Department of Electrical Engineering, IIT Madras

EE3005: Communication Systems

Tutorial #4

1. From the enclosed scanned problem sheets below(<u>*Courtesy*</u>: taken from the book "Probability, RVs, and Stochastic Processes", by A. Papoulis, 2^{nd}Ed., Chapter 3, pp-60-61,) do the following problems:

Problems 3.1 to **3.6, 3.7*, 3.9** to **3.11, 3.13*,** and **3.14***.

***** *these questions have a higher order of difficulty*

3-1 A pair of fair dice is rolled 10 times. Find the probability that "seven" will show at least once.

Answer: $1 - (5/6)^{10}$.

3-2 A coin with $P\{h\} = p = 1 - q$ is tossed n times. Show that the probability that the number of heads is even equals $0.5[1 + (q - p)^n]$.

3-3 A fair coin is tossed 900 times. Find the probability that the number of heads is between 420 and 465.

Answer: $\mathbb{G}(2) + \mathbb{G}(1) - 1 \simeq 0.819$.

3-4 A fair coin is tossed n times. Find n such that the probability that the number of heads is between $0.45n$ and $0.52n$ is at least 0.9.

Answer: $\mathbb{G}(0.04\sqrt{n}) + \mathbb{G}(0.02\sqrt{n}) \geq 1.9$, hence, $n > 4556$.

3-5 If $P(\mathscr{A}) = 0.6$ and k is the number of successes of $\mathscr{A}$ in n trials (*a*) show that $P\{550 \leq k \leq 650\} = 0.999$, and (*b*) find n such that $P\{0.59n \leq k \leq 0.61n\} = 0.95$.

3-6 A system has 1000 components. The probability that a specific component will fail in the interval (a, b) equals $e^{-a/T} - e^{-b/T}$. Find the probability that in the interval $(0, T/4)$, no more than 100 components will fail.

3-7 A coin is tossed an infinite number of times. Show that the probability that k heads are observed at the nth tossing but not earlier equals $\binom{n-1}{k-1} p^k q^{n-k}$

3-8 Show that

$$\frac{1}{x}\left(1-\frac{1}{x^2}\right)\mathrm{g}(x) < 1-\mathrm{G}(x) < \frac{1}{x}\,\mathrm{g}(x)$$

Hint: Prove the following inequalities and integrate from x to ∞

$$-\frac{d}{dx}\left(\frac{1}{x}e^{-x^2/2}\right) > e^{-x^2/2} \qquad -\frac{d}{dx}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2}\right] < e^{-x^2/2}$$

3-9 Suppose that in n trials, the probability that an event $\mathscr{A}$ occurs at least once equals P_1. Show that, if $P(\mathscr{A}) = p$ and $pn \ll 1$, then $P_1 \simeq np$.

3-10 The probability that a driver will have an accident in 1 month equals 0.02. Find the probability that in 100 months he will have 3 accidents.

Answer: About $4e^{-2}/3$.

3-11 A fair die is rolled five times. Find the probability that *one* shows twice, *three* shows twice, and *six* shows once.

3-12 Show that (3-27) is a special case of (3-39) obtained with $r = 2$, $k_1 = k$, $k_2 = n - k$, $p_1 = p$, $p_2 = 1 - p$.

3-13 Players X and Y roll dice alternately starting with X. The player that rolls *eleven* wins. Show that the probability p that X wins equals 18/35.

Outline: Show that

$$P(\mathscr{A}) = P(\mathscr{A}\mid\mathscr{M})P(\mathscr{M}) + P(\mathscr{A}\mid\bar{\mathscr{M}})P(\bar{\mathscr{M}})$$

Set $\mathscr{A} = \{X \text{ wins}\}$, $\mathscr{M} = \{$*eleven* shows at first try$\}$. Note that $P(\mathscr{A}) = p$, $P(\mathscr{A}\mid\mathscr{M}) = 1$, $P(\mathscr{M}) = 2/36$, $P(\mathscr{A}\mid\bar{\mathscr{M}}) = 1 - p$.

3-14 We place at random n particles in $m > n$ boxes. Find the probability p that the particles will be found in n preselected boxes (one in each box). Consider the following cases: (*a*) M–B (Maxwell–Boltzmann)—the particles are distinct; all alternatives are possible, (*b*) B–E (Bose–Einstein)—the particles cannot be distinguished; all alternatives are possible, (*c*) F–D (Fermi–Dirac)—The particles cannot be distinguished; at most one particle is allowed in a box.

Answer:

	M–B	B–E	F–D
$p =$	$\dfrac{n!}{m^n}$	$\dfrac{n!(m-1)!}{(m+n-1)!}$	$\dfrac{n!(m-m)!}{m!}$

Outline: (*a*) The number N of all alternatives equals m^n. The number $N_{\mathscr{A}}$ of favorable alternatives equals the $n!$ permutations of the particles in the preselected boxes. (*b*) Place the $m - 1$ walls separating the boxes in line ending with the n particles. This corresponds to one alternative where all particles are in the last box. All other possibilities are obtained by a permutation of the $n + m - 1$ objects consisting of the $m - 1$ walls and the n particles. All the $(m - 1)!$ permutations of the walls and the $n!$ permutations of the particles count as one alternative. Hence $N = (m + n - 1)!/(m - 1)!\,n!$ and $N_{\mathscr{A}} = 1$. (*c*) Since the particles are not distinguishable, N equals the number of ways of selecting n out of m objects: $N = \binom{m}{n}$ and $N_{\mathscr{A}} = 1$.

2. Let X and Y be independent, identically distributed (i.i.d) RVs with a uniform PDF between -1 and +1 (i.e., U(-1,+1)). Given Z=2Y+1, find the probability P(Z>X).

3. Consider an RV X with finite support PDF given by $f_X(x) = e^{-x}\big(U(x) - U(x-3)\big) + \alpha \cdot \delta(x-1)$ where $U(x)$ is the unit step function and $\delta(x)$ is the Dirac delta function. For what value of $\alpha \geq 0$ will this be a valid PDF? Make a rough plot of $f_X(x)$.

4. A two-sided exponential PDF is given by $f_X(x) = \gamma e^{-|x|}$. For what value of $\gamma \geq 0$ will this be a valid PDF? Now, if we use this in turn to define a PDF with finite support given by the expression $f_X(x) = \gamma e^{-|x|}\big(U(x+2) - U(x-2)\big) + \beta \cdot \delta(x)$, define β appropriately so that this becomes a valid PDF.

5. Let X be an RV with a two-sided exponential (infinite support) as in Pbm.4. Give a new RV Y=2X+3:
(a) Make a rough plot of $f_Y(y)$.
(b) What is P(Y<2) ?

6. The wattage across a R_0=1000 ohm resistor is given by $W = \frac{V^2}{R_0}$ where the voltage V is a RV which is uniform between 5V and 10V; i.e., U(5V,10V). Find and plot the PDF $f_W(w)$.

7. The conductance Y is related to the resistance X as Y=1/X. If the PDF of X is U(90ohm, 110ohm), find the PDF of the conductance Y.

8. If the RV $Y = a\,Sin(X+b),\ a > 0$, and b is a constant, then show that

$$f_Y(y) = \frac{1}{\sqrt{a^2 - y^2}} \sum_{n=-\infty}^{+\infty} f_X(x_n)\,, |y| < a, \text{where } x_n = Sin^{-1}\left(\frac{y}{a}\right) - b, \text{for } n = \cdots, -1, 0, +1, \ldots.$$

and for the case where X is uniform RV with U(-π,+π), show that only two roots exist and get the expression for the corresponding $f_Y(y)$.

9. Given that RV X has a uniform PDF U(-1,+1), find and plot the PDF of Y for each of the below transformations:
(a) Y = X U(x)
(b) Y = *sgn*(X) (where *sgn* is the "signum" function)

10. Given Y = e^X, find the PDF of Y for each of the choices below for the PDF of X:
(a) $f_X(x)$ is U(0,1)
(b) $f_X(x) = e^{-x}U(x)$

11. In a digital communication system, the samples after the matched filter are given by Y = X + N where $f_X(x) = 0.5\delta(x-A) + 0.5\delta(x+A)$ and $f_N(n) = 0.5e^{-|n|}$. Here the signal RV X and the noise RV N are statistically independent. Then, answer the following:
(a) For A=1, make a rough sketch of $f_Y(y)$.
(b) If the probability of bit error P_B is defined by the expression P(Y<0 | +A sent), what is P_B for A=1?
(c) For what value of A will P_B be equal to 0.0001? Explain.