 : $\quad \overset{P(n)}{K(n+1,n)} = F(n+1,n)\, J(n)\, F^{H}(n+1,n) + Q_1(n)$

$\rightarrow$ G. back to step 1

Predictor- Corrector form of the KF

Preliminaries :

Thm 1: If $\underset{n\times 1}{\bar{a}}$ and $\bar{b}_{m\times 1}$ are jointly Gaussian (JG), then

$$\boxed{E\left[\bar{a} \mid \bar{b}\right] = \bar{m}_a + P_{ab} \cdot P_b^{-1} \left(\bar{b} - \bar{m}_b\right)} \rightarrow 1 \checkmark$$

$\bar{m}_a = E[\bar{a}]$; $\quad P_{ab} \triangleq E\left[(\bar{a}-\bar{m}_a)(\bar{b}-\bar{m}_b)^H\right]$

Proof outline : $\quad \bar{z} = \begin{bmatrix} \bar{a} \\ \hline \bar{b} \end{bmatrix}$; $\longrightarrow$ JG

$$p(\bar{z}) = \frac{1}{\sqrt{(2\pi)^{n+m}\,|P_z|}} \exp\left\{-\frac{1}{2}(\bar{z}-\bar{m}_z)^H P_z^{-1} (\bar{z}-\bar{m}_z)\right\}$$

where $\quad P_z = \begin{bmatrix} P_a & P_{ab} \\ P_{ba} & P_b \end{bmatrix}$ & $\bar{m}_z = \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix}$;

It can be shown that

$$p(\bar{a}|\bar{b}) = \frac{1}{\sqrt{(2\pi)^n |A|}} \exp\left\{ \frac{-1}{2} (\bar{a}-\bar{m})^H A^{-1} (\bar{a}-\bar{m}) \right\}$$

where $\bar{m} = E[\bar{a}|\bar{b}] = \bar{m}_a + P_{ab} P_b^{-1} (\bar{b} - \bar{m}_b)$;

and $A = P_a - P_{ab} \cdot P_b^{-1} P_{ba}$;

Thm 2 :

Let $\bar{a}, \bar{b}, \bar{c}$ be $n\times 1$, $m\times 1$, & $r\times 1$ JG vectors.
If $\bar{b}$, & $\bar{c}$ are statistically independent, then

$$E[\bar{a}|\bar{b},\bar{c}] = E[\bar{a}|\bar{b}] + E[\bar{a}|\bar{c}] - \bar{m}_a$$

#

Thm 3 :

Let $\underset{n\times 1}{\bar{a}}, \underset{m\times 1}{\bar{b}}, \underset{r\times 1}{\bar{c}}$ be JG. If $\bar{b}$ and $\bar{c}$ are not statistically independent, then $\tilde{c} \perp \bar{b}$

(i) $E[\bar{a} \mid \bar{b}, \bar{c}] = E[\bar{a} \mid \bar{b}, \tilde{c}]$ where $\tilde{c} = \bar{c} - E[\bar{c} \mid \bar{b}]$

(ii) $E[\bar{a} \mid \bar{b}, \bar{c}] = E[\bar{a} \mid \bar{b}] + E[\bar{a} \mid \tilde{c}] - \bar{m}_a$;

Predictor-Corrector form of the KF : $\quad F(n, n-1) \longrightarrow F$

$$\begin{bmatrix} \text{State Eqn:} & \underset{n\times 1}{\bar{x}}(n) = F\,\bar{x}(n-1) + \bar{v}_1(n-1) \\ \text{Meas. Eqn:} & \bar{y}(n) = C(n)\,\bar{x}(n) + \bar{v}_2(n) \end{bmatrix} \quad \underset{r\times n}{}$$

$\longrightarrow$ If $\bar{x}(0) \rightarrow$ Gaussian

$$\bar{x}(n) = F(n,0)\,\bar{x}(0) + \sum_{i=0}^{n-1} F(n, i+1)\,\bar{v}_1(i)$$

Gaussian $\overset{JG}{=}$

(*) $\underbrace{\begin{bmatrix} \bar{x}(n) \\ \bar{y}(n) \end{bmatrix}}_{JG} = \underbrace{\begin{bmatrix} [F] & [I] & [0] \\ [CF] & [C] & [\underline{I}] \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \bar{x}(n-1) \\ \bar{v}_1(n-1) \\ \bar{v}_2(n) \end{bmatrix}}_{JG}$

(*) $\bar{\alpha}(n) = \bar{y}(n) - \underbrace{\hat{y}(n \mid Y^{n-1})}_{E[\bar{y}(n) \mid Y^{n-1}]}$

$y(n), Y^{n-1}$ are JG

- $\bar{\alpha}(n)$ & $\bar{y}(n)$ are JG
- $\bar{\alpha}(n) \perp \hat{y}(n \mid Y^{n-1})$
- $E[\bar{\alpha}(n)] = \bar{0}$;

$\bar{\alpha}(n) \rightleftharpoons \bar{y}(n)$

(*) Early Result

By conditioning both sides of the State Equation with Y^n, and taking expectations, we get

filtered state

$$\boxed{\hat{x}(n+1 \mid Y^n) = F(n+1, n) \cdot \hat{x}(n \mid Y^n)} \longrightarrow \text{1a}$$

predicted state

Subtract 1a from both sides of the state Eq.

$$\underbrace{x(n+1) - \hat{x}(n+1 \mid Y^n)}_{\triangleq\ \epsilon(n+1,\, n)} = F \cdot \left[\underbrace{x(n) - \hat{x}(n \mid Y^n)}_{\to\ \epsilon(n,\, n)} \right] + v_1(n)$$

predicted error vector

filtered error vector

4 terms

$$\boxed{\underbrace{K(n+1,n)}_{\text{predicted error covariance}} \overset{E[\epsilon\epsilon^H]}{} = F \cdot \underbrace{K(n,n)}_{\text{filtered error covariance}} \cdot F^H + Q_1(n)} \rightarrow \text{1.B}$$

Predictor - Corrector form

$$E\left[\bar{x}(n) \mid Y^n\right] = E\left[\bar{x}(n) \mid Y^{n-1}, \bar{y}(n)\right]$$

$\bar{\alpha}(n)$

$$= E\left[\bar{x}(n) \mid Y^{n-1}, \bar{\alpha}(n)\right]$$

$$= \underbrace{E\left[\bar{x}(n) \mid Y^{n-1}\right]}_{\text{predicted estimated from time } n-1} + E\left[\bar{x}(n) \mid \bar{\alpha}(n)\right] - m_x(n)$$

i.e.,

$$\boxed{\hat{x}(n|n) = \hat{x}(n|n-1) + \underbrace{E\left[\bar{x}(n) \mid \bar{\alpha}(n)\right]} - \bar{m}_x(n)}$$

(2-1)

$\bar{x}, \bar{\alpha} \to$ are JG (?)

$$E\left[\bar{x}(n) \mid \bar{\alpha}(n)\right] = \bar{m}_x(n) + P_{x\alpha}(n,n)\, \underbrace{P^{-1}_{\alpha\alpha}(n)}_{R^{-1}(n)} \left(\bar{\alpha}(n) - \underbrace{\bar{m}_\alpha(n)}_{0}\right)$$

$$= \left(\bar{m}_x(n) + P_{x\alpha}(n,n) \cdot R^{-1}(n)\, \bar{\alpha}(n)\right)$$

$$\boxed{\hat{x}(n|n) = \hat{x}(n|n-1) + \underbrace{P_{x\alpha}(n,n) R^{-1}(n)}_{G(n)} \cdot \bar{\alpha}(n)} \quad (2b)$$

$G(n) \leftarrow$ kalman Gain

$$P_{x\alpha}(n,n) = E\left[(\bar{x} - \bar{m}_x)(\bar{\alpha} - \bar{m}_\alpha)^H \right]$$

$$= K(n,n-1)\, C^H(n)$$

so

$$\boxed{G(n) = K(n,n-1)\, C^H(n) \cdot R^{-1}(n)} \rightarrow (3)$$

It can be shown that

$$\boxed{K(n, n) = \left[I - G(n)\, C(n) \right] K(n, n-1)} \quad (4)$$

Filtered error covariance

Predicted error covariance

EKF

H_∞ KF

Robust KF

Self-tuned KF → scented KF

$$\bar{x}(n) \Longrightarrow \overset{A}{\bar{F}}\, \bar{x}(n-1) + \overset{B}{\tilde{\cdot}}\, \bar{v}_1(n) \quad Q_1$$

$$\bar{y}(n) = \bar{C}\, \bar{x}(n) + \overset{D}{\cdot}\, \bar{v}_2(n) \quad Q_2$$