A Brief Note on Maxwell's Equations

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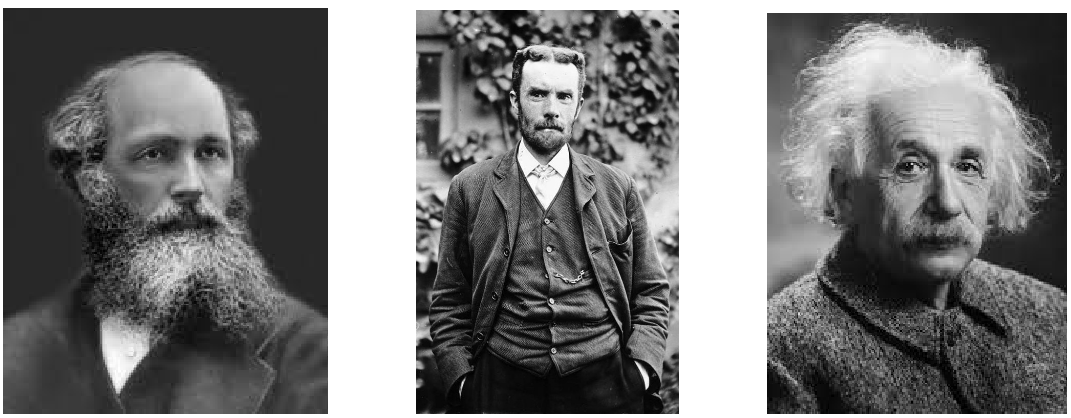
Abstract: The combined mathematical representation of Gauss' laws of electricity and magnetism, Ampere's circuital law, and Faraday's law is known as "Maxwell's Equations". It is one of the important milestones in the human history and was championed by the great Scottish Scientist James Clerk Maxwell in 19th Century (1860 -1871). In this note, we will quickly discuss about the important terms used in Maxwell's Equations, their role in understanding electromagnetism and its versatile applications.

1. Introduction



(a) (b) (c)

Fig. 1: (a) James Clerk Maxwell (13th June 1831 - 5th November 1879); (b) Oliver Heaviside (18th May 1850 -3rd February 1925); and (c) Albert Einstein (14th March 1879 - 18th April 1955).

As we learn from any standard text book on "Electromagnetic Fields", the Maxwell's Equations in differential form are:

$$\vec{\nabla}.\vec{D} = \rho_v; \qquad \vec{D} = \varepsilon\vec{E} \tag{1a}$$

$$\vec{\nabla}.\vec{B} = 0; \qquad \vec{B} = \mu\vec{H} \tag{1b}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{1c}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_c + \vec{J}_d; \qquad \vec{J}_c = \sigma\vec{E} \qquad \vec{J}_d = \frac{\partial\vec{D}}{\partial t} \tag{1d}$$

where $\vec{\nabla} \equiv \vec{a}_x\frac{\partial}{\partial x} + \vec{a}_y\frac{\partial}{\partial y} + \vec{a}_z\frac{\partial}{\partial z}$ (in Cartesian coordinates, x,y,z); $\vec{\nabla} \equiv \vec{a}_\rho\frac{\partial}{\partial\rho} + \vec{a}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{a}_z\frac{\partial}{\partial z}$ (in Cylindrical coordinates, ρ,ϕ,z); and $\vec{\nabla} \equiv \vec{a}_r\frac{\partial}{\partial r} + \vec{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$ (in Spherical coordinates, r,θ,ϕ); ρ_v free charge density; ε,μ and σ are permittivity, permeability, and conductivity of the medium, respectively; $\vec{E}$ and $\vec{H}$ are the electric and magnetic field strengths, respectively; $\vec{D}$ is known as displacement vector or electric flux density and $\vec{B}$ is the magnetic flux density; $\vec{J}_c$

is the conduction current density, whereas $\vec{J}_d$ is the "displacement current density". All the four equations were known before Maxwell as "Laws": (1a) Gauss' Law for electrostatics, (1b) Gauss' Law for magnetostatics, (1c) Faraday's Law, and (1d) Ampere's Circuital Law, but without the second term representing $\vec{J}_d$. This term was added by Maxwell and subsequently proved that "light is an electromagnetic wave".

However, to justify the correction term ($\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$) in Ampere's circuital law Maxwell formulated twenty differential equations (vector calculus was not know in his time). It was Oliver Heaviside who in fact reduced the Maxwell's twenty equations into four by using vector calculus (during his intensive studies on transmission lines). Riding on Maxwell's Equations, Albert Einstein discovered his "Theory of Relativity" and thus he also popularized Maxwell's Equations.

2. Historical Landmarks

The important landmarks related to Maxwell's Equations are:

1. 1785: Coulomb's Law is published
   Like charges repel and unlike charges attract with a force defined by:
   $$F = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2} \tag{2}$$
   where q_1 and q_2 are two point charges (+ve or -ve) separated by a distance r. The force on an arbitrary charge placed at $\vec{r}$ can also be expressed in terms of electric field $\vec{E}(r)$ due to another point charge Q at $\vec{r_0}$:
   $$\vec{F} = q\vec{E} = q\frac{Q(r_0)}{4\pi\varepsilon r^2}\frac{\vec{r}-\vec{r_0}}{|\vec{r}-\vec{r_0}|}; \qquad r = |\vec{r}-\vec{r_0}| \tag{3}$$
   Thus electric field $\vec{E}$ ($Volt/m$) at $\vec{r}$ due to point charge Q at $\vec{r_0}$ is given by $\vec{E}(r) = \frac{Q(r_0)}{4\pi\varepsilon r^2}\frac{\vec{r}-\vec{r_0}}{|\vec{r}-\vec{r_0}|}$; ε is the permittivity of the medium ($Farad/m$).

2. 1812: Poisson's Law is published
   Poisson' law is just the modified Laplace's equation $\nabla^2\phi(x,y,z) = 0$, for electrostatic potential ϕ:
   $$\nabla^2\phi(x,y,z) = -\frac{\rho_v(x,y,z)}{\varepsilon} \tag{4}$$
   where $\rho_v(x,y,z)$ is the free volume charge density at $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$

3. 1813: Gauss' Divergence Theorem is discovered
   The outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface.
   Which again can be re-stated intuitively:
   The sum of all sources minus the sum of all sinks gives the net flow out of a region.
   Mathematically, it can be represented for an arbitrary vector field $\vec{A}(x,y,z)$:
   $$\int_V \vec{\nabla}\cdot\vec{A}dv = \oint_s (\vec{A}\cdot\hat{n})ds \tag{5}$$
   The above theorem can be extended to electrostatic displacement vector or electrical flux density $\vec{D}$ and free volume charge density ρ_v (can exist either as "Positive" or "Negative" charge):
   $$\int_V \vec{\nabla}\cdot\vec{D}dv = \oint_s (\vec{D}\cdot\hat{n})ds = \int_v \rho_v dv \qquad \Rightarrow \vec{\nabla}\cdot\vec{D} = \rho_v \tag{6}$$
   Similarly, it can be written for magnetic flux density $\vec{B}$ (no free magnetic mono-pole exists in nature or not yet discovered):
   $$\vec{\nabla}\cdot\vec{B} = 0 \tag{7}$$
   In the above equations, electric flux density $\vec{D}$ and magnetic flux density $\vec{B}$ are defined by:
   $$\vec{D} = \varepsilon\vec{E}; \qquad \vec{B} = \mu\vec{H} \tag{8}$$
   where, μ is the permeability of the medium ($Henry/m$) and $\vec{H}$ is the magnetic field strength (Amp/m); $1\ Wb = 1\ Henry \times 1\ Amp$, $1\ Tesla = 1\ Wb/m^2$.

4. 1820: H. C. Orsted discovers that an electric current creates a magnetic field
   Oersted's experiment proved that *Every electric current has a magnetic field surrounding it.*

5. 1820: Andre'-Marie Ampere's work founds electrodynamics; Biot-Savart Law is discovered
   The magnetic field at any position $\vec{r}$ due to current element $i\vec{dl}$ at a position r_0 is defined by:

$$d\vec{B} = \frac{\mu}{4\pi}\frac{id\vec{l} \times \vec{a}_r}{r^2} \tag{9}$$

   Where $r = |\vec{r} - \vec{r}_0|$ and $\vec{a}_r = \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|}$

6. 1826: Ampere's Circuital Law is published
   It states the relationship of *Integrated magnetic field around closed loop to the electric current passing through the loop.* Mathematically,

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} \tag{10}$$

   Which can be further re-stated using Stoke's theorem $\oint \vec{A} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{A}).d\vec{S}$:

$$\int_s (\vec{\nabla} \times \vec{H}).d\vec{S} = \int_s \vec{J} \cdot d\vec{S} \qquad \Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} \tag{11}$$

7. 1831: Faraday's Law is published
   The law states that *When the magnetic flux linking to a circuit changes, an electromotive force is induced in the circuit proportional to the rate of change of the flux linkage.*
   Mathematically, it can be written using Stoke's theorem:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S} \qquad \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

8. 1856: James Clerk Maxwell publishes "On Faraday's lines of force"

9. 1861: Maxwell publishes "On physical lines of force"

10. 1865: Maxwell publishes "A dynamical theory of the electromagnetic field"

11. 1873: Maxwell publishes *Treatise on Electricity and Magnetism*
    The important contribution:

$$\vec{J}_{tot} = \vec{J}_c + \vec{J}_d = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t} \tag{13}$$

    This eventually leads to a correction of Ampere's Circuital Law (in compact form after Heaviside):

$$\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \tag{14}$$

    Without the correction term known as displacement current density $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$, it would not be possible to explain time varying current flow through a capacitor! The above equation (with correction term), we can prove that the conduction current density $\vec{J}_c$ in the conductor converts into an equal amount displacement current density $\vec{J}_d$ inside the capacitor due to time varying electric field across it:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}_c + \vec{\nabla} \cdot \vec{J}_d \qquad \Rightarrow \vec{\nabla} \cdot \vec{J}_c = -\vec{\nabla} \cdot \vec{J}_d \tag{15}$$

12. 1885: Oliver Heaviside publishes a condensed version of Maxwell's equations, reducing the equation count from twenty to four well-known Maxwell's Equations.
    I remember my first look at the great treatise of Maxwell's when I was a young man, I saw that it was great, greater and greatest, with prodigious possibilities in its power. I was determined to master the book and set to work. I was very ignorant. I had no knowledge of mathematical analysis (having learned only school algebra and trigonometry which I had largely forgotten) and thus my work was laid out for me. It took me several years before I could understand as much as I possibly could. Then I set Maxwell aside and followed my own course. And I progressed much more quickly..... It will be understood that I preach the gospel according to my interpretation of Maxwell. Maxwell himself could not get an opportunity to see Maxwell's Equations!

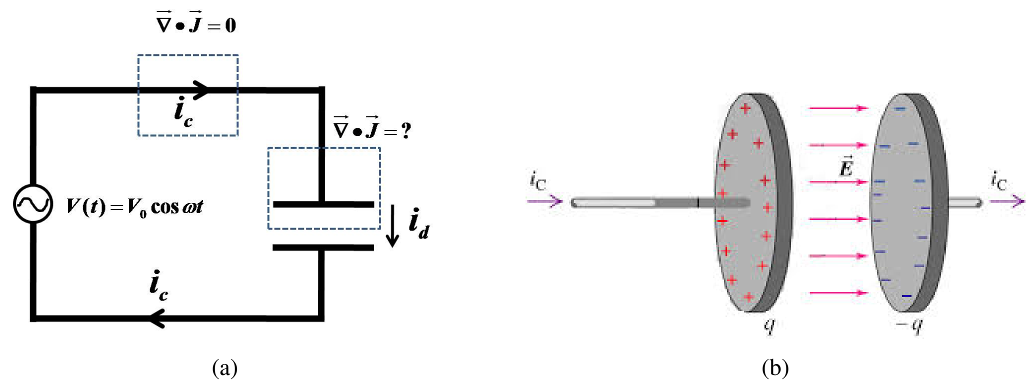


Fig. 2: (a) Illustration of conduction current and displacement current for time varying source (electric field); (b) Time varying electric field loops (perpendicular to electric field lines) can cause time varying magnetic field which eventually the source of a current (known as displacement current) inside the capacitor following Faraday's law.

13. 1888: Heinrich Hertz discovers radio waves

14. 1940: Albert Einstein popularizes the name 'Maxwell's Equations'
    Albert Einstein said: *The special theory of relativity owes its origins to Maxwell's equations of the electromagnetic field.* Einstein also said: *Since Maxwell's time, physical reality has been thought of as represented by continuous fields, and not capable of any mechanical interpretation. This change in the conception of reality is the most profound and the most fruitful that physics has experienced since the time of Newton.*

15. 1966: Kane Yee introduces finite-difference time domain (FDTD) methods to solve Maxwell's Equations. The method is widely used in integrated optical and nano-photonics device simulation/modeling.

3. Continuity Equations

Taking divergence in both sides of 4th Maxwell's Equation, we can derive:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}_c + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D}) = 0 \tag{16}$$

Now using 1st Maxwell's Equation, i.e. $\vec{\nabla} \cdot \vec{D} = \rho_v$, the above equation boils down to the necessary continuity equation or the law of charge conservation:

$$\vec{\nabla} \cdot \vec{J}_c + \frac{\partial \rho_v}{\partial t} = 0 \tag{17}$$

Now let us try to understand this equation using a thought experiment. Assume a finite amount of like charges with an initial density of ρ_0 is injected inside an infinitely extended medium characterized by its conductivity σ, permittivity ε, and permeability $\mu = \mu_0$. If all the injected charges are assumed to be localized initially around $\vec{r} = \vec{r}_0$, what will happen to those charges as a function of time? Find the charge density $\rho(t)$ at $\vec{r} = \vec{r}_0$.

We can use the continuity equation as following:

$$\vec{\nabla} \cdot \vec{J}_c + \frac{\partial \rho_v}{\partial t} = 0 \Rightarrow \frac{\sigma}{\varepsilon}\vec{\nabla} \cdot \vec{D} + \frac{\partial \rho_v}{\partial t} = 0 \Rightarrow \frac{d\rho_v}{dt} = -\frac{\sigma}{\varepsilon}\rho_v \Rightarrow \rho_v(t) = \rho_0 e^{-\frac{\sigma}{\varepsilon}t} \tag{18}$$

Can you now relate the above equation to discharging phenomena of a capacitor?

From Ohm's Law, we have:

$$R = \frac{V}{I} = \frac{-\int \vec{E}.d\vec{l}}{\sigma \int \vec{E}.d\vec{S}} \tag{19}$$

Again from the definition of capacitance, we know:

$$C = \frac{Q}{V} = \frac{\int \vec{D}.d\vec{S}}{-\int \vec{E}.d\vec{l}} = \frac{\varepsilon \int \vec{E}.d\vec{S}}{-\int \vec{E}.d\vec{l}} \tag{20}$$

Thus we get $RC = \frac{\varepsilon}{\sigma}$, which is nothing but the time constant of capacitor or relaxation time τ_r of the medium. For ideal dielectric medium $\sigma \to 0$ and hence $\tau \to \infty$. And for a good conductor $\sigma \to \infty$ and hence $\tau \to 0$, i.e. no free charge can exist inside a conductor. It also explains why electric field inside an ideal conductor is zero and no tangential component of electric field can exist in the surface of a conductor.

4. Wave Equations

Taking curl of 1st curl equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) \Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}) \tag{21}$$

Assuming charge free ($\rho_v = 0$) homogeneous dielectric medium ($\sigma = 0$), the above equation boils down to:

$$\nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \Rightarrow \nabla^2 \vec{E} - \frac{1}{v_p^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{22}$$

and similarly, we can derive another equation for magnetic field $\vec{H}$:

$$\nabla^2 \vec{H} - \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \Rightarrow \nabla^2 \vec{H} - \frac{1}{v_p^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{23}$$

These equations are known as wave equations as the solutions for both the fields are traveling waves with phase velocity given by $v_p = \frac{1}{\sqrt{\mu \varepsilon}}$. For free space the equations can be written as:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0; \qquad \nabla^2 \vec{H} - \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{24}$$

5. EM Wave Solutions

The EM wave equations are essentially derived for electric field and magnetic field vectors. One can easily decompose them into three scalar wave equations for $E_x(x,y,z,t)$, $E_y(x,y,z,t)$, and $E_z(x,y,z,t)$; and three more scalar wave equations for $H_x(x,y,z,t)$, $H_y(x,y,z,t)$, and $H_z(x,y,z,t)$. However, without any loss of generality, we can express the wave equations in 1D (say, all the field components depend only on z and t):

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0; \qquad \frac{\partial^2 \vec{H}}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{25}$$

The above equations can be further simplified into scalar equations if we assume/align the field parallel to one of the axes in Cartesian coordinate system, say $\vec{E}(x,y,z,t) \equiv E_x(z,t)\vec{a}_x$ and the corresponding wave equation for electric field can be written as:

$$\frac{\partial^2 E_x(z,t)}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2 E_x(z,t)}{\partial t^2} = 0 \tag{26}$$

The most general solution of the above partial differential equation is found to be as following:

$$E_x(z,t) = f(v_p t - z) + g(v_p t + z) \tag{27}$$

The first term is a function of $v_p t - z$, represents a traveling wave along +ve z-direction, whereas the second term is a function of $v_p t + z$, stands for another traveling wave along -ve z-direction. It must be noted that the traveling wave can exist even in free-space and source free region, with a phase velocity $v_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \times 10^8\ m/s$. Fig. 3 shows the schematic representation of an arbitrary EM signal generation and its 1D propagation in free space. Please note that signal shape won't be retained as it propagates in a dispersive medium (the electromagnetic parameters ε and μ are frequency dependent). This is because the EM signal $s(t)$ may be decomposed into its frequency components:

$$s(t) = \sum w_i \cos(\omega_i t + \delta_i) \tag{28}$$

where, w_i and δ_i are the weightage and phase-delay for i-th frequency component $\omega_i = 2\pi f_i$. Assuming the signal radiating only along $+z$ direction, we can directly write the space and time dependent signal as:

$$s(z,t) = \sum w_i cos(\omega_i t - \beta_i z + \delta_i) = \sum w_i cos[\beta_i(v_{pi} t - z) + \delta_i] \tag{29}$$

where,

$$v_{pi}(\omega_i) = \frac{\omega_i}{\beta_i} = \frac{1}{\sqrt{\mu(\omega_i)\varepsilon(\omega_i)}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \cdot \frac{1}{\sqrt{\mu_r(\omega_i)\varepsilon_r(\omega_i)}} = \frac{c}{\sqrt{\mu_r(\omega_i)\varepsilon_r(\omega_i)}} \tag{30}$$

Thus we can easily inferred that the shape of $s(z,t)$ won't be the same as $s(0,t)$ in a dispersive medium.

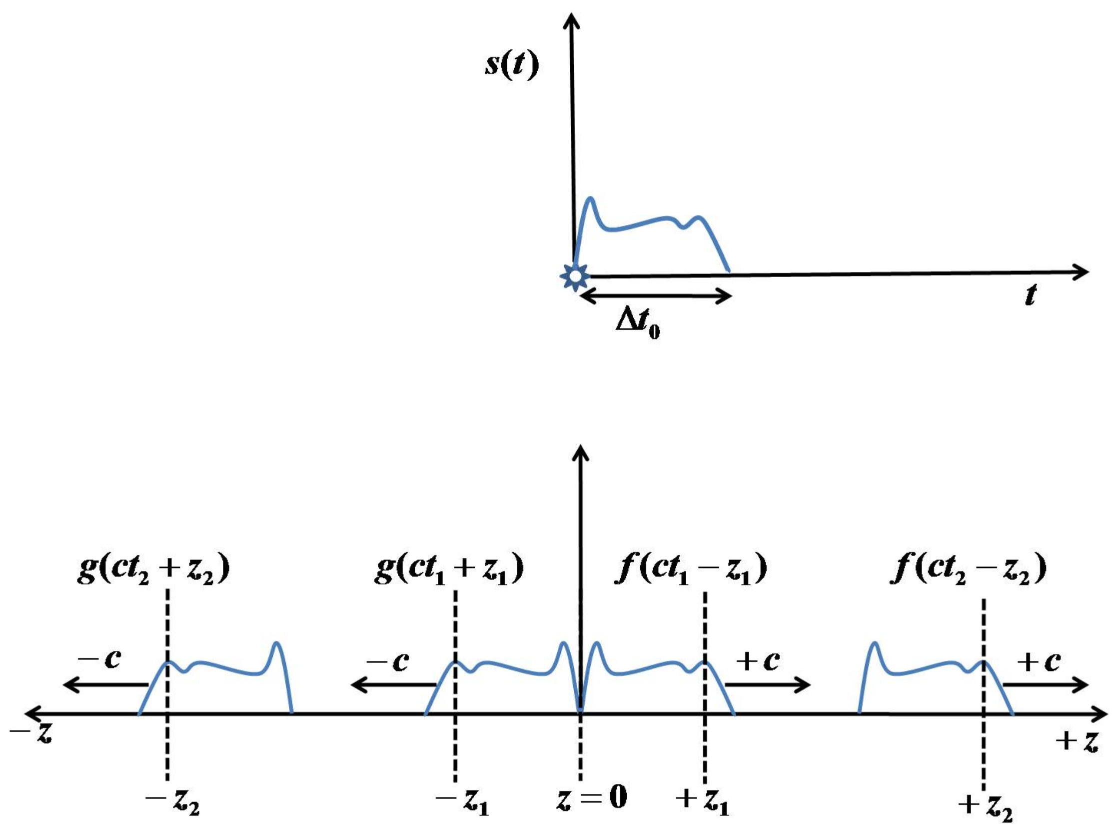


Fig. 3: Top cartoon showing an arbitrary EM signal $s(t)$ generated at $z = 0$; Bottom cartoons showing the generated signal being radiated along $+z$ and $-z$ directions (1D propagation in free space); $t_2 > t_1$ and $|z_2| > |z_1|$.

6. Harmonic Plane Wave Solution in Lossy Medium

Let us try to find a plane wave solution (simplest form of electromagnetic wave) in a charge free ($\rho_v = 0$) homogeneous medium. Using Maxwell's Equations, we can derive the wave equation for a medium characterized by electromagnetic parameters ε, μ and σ:

$$\nabla^2\vec{E} - \mu\sigma\frac{\partial \vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} = 0; \qquad \nabla^2\vec{H} - \mu\sigma\frac{\partial \vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{31}$$

Assuming the EM wave originates from a harmonic oscillator (antenna), with angular oscillation frequency $\omega(=2\pi f)$, the resulting electric and magnetic fields will be varying sinusoidal ($\vec{E},\vec{H} \sim e^{j\omega t}$). Therefore, we can describe the electric and magnetic fields in Cartesian coordinate system:

$$\vec{E} = \vec{E}_s(x,y,z)e^{j\omega t}; \qquad \vec{H} = \vec{H}_s(x,y,z)e^{j\omega t} \tag{32}$$

Thus the above equations can be written in frequency domain by substituting $\frac{\partial}{\partial t} \equiv j\omega$:

$$\nabla^2\vec{E}_s - \gamma^2\vec{E}_s = 0; \qquad \nabla^2\vec{H}_s - \gamma^2\vec{H}_s = 0 \tag{33}$$

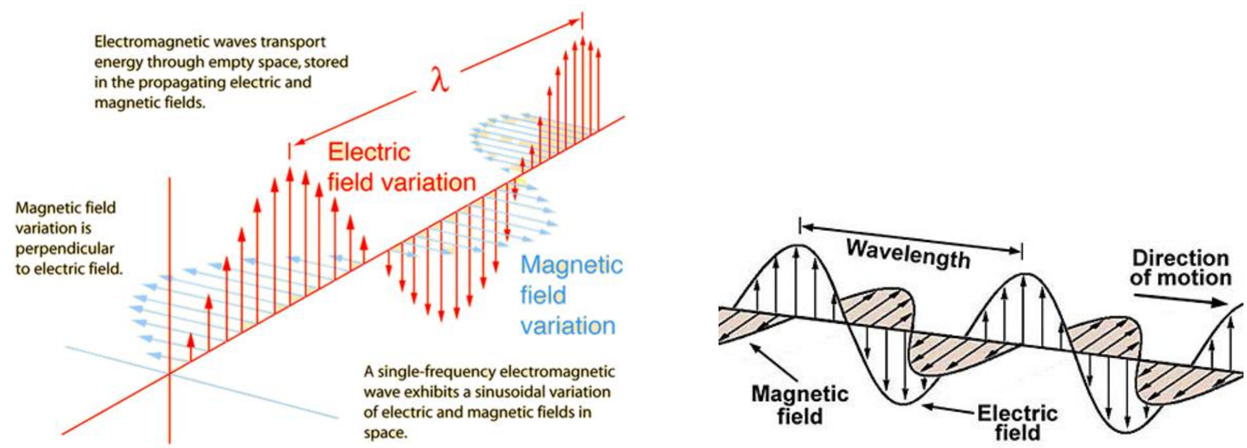
Where, $\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ and the loss co-efficient α and phase constant β can be derived as:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2}-1\right]}; \qquad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2}+1\right]} \tag{34}$$

Using above mentioned frequency domain wave equations, we can solve for a 1D electromagnetic plane wave solution propagating along +ve z-axis:

$$\vec{E}(z,t) = E_0e^{-\alpha z}e^{j(\omega t-\beta z)}\vec{a}_x; \qquad \vec{H}(z,t) = H_0e^{-\alpha z}e^{j(\omega t-\beta z)}\vec{a}_y \tag{35}$$

with $H_0 = E_0/\eta$ and $\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}}$ (intrinsic impedance). The phase velocity of EM wave is defined by $v_p = \frac{\omega}{\beta}$. Thus in free space ($\sigma = 0$, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$), the phase velocity of the EM wave $v_p = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \approx 3\times 10^8 m/s$ (the speed of light in free space). Please note that if a medium is characterized by a complex η (e.g., when $\sigma \neq 0$), it can be represented in phasor form $\eta = |\eta|e^{-j\theta_\eta}$ along with $\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$. In such situation electric field-wave and magnetic field-wave are out of phase by θ_η, however they are always intrinsically coupled.



(a) (b)

Fig. 4: (a) Illustration of EM plane wave snap-shot with electric field and magnetic field are in phase; (b) Illustration of EM plane wave snap-shot with electric field and magnetic field are out of phase $\pi/2$.

7. Application Examples

Find examples of electronic/photonic devices, circuits or systems where the knowledge of EM waves is not required for their accurate or nearly-accurate designs and predictable operations. You may take this assignment for entire course of your academic/professional career....

8. Any Question? Well, I have few.....Please answer the following:

1. Have you gone through entire note line-by-line? If so, how much time have you taken?
2. How much is your comfort level to understand and visualize vector calculus? Your answer should be in the scale of 0 to 10.
3. Do you find anything wrong in this note? Please point out those only if they are significant.
4. Have you learned anything new from this note? If so, please write in brief.
5. Assume that you are immersed in river-water along with your life-support system. Your friend is trying to locate you from the bank of the river using red-color laser pointer. Will you see changed color of the laser light inside river-water? Reason your answer.
6. Derive Poisson's equation using Maxwell's Equations in point form. Do you know that the Poisson's Equation is widely used in semiconductor device modeling? Provide one such application example.
7. Please be referred to the Fig.5. Can you formulate a question which can be solved analytically using the learning outcome from this note on Maxwell's Equations? If so, please write a brief description to solving the said problem.

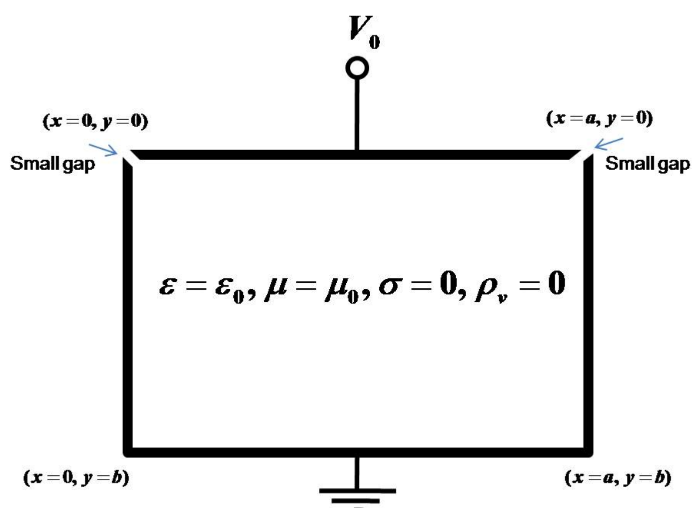


Fig. 5: Cross-sectional view (in $x-y$ plane) of infinitely extended (in z-direction) metallic box in which the lid is maintained with a steady-state potential of V_0. The coordinates of the four corners and the medium inside the box are known as indicated.