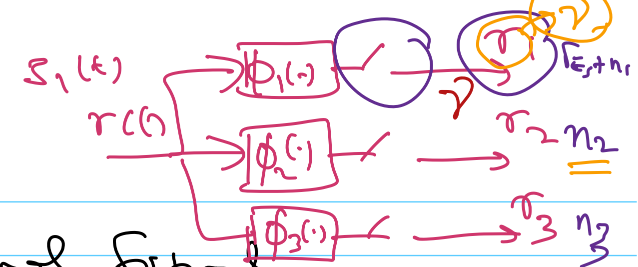
Zoom Class #2
Sept. 27, 2025

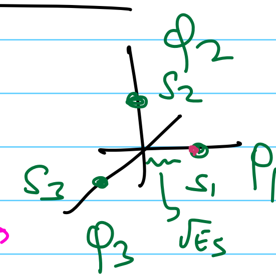
EE 4140



P(e) for M-ary orthogonal signals

$\boxed{N = M}$; $s_i(t) = \sqrt{E_s}\,\phi_i(t)$

$M = 3 = N$



$\bar{s}_1$ was sent

$\bar{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$$\bar{r} = \bar{s}_1 + \bar{n} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$\sqrt{E_s} + n_1 = \nu$

$$P(c \mid s_1) = P[\bar{r} \in Z_1 \mid s_1]$$

$$= \int_{-\infty}^{+\infty} \underbrace{P[\bar{r} \in Z_1 \mid s_1, r_1 = \nu]}_{①}\; \underbrace{p[r_1 = \nu \mid s_1]}_{②}\, d\nu$$

Proof:

$$P(\bar{r} \in Z_1 \mid s_1) = \frac{P(\bar{r} \in Z_1, s_1)}{P(s_1)}$$

$$= \frac{1}{\cancel{P(s_1)}} \int_{-\infty}^{+\infty} P(\bar{r} \in Z_1, s_1 \mid r_1 = \nu)\, \cancel{p(r_1 = \nu)}\, d\nu$$

$$\frac{P(\bar{r} \in Z_1, s_1, r_1 = \nu)}{\cancel{p(r_1 = \nu)}}$$

$$= P(\bar{r} \in Z_1 \mid s_1, r_1 = \nu) \cdot \underbrace{p(s_1, r_1 = \nu)}_{p(r_1 = \nu \mid s_1)\, \cancel{P(s_1)}}$$

① $\rightarrow P[n_2 < \nu \text{ and } n_3 < \nu]$

$\sqrt{E_s} + n_1 = \nu$

$\int_{-\infty}^{\nu} f_N(\alpha)\, d\alpha$

② $\rightarrow p(r_1 = \nu \mid s_1) = f_N(\nu - \sqrt{E_s})$

∴ For M-ary orthogonal signals,

$$P_C = P(C|s_1) = \int_{-\infty}^{+\infty} f_N(\nu - \sqrt{E_s}) \left(1 - \int_{\nu}^{\infty} f_N(\alpha)\, d\alpha\right)^{N-1} d\nu$$

$(P(\bar{r} \in Z_1 | s_1))$

$q(\nu)$

P(e) for Bi-Orthogonal Signals

Here, $M = 2N$; $s_1(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + 2\pi f_1 t)$ $(f_c + f_1)$

$s_2(t) = -s_1(t)$;

$M = 6$; $N = 3$

$s_1 \to$ was sent

$$\bar{r} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

r_1: $\sqrt{E_s} + n_1$, r_2: n_2, r_3: n_3

$$P(C|s_1) = P(\bar{r} \in Z_1 | s_1)$$

$$= \int_{-\infty}^{+\infty} P(\bar{r} \in Z_1 | s_1, r_1 = \nu > 0)\, P(r_1 = \nu > 0 | s_1)\, d\nu$$

$1 - 2q$ $\quad -\nu \quad \nu$

$$P[-\nu < n_2 < \nu, \; -\nu < n_3 < \nu]$$

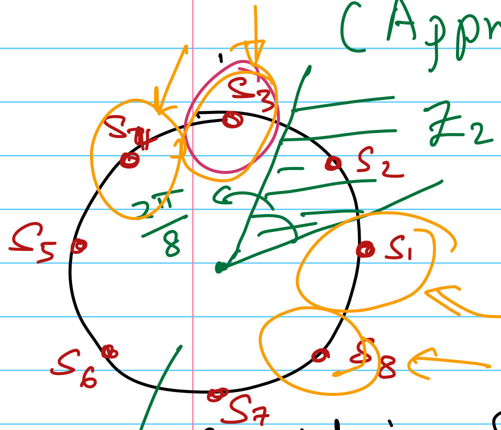
$$\int_0^{\infty} f_N(\nu - \sqrt{E_s})\, d\nu$$

$$\boxed{P(C) = P(C|s_1) = \int_0^{\infty} f_N(\nu - \sqrt{E_s}) \left(1 - 2\int_{\nu}^{\infty} f_N(\alpha)\, d\alpha\right)^{N-1} d\nu}$$

$M/2$

Union Bound on $P(e)$

(Approximation)



"Pair-wise" error events

$$P(e) = \sum_{i=1}^{M} P[\bar{r} \notin Z_i \mid s_i]\, P[s_i]$$

Considering S_1 sent

"Distance" spectrum

$\{d_{21}, d_{23}\}$

$\{d_{24}, d_{28}\}$

$$P[\bar{r} \notin Z_1 \mid S_1] = P[\underbrace{\mathcal{E}_{12} \cup \mathcal{E}_{13} \cup \mathcal{E}_{14} \dots \mathcal{E}_{1M}}_{M-1 \text{ terms}}]$$

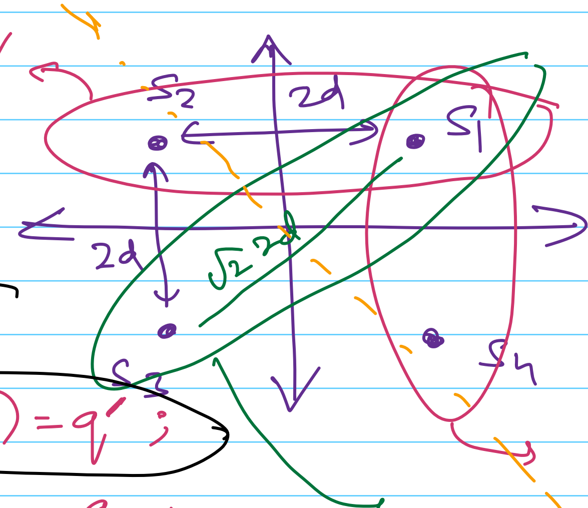
$\mathcal{E}_{1k} \rightarrow$ is the error event that "S_1" gets decoded as "S_k"

$P(A \cup B) \le P(A) + P(B) - \cancel{P(A \cap B)}$

$$\le \sum_{k=2}^{M} P[\mathcal{E}_{1k}]$$

Example

QPSK



Recall:

$P(c) = (1-q)^2$

$P(e) = 1 - P(c)$

$= 2q - q^2$

$q \approx \frac{d}{N_0}$

$q' \approx \frac{\sqrt{2}d}{N_0}$

$P(\mathcal{E}_{13}) = q'$;

$P(\mathcal{E}_{14}) = q$;

$P(\mathcal{E}_{12}) = q$;

Union Bound Argument

$$P(e) < q + q + q' = \underbrace{2q + q'}_{P_{UB}(e)}$$

$$P(e) < P_{UB}(e)$$

"Intelligent" union Bound where we refer only the "nearest neighbour" symbols when computing the union bound

$$P_{IUB}(e) \longrightarrow 2q > 2q - q^2$$