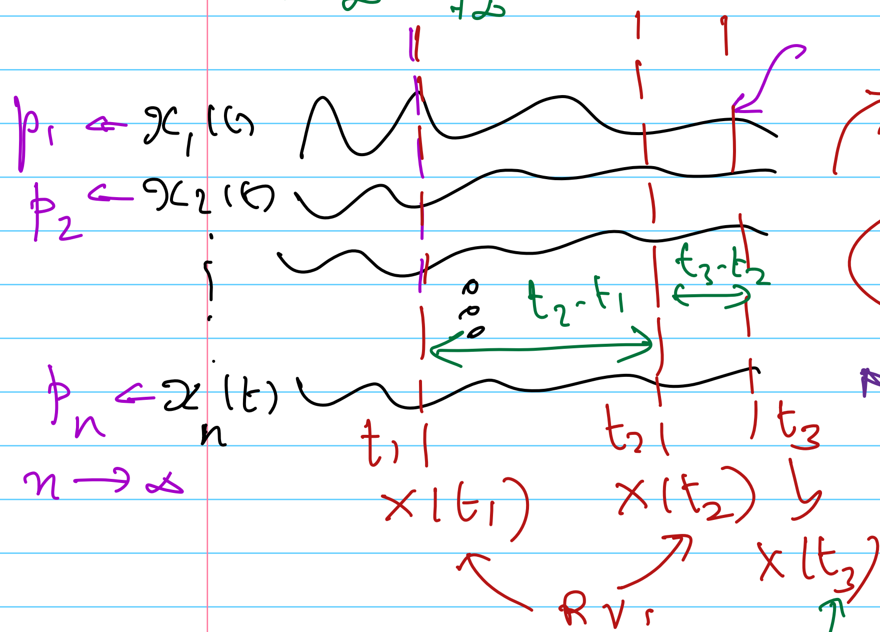
Random Processes

April 13, 2025

→ stationary ?



$f_{\vec{X}(t)}(\vec{x})$ $\quad \vec{x} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \end{bmatrix}$

joint pdf

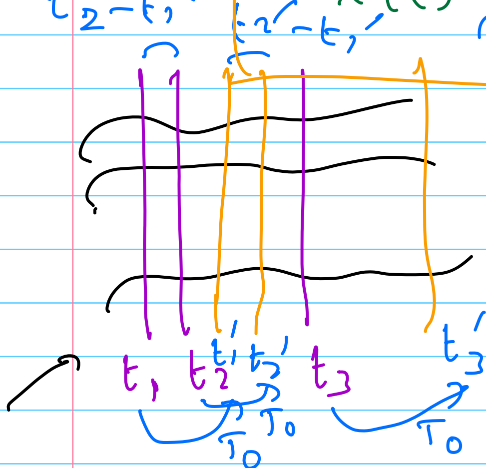
t_1, t_2, t_3 → t_l $\quad l \to \infty$

1. Strict Sense Stationarity (SSS)

RP is SSS only if $f_{\vec{X}(t)}(\vec{x})$ is invariant to translation in time

$$f_{\vec{X}(t)}(\vec{x}) = f_{\vec{X}(t+T_0)}(\vec{x}) \quad \forall T_0$$

$t_2 - t_1$ $\quad t_2' - t_1'$

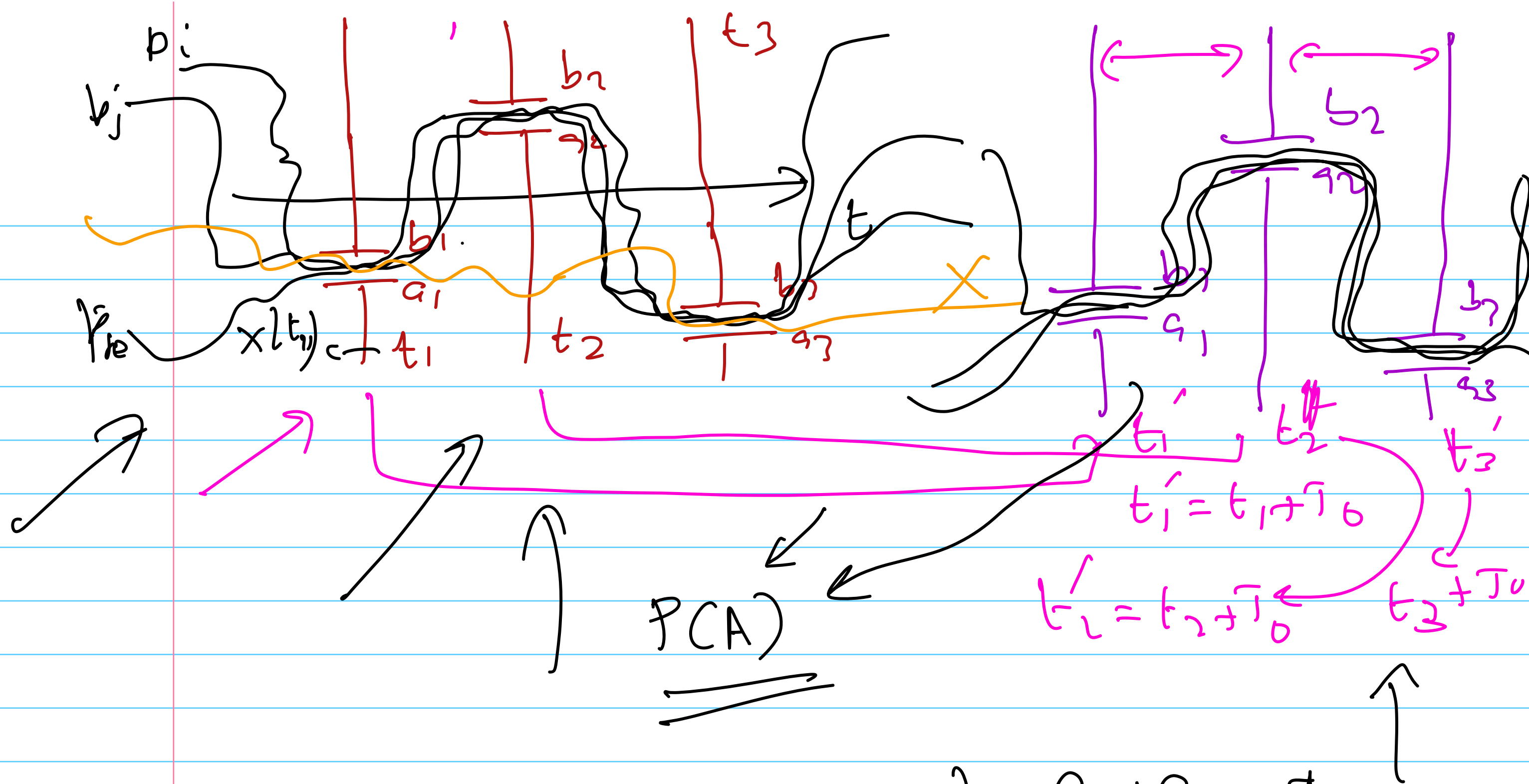


then

$$P(A) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f_{\vec{X}(t)}(\vec{x})\, d\vec{x}$$

eg: Let A be an event

$$A = \left\{ s : a_1 < X(t_1) \le b_1,\ a_2 < X(t_2) \le b_2,\ a_3 \le X(t_3) \le b_3 \right\}$$



→ sample functions could be different

In most practical cases, SSS is impossible to know or estimate !!

Wide Sense Stationarity (WSS)

(a) Mean of $X(t)$ $\quad t = t_i$

$$m_X(t_i) \triangleq E[X(t_i)] = \int x \, f_{X_i}(x)\, dx$$

(b) Auto-Correlation function (acf)

$$R_X(t_i, t_j) \triangleq E[X(t_i)\, X(t_j)] = \iint x\, y\, f_{X_i, X_j}(x, y)\, dx\, dy$$

(c) Auto Covariance function

$$C_X(t_i, t_j) \triangleq E\left[\left(X(t_i) - m_X(t_i)\right) \cdot \left(X(t_j) - m_X(t_j)\right) \right]$$

$= R_x(t_i, t_j) - m_x(t_i) m_x(t_j)$

$x(t)$ is **WSS** $\Updownarrow$ **iff**

(a) $m_x(t) = m_x \quad \forall t$

(b) $R_x(t_i, t_j) = R_x(t_i + T_0, t_j + T_0)$ $\quad T_0 = -t_j$

$= R_x(\underbrace{t_i - t_j}_{\tau}, 0)$

$$\boxed{R_x(\tau) = E[x(t+\tau)x(t)] = E[x(t)x(t-\tau)]} = \boxed{R_x(\tau)} \quad \tau = t_i - t_j$$

"t" is not relevant

Properties of $R_x(\tau)$ ← **WSS**

Real r.p.

(*) $R_x(\tau) = R_x(-\tau) \longrightarrow$ even fn:

$E[x^*(t)x(t-\tau)]$

(*) Mean square value $E[x^2(t)] = R_x(0)$

(*) $|R_x(\tau)| \leq R_x(0)$

↳ Proof: $E\left[\left(x(t) \pm x(t-\tau)\right)^2\right] \geq 0$

$\underbrace{E(\cdot)}_{R_x(0)} + \underbrace{E(\cdot)}_{R_x(0)} \pm 2\underbrace{E[x(t)x(t-\tau)]}_{R_x(\tau)}$

(*) Variance of the RP

$E[(x(t) - m_x(t))^2] = C_x(0) = R_x(0) - m_x^2$

Auto covariance → **WSS** $\boxed{C_x(t, t-\tau) = R_x(\tau) - m_x^2}$ $\quad C_x(\tau)$

$\boxed{C_x(\tau) = R_x(\tau)}$ $= E[(x(t) - m_x)(x(t-\tau) - m_x)]$

Cyclostationary RP (periodically stationary)

→ if a RP is stationary $\begin{cases} \to \text{WSS} \\ \to \text{SSS} \end{cases}$ for every <u>integral</u> multiple of a constant, say T

SS Cyclostationary

$$f_{\vec{X}(t)}(x_1,\dots,x_n) = f_{\vec{X}(t+mT)}(x_1,\dots,x_n)$$

"T"

$m = 0, \pm 1, \pm 2, \dots$

WS Cyclostat

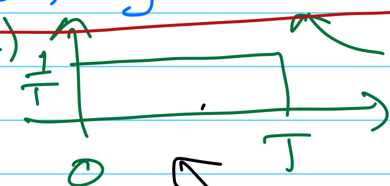
m_x:

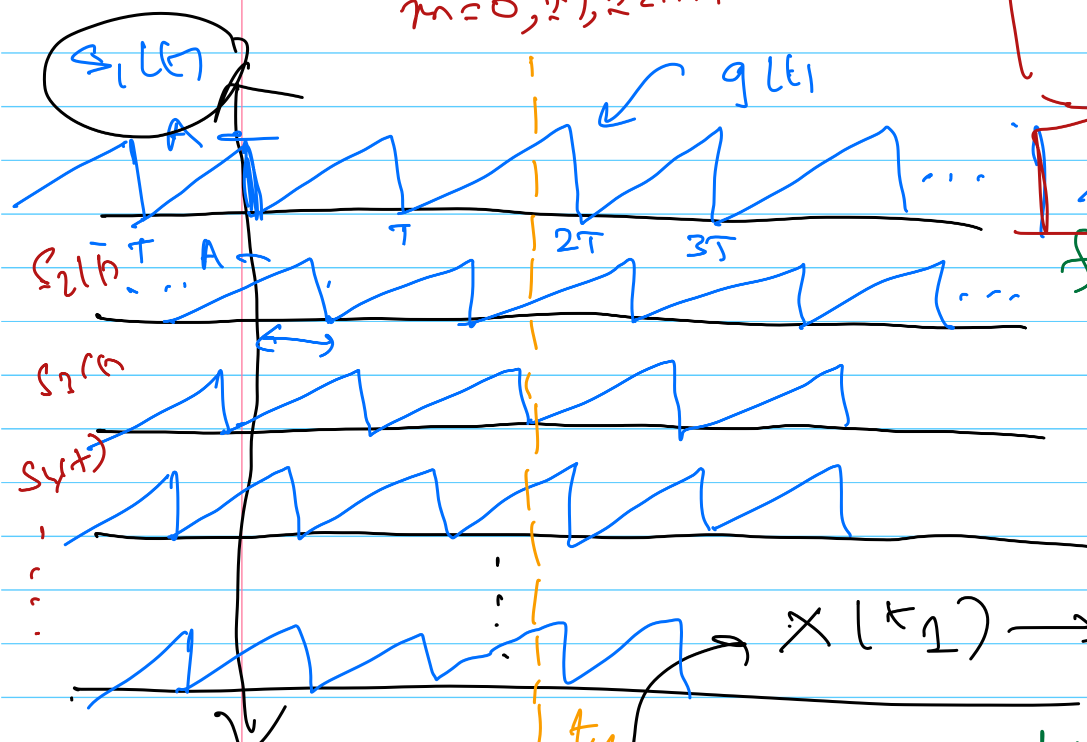
$$E[X(t)] = E[X(t+mT)]$$

$$E[X(t_1)X(t_2)] = E[X(t_1+mT)X(t_2+mT)]$$

$$\boxed{R_X(\tau) = R_X(mT+\tau)}$$

$$\boxed{X(t) = g(t+\Theta)}$$

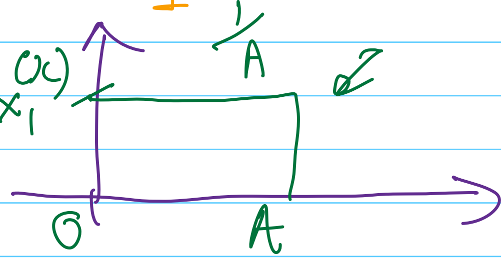




t_i

√ SSS

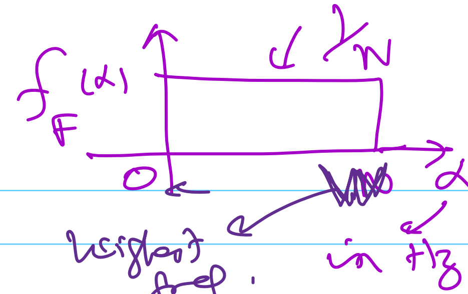
√ WSS

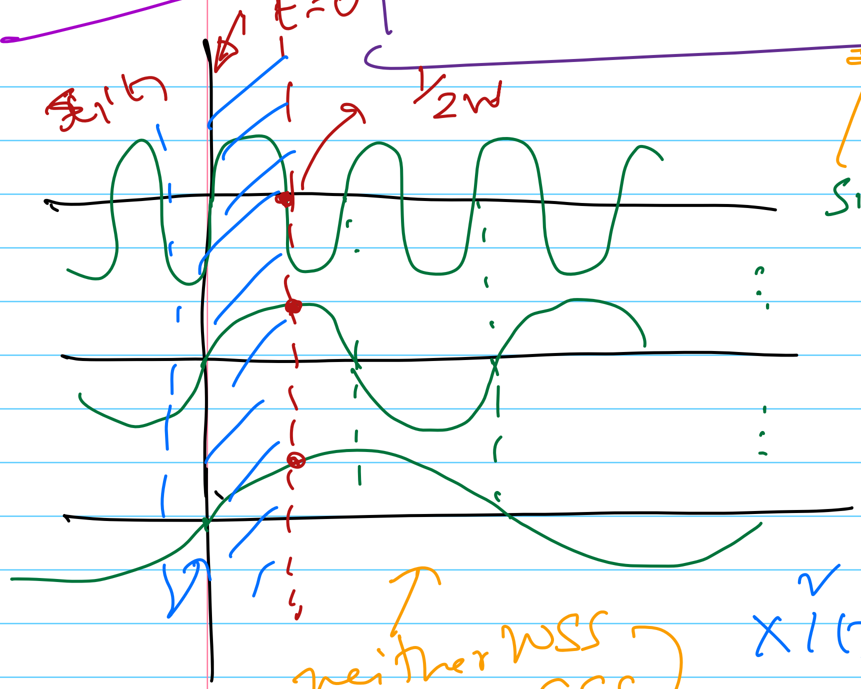


"one" dimensional pdf. $\forall t$

$$m_x(t) = m_x = A/2 \quad \forall t$$

Example : $\boxed{X(t) = \sin 2\pi F t}$





$\sin 2\pi W t$

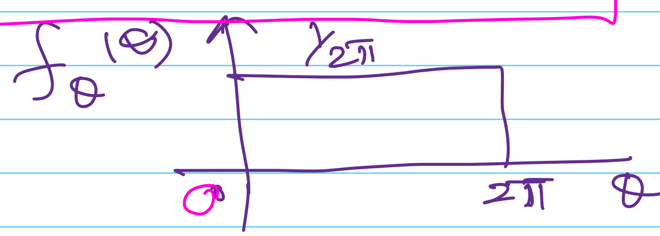
$\sin 2\pi \left(\frac{W}{2}\right) t$

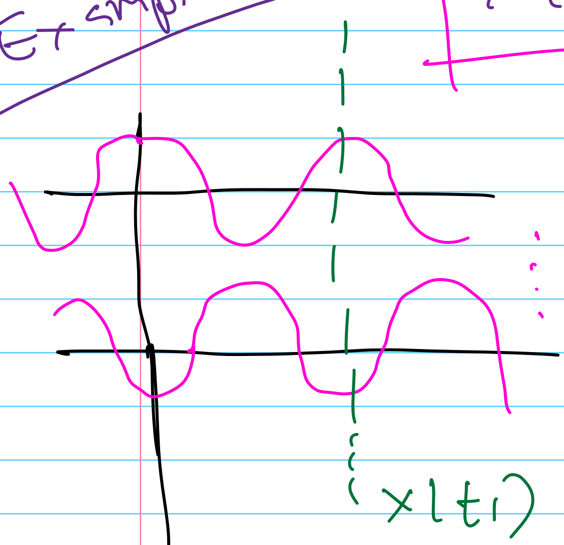
$\sin 2\pi \left(\frac{W}{4}\right) t$

$$x(t_1) = \begin{cases} 0, & t = 0 \\ (+)ve, & 0 < t < \frac{1}{2W} \\ (-)ve, & \frac{-1}{2W} < t < 0 \end{cases}$$

Example :

$\boxed{X(t) = a \cos(\omega_c t + \theta)}$





$x(t_1) \longrightarrow f_{x_1}(t)$? ——