EE 3005 – RP #3

$X(t)$ & $Y(t)$ $\longrightarrow$ Cross Correlation & Cross SD

Auto

(A) If $X(t)$ & $Y(t)$ are RPs

$$\rightarrow \quad R_{X,Y}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

$$R_{Y,X}(t_1, t_2) = E[Y(t_1)\, X(t_2)]$$

(*) If $X(t)$ & $Y(t)$ are jointly WSS then

$$R_{X,Y}(t_1, t_2) = R_{X,Y}(t_1 - t_2) = R_{XY}(\tau)$$

t_2 t_1

(*) This is not an even fn: in general

i.e., $R_{XY}(\tau) \neq R_{XY}(-\tau)$ $\leftarrow$

however $\quad R_{XY}(\tau) = R_{YX}(-\tau)$ ✓ $\leftarrow$

(*) $R_{XY}(0)$ is not necessarily the max. value.

$\longrightarrow R_X(0) \geq R_X(k)$

(*) Two RPs are mutually orthogonal

if $\quad R_{XY}(t_1, t_2) = 0 \quad \forall\, t_1, t_2$

(*) Two RPs are uncorrelated ✓

Cross Covariance $\quad C_{X,Y}(t_1, t_2) = R_{X,Y}(t_1, t_2) - m_X(t_1)\, m_Y(t_2)$

$= 0$

$\forall\, t_1, t_2$

$$\rightarrow \quad E[X(t_1)\, Y(t_2)] = E[X(t_1)]\, E[Y(t_2)]$$

(*) Cross Spectral Density.

$$R_{xy}(\tau) \rightleftharpoons S_{xy}(f)$$

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Real $x(t)$ & $y(t)$

$R_{xy}(\tau)$ | $S_{xy}(f)$

$\rightarrow$ real $\longrightarrow$ even

not even in general. $\longrightarrow$ not real in general

(*) $S_{xy}(f) = S^*_{xy}(-f)$ (?) Complex fn.

(*) $S_{xy}(f) = S_{yx}(-f)$ $\longleftarrow$

Applications

(A)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Stable LTI

WSS

$R_x(\tau)$ ✓ $\quad r_h(\tau)$ ✓ $\quad R_y(\tau)$

$$\boxed{S_y(f) = H(f)\,H^*(f)\,S_x(f)} \rightarrow (1)$$

o/p PSD

$$= |H(f)|^2\,S_x(f)$$

$$\rightarrow S_x(f) \longrightarrow \boxed{H(f)} \xrightarrow{\overbrace{S_x(f)\cdot H(f)}^{\checkmark\ S_{yx}(f)}} \boxed{H^*(f)} \longrightarrow S_x(f)\cdot|H(f)|^2$$

$$\rightarrow R_x(\tau) \qquad h(t) \qquad \checkmark\ R_{yx}(\tau) \qquad \underline{\underline{h^*(-t)}} \qquad R_y(\tau)$$

Matched filter

Exercise: Prove that

(a) $R_{xy}(\tau) = R_x(\tau) * h(-t)$

(b) $R_{yx}(\tau) = R_x(\tau) * h(t)$

B Let $\boxed{Z(t) = X(t) + Y(t)}$ where $X(t)$ & $Y(t)$ are individually & jointly WSS and $m_x = 0 = m_y$

RTF: $S_Z(f)$

$$R_Z(t+\tau, t) = E[Z(t+\tau)\,Z(t)]$$

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

WSS

$$\boxed{S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f)}$$

(*) $X(t)$ & $Y(t)$ are uncorrelated (jointly WSS)

$$R_{XY}(\tau) = 0 = R_{YX}(\tau)$$

$$\boxed{S_Z(f) = S_X(f) + S_Y(f)}$$

(*)

$$X(t) \longrightarrow \boxed{h_1(t)} \longrightarrow V(t)$$

$$Y(t) \longrightarrow \boxed{h_2(t)} \longrightarrow Z(t)$$

Given $X(t)$ & $Y(t)$ are jointly WSS, we can show that

$$\boxed{S_{VZ}(f) = H_1(f) \cdot H_2^*(f)\, S_{XY}(f)}$$ 2

(*)

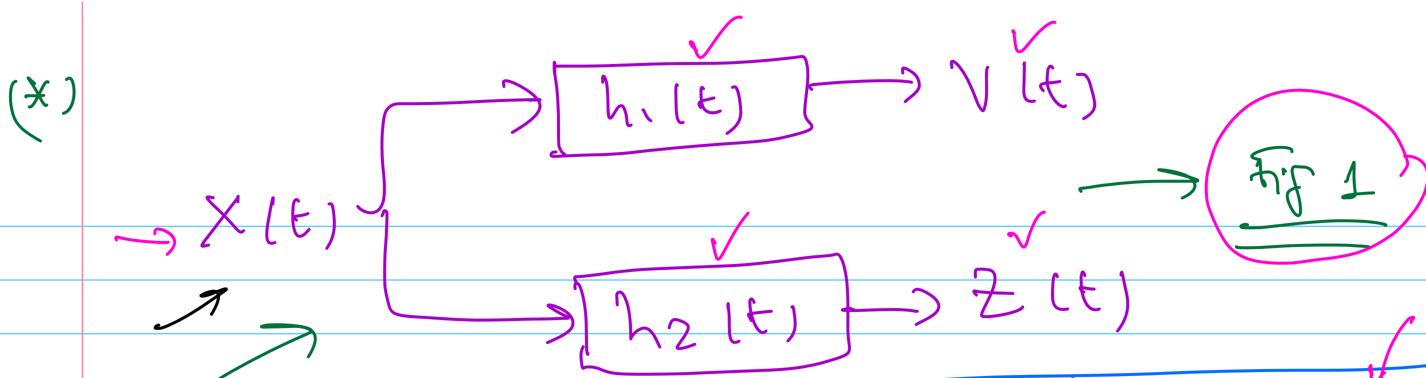


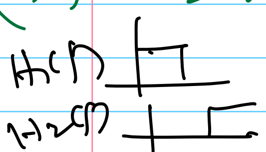
Fig 1

$$\boxed{S_{VZ}(f) = H_1(f)\, H_2^*(f)\, S_X(f)}$$

(*) In the above (Fig 1) if $h_1(t) = h_2(t) = h(t)$

$S_V(f)$ or $S_Z(f)$ $\quad S_{VZ}(f) = |H(f)|^2 S_X(f)$

(*) In Fig 1 if $H_1(f) \cdot H_2^*(f) = 0 \quad \forall f$



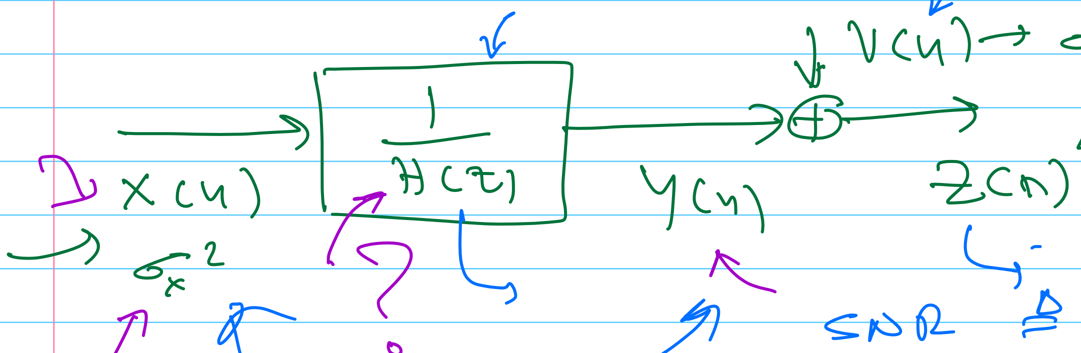
i.e., $h_1(t)$ & $h_2(t)$ have non-overlapping spectrum

$$R_{VZ}(\tau) = R_{ZV}(\tau) = 0$$

V(t) & Z(t) are orthogonal

$C_{VZ}(\tau) = 0 \implies$ uncorrelated

Auto Regressive Process → WSS



$$SNR \triangleq \frac{\sigma_x^2 (\cdot) 1}{\sigma_v^2}$$

$$\sum_{i=1}^{3} a_i y(n-i) + x(n) \sim \frac{1}{\sigma_v^2}$$

$$y(n) = a_1 y(n-1) + x(n) \rightarrow ③$$

$z(n) = y(n) + v(n)$

Yule-Walker Eqn.

$y(n-2) \leftarrow$

$y(n-3) \leftarrow$

③ × $y(n-1)$ $\triangle$ take $E(\cdot)$

$$\underbrace{E[y(n)y(n-1)]} = a_1 E[y(n-1)y(n-1)] + \underbrace{E[y(n-1)x(n)]} \rightarrow 0$$

$y(n)$

$$\boxed{R_y(1) = a_1 R_y(0)}$$

$$[\ ] = [R][a_1\ a_2\ a_3]^T$$

3×1 3×3

Toeplitz

a_1

$$\hat{a}_1 = \frac{R_y(1)}{R_y(0)} \rightarrow \frac{\hat{R}_y(1)}{\hat{R}_y(0)}$$

N

$$\hat{R}_y(k) = \sum_{n=k}^{N} x(n)\,x(n-k)$$

Time-averaged acf.

$\left(\frac{1}{N-k}\right)$

$$\text{error}\left(\hat{a}_1 - a_1\right)^2$$

N_0

$N \times N$

$R \longrightarrow R^{-1}$ N^3

Toeplitz

Levinson–Durbin Algorithm $\sim \underline{\underline{N^2}}$