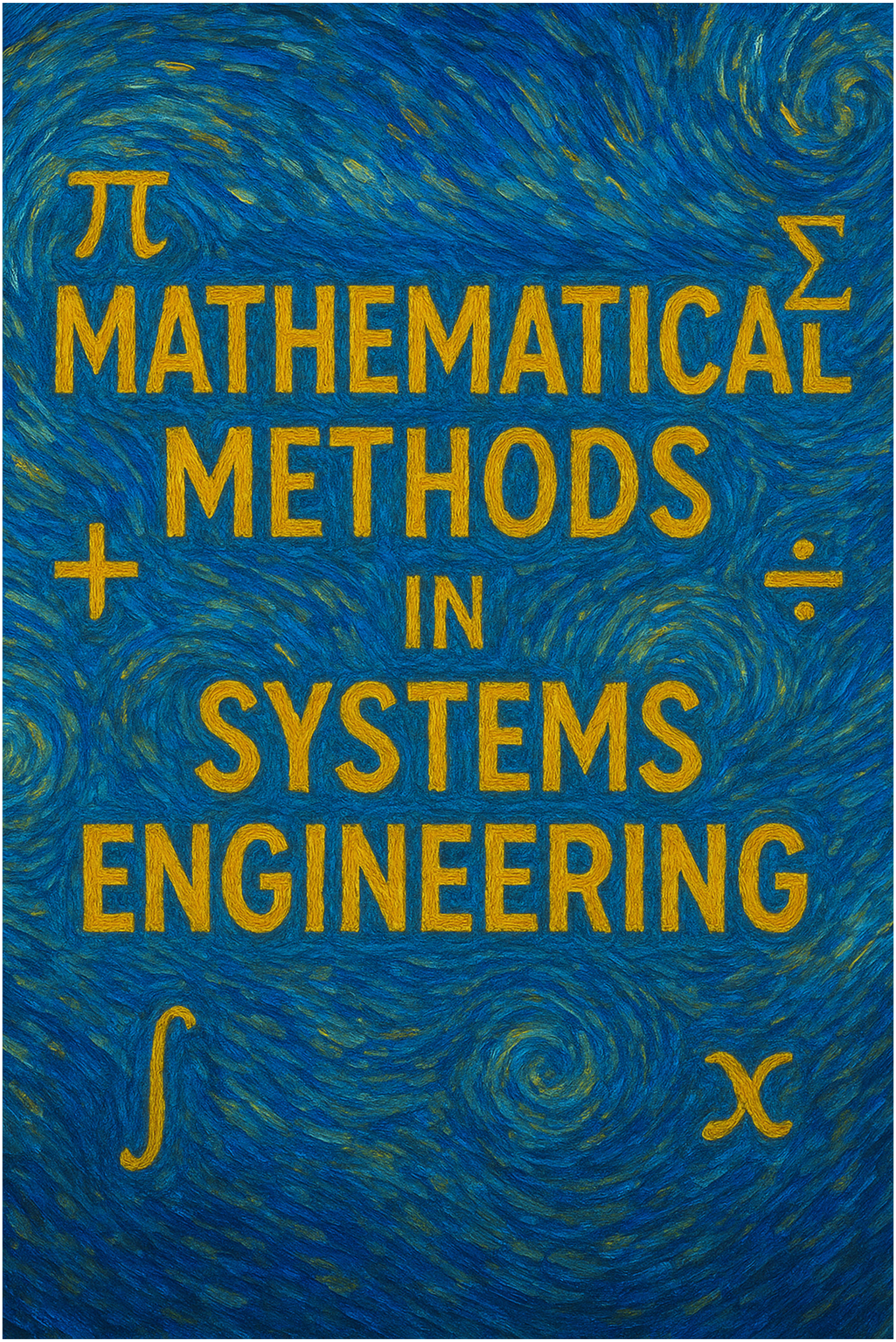

π
Σ
MATHEMATICAL
METHODS
+
÷
IN
SYSTEMS
ENGINEERING
∫
x

2

Mathematical Methods in Systems Engineering

Arun D Mahindrakar

Professor
Control and Optimization
Department of Electrical Engineering
Indian Insttitute of Technology Madras, India
July-2025

PREFACE

This lecture notes on Mathematical Methods in Systems Engineering is designed to provide a foundation for mathematical analysis arising in control theory and optimization. The emphasis is on developing a precise and logical framework for analysis.

We begin with the language of point set topology, introducing concepts such as open and closed sets, limit points, and compactness, which form the backbone of modern analysis. From there, we explore the notions of supremum and infimum, establishing the completeness property of the real numbers—a key ingredient for convergence arguments.

A detailed study of sequences and series follows, where we examine various modes of convergence, including pointwise and uniform convergence, and their implications for function approximation. We then turn to continuity and related concepts such as uniform continuity and Lipschitz conditions, grounding them in an intuitive yet mathematically rigorous setting.

Throughout, the focus remains on clarity of thought and logical precision, with proofs and examples chosen to reinforce understanding. The intention is not merely to equip students with tools, but to cultivate the habit of thinking mathematically—formulating precise definitions, constructing counterexamples, and applying abstract concepts to concrete problems.

It is hoped that this notes will serve as a gateway to more advanced mathematical studies and provide a solid analytical foundation for applications in engineering and the sciences.

The lecture notes and problems are based on the material collected from [1, 2, 3, 4, 5, 6, 7].

ACKNOWLEDGEMENTS

The assignmenets were contributed by my Doctoral students C. Ramaseshan and Anik Kumar Paul.

CONTENTS

Preface **3**

Acknowledgements **5**

Contents **7**

List of Figures **11**

List of Tables **13**

Lecture-1 **15**

1.1 Real space 15

1.2 Field and Vector space 17

Lecture-2 **21**

2.1 Interior points, open set 21

2.1.1 Finite intersection, arbitrary union and arbitrary intersection of open sets 22

Lecture-3 **23**

3.1 Closed sets 23

3.2 Topological properties of open and closed sets 24

Lecture-4 **27**

4.1 Boundary of a set, bounded and compact set 27

Lecture-5 **31**

5.1 Dense and Connected sets 31

Lecture-6 **33**

6.1 Supremum and Infimum of a set 33

Lecture-7 **37**

7.1 Properties of Supremum and Infimum 37

7.1.1 Scaling of a set 37

7.1.2 Set Addition and Set Subtraction 38

Lecture-8 **43**

8.1 Pre-image of a subset and inverse of a function 43

Lecture-9 **45**

9.1 Cardinality of sets 45

Lecture-10 **51**

10.1 Sequences in $\mathbb{R}$ 51

Lecture-11 **55**

11.1 Limit Theorems . . . 55
Lecture-12 **59**
12.1 Monotonic sequences . . . 59
Lecture-13 **63**
13.1 The Squeeze/Sandwhich Rule . . . 63
13.2 Subsequences . . . 64
Lecture-14 **69**
14.1 Bolzano-Weierstrass Theorem . . . 69
Lecture-15 **71**
15.1 Limit superior and limit inferior . . . 71
Lecture-16 **77**
16.1 Arithmetic on the Extended Real Line . . . 77
16.2 Some more results on limsup and liminf . . . 78
Lecture-17 **83**
17.1 Series . . . 83
17.1.1 Conditional Convergent Series . . . 83
Lecture-18 **87**
18.1 Sums of non-negative numbers . . . 88
Lecture-19 **91**
19.1 Tests for convergence . . . 91
19.1.1 Ratio Test (D'Alembert) . . . 91
19.1.2 Integral test . . . 95
Lecture-21 **97**
21.1 Conditionally convergent . . . 97
Lecture-21 **99**
21.1 Limits . . . 99
21.1.1 Sequential characterization of limits . . . 99
Lecture-22 **103**
22.1 Divergence Criterion: . . . 103
Lecture-23 **107**
23.1 Continuity . . . 107
23.1.1 Uniform Continuity . . . 110
23.2 Lipschitz Functions . . . 113
Lecture-24 **117**
24.1 Converse, Inverse, Contrapositive and Negation statements . . . 117
Lecture-25 **121**
25.1 Differentiability . . . 121
25.1.1 Basic Properties: . . . 122
Lecture-26 **125**
26.1 $\epsilon - \delta$ interpretation of derivative . . . 125
26.1.1 Smooth and piecewise smooth functions . . . 126

Lecture-27 **131**
27.1 Generalization of the derivative . . . 131
27.1.1 Derivative of a real-valued function . . . 131
Lecture-28 **139**
28.1 Mean value theorem . . . 139
Lecture-29 **151**
29.1 Implicit function Theorem . . . 151
Lecture-30 **157**
30.1 Riemann Integration . . . 157
30.1.1 Continuity and Boundedness . . . 157
30.2 Lower and Upper Riemann Sums . . . 159
Lecture-31 **169**
31.1 Sequential characterization of Riemann integral . . . 169
Lecture-32 **171**
2.1 Properties of the Integral . . . 171
Lecture-33 **189**
33.1 Metric spaces . . . 189
Lecture-34 **193**
34.1 Infinite dimensional metric spaces . . . 193
34.1.1 Sequence space S . . . 193
34.1.2 l^p space . . . 194
Assignment-1 **199**
1.1 Point set topology . . . 199
Assignment-2 **203**
2.1 Suprema and Infima . . . 203
Assignment-3 **207**
3.1 Sequences . . . 207
Assignment-4 **213**
4.1 Series . . . 213
Assignment-5 **215**
5.1 Limits and continuity . . . 215
Assignment-6 **219**
1 Riemnann Integration . . . 219
Assignment-7 **221**
7.1 Metric Spaces . . . 221
Bibliography **223**

LIST OF FIGURES

2.1 Set subtraction 22
9.2 Class strength 45
9.3 Counting scheme 46
9.4 Counting scheme for the set of integers 47
15.5 Ex. $a_n = (-1)^n \frac{n+1}{n}$: Even though the sequence is not converging, the limsup and liminf exists 75
21.6 Various sets 100
23.7 $f(x) = -|x|^{3/2}\text{sign}(x)$ 114
23.8 Piece-wise continuous function 115
25.9 Piecewise function 123
28.10 Set S 141
28.11 Image of F for S 142
29.12 Unique function g 153
29.13 Discontinuous function h 153
29.14 Nonexistence of inverse 154
34.15 Overestimate of the area 195

LIST OF TABLES

5.1 Separable and not separable sets. 32
26.2 Resolving inderminate forms using L'Hôpital's Rule 129

LECTURE-1

1.1 Real space

We begin with a formal definition of real spaces.

Definition 1.1.1: Let $n > 0$ be an integer. An ordered set of n real numbers $(x_1, x_2, \ldots, x_n)$ is called an n-dimensional point or a vector with n components. The number x_k is called the k^{th} coordinate of the point x or k^{th} component of the vector x. The set of all n-dimensional points is called n-dimensional Euclidean space and denoted by $\mathbb{R}^n$.

The real line is an example of a 1-dimensional space and the real plane is a 2-dimensional space. A typical vector in $\mathbb{R}^2$ is denoted by $[x_1 \;\; x_2]^\top$ or by the tuple (x_1, x_2).

The algebraic operations on $\mathbb{R}^n$ are carried out using the following rules.

1. For $x, y \in \mathbb{R}^n$, $x = y$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

2. For $x, y \in \mathbb{R}^n$,

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

3. For $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\alpha\, x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

4. $0 = [0\ 0\ 0\ \cdots\ 0]^\top$.

5. For $x, y \in \mathbb{R}^n$, the scalar dot product or inner product is

$$x.y = \sum_{i=1}^{n} x_i y_i.$$

6. The length of a vector is captured by the norm of a vector. The norm function $||(\cdot)|| : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ satisfies the following axioms:

   A1 $||\alpha x|| = |\alpha|\ ||x||$ (positive homogeneity).

   A2 $||x + y|| \leq ||x|| + ||y||$ (triangle inequality or subadditivity).

   A3 $||x|| = 0 \Longleftrightarrow x = 0$ and $||x|| > 0$ whenever $x \neq 0$ (positive definiteness).

   There is class of p-norms defined by

$$||x||_p = \left(|x_1|^p + \ldots + |x_n|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

   and

$$||x||_\infty = \max_i |x_i|.$$

$$||x||^2 =< x, x >= x_1^2 + x_2^2 + \ldots x_n^2$$

   where, the operator $< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is used to denote the dot product between two vectors in $\mathbb{R}^n$. All p-norms are equivalent in the sense that if $|| \cdot ||_{\gamma_1}$ and $|| \cdot ||_{\gamma_2}$ are two different p-norms, then there exists two constants $c_1, c_2 > 0$ such that the following holds

$$c_1 ||x||_{\gamma_1} \leq ||x||_{\gamma_2} \leq c_2 ||x||_{\gamma_1}\ \forall\, x \in \mathbb{R}^n.$$

Lemma 1.1.2: Cauchy-Schwartz inequality

Let x, y belong to $\mathbb{R}^n$. Then $| < x, y > | \leq ||x||\ ||y||$ and $| < x, y > | = ||x||\ ||y||$ if and only if the elements x, y are linearly dependent.

Instead of $\mathbb{R}^n$, we can consider a more general vector space X on which we can define the notion of distance. This leads to the definition of *normed linear space*.

1.2 Field and Vector space

Definition 1.2.1: Field
Let F be a set along with two binary operations '$+$' and '$\cdot$', called the addition and multiplication. Then the triple $(F, +, \cdot)$ is called Field if the following axioms hold for all $a, b, c \in F$.

- $a + b \in F$ **Closure under addition**
- $a + b = b + a$ **Commutative under addition**
- $a + (b + c) = (a + b) + c$ **Associative under addition**
- $a + 0 = a$ **Existence of additive identity element**
- $a + (-a) = 0$ **Existence of additive inverse**
- $ab = ba$ **Commutative property under multiplication**
- $a(bc) = (ab)c$ **Associative property under multiplication**
- $1.a = a$ **Existence of identity element $1 \in F$**
- $aa^{-1} = 1$ **Existence of multiplicative inverse $a^{-1} \in F$ for every $a \neq 0$**
- $a(b + c) = ab + ac$ **Distributivity of multiplication over addition**
- $0 \neq 1$ **Distinct additive and multiplication identites**

Example 1.2.1. *The set of real numbers is a Field and so also the set of complex numbers*

Example 1.2.2. *The set of rationsls is a Field*

Example 1.2.3. *The set of integers is not a Field*

Definition 1.2.2: Vector space

Let V be a set on which two operations (vector addition, denoted by $+$ and scalar multiplication, denoted by $\cdot$) are defined. Then if the following axioms hold for all $u, v, w \in V$ and $a, b, c \in F$, we call V a vector space over the Field F.

- $u + v \in V$ Closure under vector addition
- $u + v = v + u$ Commutative property
- $u + (v + w) = (u + v) + w$ Associative property
- $u + 0 = u$ Existence of additive identity vector $0 \in V$
- $u + (-u) = 0$ Existence of additive inverse $-u \in V$
- $a\,u \in V$ Closed under scalar multiplication
- $a(bu) = (ab)u$ Associative property of scalars
- $1.u = u \quad 1 \in F$
- $(a+b)u = au+bu$ Distributivity of scalar multiplication with over scalar addition
- $a(u + v) = au + av$ Distributivity of scalar multiplication with over vector addition

Examples of vector spaces over the reals.

Example 1.2.4. *$V = \mathbb{R}^n$ and the scalars be real numbers*

Example 1.2.5. *$V = \mathbb{C}$ and the scalars be complex numbers*

Example 1.2.6. *$V = \{f : \mathbb{R} \to \mathbb{R} : \frac{df}{dx} exists\}$ over the real numbers.*

Example 1.2.7. *$V = P_n$, the set of all real polynomials of degree at most $n \geq 0$ over the real numbers.*

Example 1.2.8. *The set of all $n \times n$ matrices, with the usual opearation of matrix addition and multiplication, with real entries, is a vector space over the reals*

Definition 1.2.3: A normed linear space is an ordered pair $(X, ||\cdot||)$ where X is a linear vector space and $||\cdot|| : X \longrightarrow \mathbb{R}_{\geq 0}$ is real-valued function on X such that the following

A1. $||x|| \geq 0 \;\; \forall\, x \in X$**;** $||x|| = 0$ if only if $x = 0$**, the zero vector in** X**.**

A2. $||\alpha x|| = |\alpha|||x|| \; \forall\, x \in X$ **and** $\alpha \in \mathbb{R}$ or $\mathbb{C}$**.**

A3. $||x + y|| \leq ||x|| + ||y|| \;\; \forall\, x, y \in X$**.**

Example 1.2.9. *Show that the function defined as* $||x||_M = \sqrt{x^\top M x}$*, where* M *is a positive definite matrix, is a valid norm.*

Often we encounter statements like necessary and sufficient conditions. Let us understand what each of them means.

- Necessary: A is a necessary for B if $B \implies A$. But A alone does not imply B.
- Sufficiency: A is a sufficient condition for B if $A \implies B$. If A holds, then B must be true.
- Necessary and Sufficient: $A \Leftrightarrow B$.

LECTURE-2

2.1 Interior points, open set

Definition 2.1.1: Interior point

Let $S \subseteq \mathbb{R}^n$. A point $s \in S$ is said to be an interior point of S if there exists an open n-ball of some radius $r > 0$ with center s such that all of its points belong to S. The set of all interior points of S is denoted by $\text{int}(S)$.

Definition 2.1.2: Open set

A set S $\subseteq \mathbb{R}^n$ is open if all its points are interior points.

The set of all interior points of a set $A \subset \mathbb{R}^n$ is denoted by $\text{int}(A)$. In $\mathbb{R}$, the open interval is defined as $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

Lemma 2.1.3

An open interval $(a, b) \subset \mathbb{R}$ is an open set.

Proof. Let $x_0 \in (a, b)$. Define $\epsilon < \min\{x_0 - a, b - x_0\}$. Then the ball $(x_0 - \epsilon, x_0 + \epsilon) \subset (a, b)$. Hence, x_0 is an interior point of (a, b) and since x_0 was chosen arbitrarily, every point of (a, b) is an interior point. Thus, the interval (a, b) is an open set. □

Example 2.1.1. $(-1, 3) \subset \mathbb{R}$ *is an open set.*

Very often we need the notion of neighbourhood of a point.

Definition 2.1.4: Open neighbourhood

Let $x \in \mathbb{R}^n$. A neighbourhood $U \subset \mathbb{R}^n$ of x is an open set containing x.

The complement of a set is defined using set subtraction.

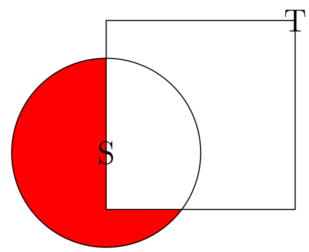


Figure 2.1: Set subtraction

Definition 2.1.5: set complement

Let $S, T \subset \mathbb{R}^n$. Then, the set subtraction, denoted by $S \setminus T$ is the set of all points S with the points common to T removed, as shown in Figure 2.1.Given set $S \subset \mathbb{R}^n$. Then, $\mathbb{R}^n \setminus S$ is called the complement of S and denoted by S^c.

Example 2.1.2. *Let $S \subset \mathbb{R}$ be $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Then* $\text{int}(S) = (-\infty, -1) \cup (1, 2)$ *and $S^c = (2, 3) \cup (3, \infty)$.*

2.1.1 Finite intersection, arbitrary union and arbitrary intersection of open sets

The arbitrary intersection of open sets need not be open. For example the set $S = \cap_{n=1}^{\infty} S_n$, where $S_n = (-\frac{1}{n}, \frac{1}{n})$. Then, $\cap_{n=1}^{\infty} S_n = \{0\}$. S is a singleton set which is not an open set.

Lemma 2.1.6

Arbitrary union of open sets is open

Proof. Let $S = \cup_{i=1}^{\infty} S_i \subseteq \mathbb{R}^n$, where each S_i is open. To show that S is an open set. Let $s \in S$. Then $s \in S_i$ for every $i \geq 1$. Since S_i is open, there exists a open ball $B(s, r)$ s.t $B(a, r) \subset S_i$. This implies $B(a, r) \subset \cup_{i=1}^{\infty} S_i$. This establishes that s is an interior point of S. Since $s \in S$ was chosen arbitrarily, $S = \text{int}(S)$. Hence, S is open subset. □

Example 2.1.3. *Show that* $\text{int}(S_1 \cap S_2) = \text{int}(S_1) \cap \text{int}(S_2)$

Example 2.1.4. *Show that* $\text{int}(S_1) \cup \text{int}(S_2) \subset \text{int}(S_1 \cup S_2)$

.

Example 2.1.5. *Show that finite intersection of open sets is open*

.

LECTURE-3

3.1 Closed sets

Definition 3.1.1: Closed set

A set S in $\mathbb{R}^n$ is said to be closed if its complement $\mathbb{R}^n \setminus S$ is open.

The notation $A \setminus B$ is read as A minus B, consists of all points of A but excluding those that are common to A and B. A collection of closed sets have the property that a union of finite collection of closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed. Closed set can be defined in terms of adherent points accumulation point/s, which are defined as follows.

Definition 3.1.2: Adherent point

Let S be a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, not necessarily in S. Then x is said to be adherent to S if for every $r > 0$ the n-ball $B(x, r)$ contains at least one point of S.

The set of all adherent points of a set S is called the closure of S and is denoted by $\bar{S}$. Some points adhere to S because every ball contains points of S distinct from x and this leads to the notion of accumulation point.

Definition 3.1.3: Accumulation or limit point

If A is a subset of $\mathbb{R}^n$ and if $x \in \mathbb{R}^n$, then x is said to be a limit point or accumulation point of A if every neighbourhood of x contains at least one point of A distinct from x.

We say x is an accumulation point of S if x adheres to $S \setminus x$. Closed set can be alternatively defined as

- A set is closed if and only if it contains all its adherent points.
- A set is closed if and only if $S = \bar{S}$.
- A set $A \subset \mathbb{R}^n$ is closed if and only if, it contains all its accumulation points.

The following Lemma brings out the relation between the closure of a set and the set of its accumulation points.

Lemma 3.1.4

Let A' denote the set of all accumulation points of a set A in $\mathbb{R}^n$, and $\bar{A}$ the set of all adherent points of A. Then, $\bar{A} = A \cup A'$.

Proof. Let $x \in A \cup A'$. Then $x \in A$ or $x \in A'$. If $x \in A$, then $x \in \bar{A}$. Since $A \subset \bar{A}$. Now if $x \in A'$, then every neighbourhood of x intersects A (in a point different from x). Therefore $x \in \bar{A}$.

Conversely, if $x \in \bar{A}$. Then if $x \in A$, then it immediately follows that $x \in A \cup A'$. If $x \notin A$, then every neighbourhood $B(x,r)$ contains atleast one element of A, which implies $B(x,r) \cap A \neq \emptyset \implies x \in A'$. Therefore $x \in A \cup A'$. □

On the real line, closed sets are denoted by closed intervals $[a,b] \triangleq \{x \in R : a \leq x \leq b\}$, while in $\mathbb{R}^2$ they are represented by closed disk $\{(x_1, x_2) : \mathbb{R}^2 : x_1^2 + x_2^2 \leq r\}$.

Example 3.1.1.

Let $A = (0,1] \subset \mathbb{R}$. Then 0 is the limit point of A and so is every point of $[0,1]$ a limit point of A.

Example 3.1.2.

Consider the set $A = \{1/n : n \in Z^+\}$, where Z^+ is the set of positive integers. The only limit point of A is 0.

3.2 Topological properties of open and closed sets

1. Arbitrary intersection of open sets is not open. Finite intersection of open sets is open.

2. Arbitrary union of open sets is open.

3. Arbitrary union of closed sets is not closed. Finite union of closed sets is closed.

4. Arbitrary intersection of closed sets is closed.

   Example 3.2.1. *Counterexample to show that arbitrary intersection of open sets is not open.*

$$S = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$$

   which is not an open set.

   Example 3.2.2. *Counterexample to show that arbitrary union of closed sets is not closed.*

$$P = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n} \right] = (0,1).$$

   which is not a closed set.

Proposition 3.2.1

The intersection of finite number of open sets is open.

Proof. Let $A_1, \ldots, A_N$ be open sets in $\mathbb{R}^n$. To show that $A = \cap_{i=1}^N A_i$ is open. Let $x \in A$. Then, $x \in A_i$ for every $i \in \{1, \cdots, N\}$. But A_i is open and so there exists a neighbourhood $B(x, r_i) \subset A_i$. Let $r = \min\{r_1, \ldots, r_2\}$. Then, $B(x,r) \subset B(x, r_i) \subset A_i$ for every $i \in \{1, \ldots, N\}$. Hence, $B(x,r) \subset \cap_{i=1}^N A_i = A$. Since $x \in A$ was arbitrary, the claim holds. □

Proposition 3.2.2

The union of any collection of open sets is open.

Proof. Suppose $\{S_i\}$ is a collection of open sets, indexed by $i \in I$. Let $S = \cup_{i \in I} S_i$. To show S is open. Let $x \in S$. Then $x \in S_i$ for some $i \in I$. But, S_i is open, so $\exists\, r > 0$ such that $B(x,r) \subset S_i \subset \cup_{i \in I} S_i$. Since $x \in S$ was arbitrary, every point in S is an interior point. Thus, S is open. □

Proposition 3.2.3

An arbitrary intersection of closed sets is closed and finite union of closed sets is closed.

Proof. Let $\{A_i, i \in I\}$ be an arbitrary collection of closed sets. By De Morgan's law $(\cap_{i \in I} A_i)^c = \cup_{i \in I} A_i^c$ which is open, since arbitrary union of open sets is open. Thus, $\cap_{i \in I} A_i$ is closed. Next, to show the finite union of closed sets is closed. Let $A_1, \ldots, A_N$ be closed sets, let $A = \cup_{i=1}^N A_i$ To show A is closed. Consider $(\cup_{i=1}^N A_i)^c = \cap_{i=1}^N A_i^c$. Since, finite intersection of open sets is open, $\cap_{i=1}^N A_i$ is closed. □

Lemma 3.2.4

The closure of a set $A \subset \mathbb{R}^n$ is the smallest closed set containing A.

Proof. The closure of the set is given by $\bar{A} = \cap_{i \in I} C_i$, where C_i is closed and $A \subseteq C_i$. Let H be any closed set that contains A. Then H is an element of the family $\{C_i : i \in I\}$. Thus $\bar{A} \subseteq \cap_{i \in I} C_i \subseteq H$. Since H was arbitrary closed set, the claim follows. □

LECTURE-4

4.1 Boundary of a set, bounded and compact set

Definition 4.1.1: Boundary of a set $S \subseteq \mathbb{R}^n$

The boundary of a set S, denoted by ∂S or $\text{bnd}(S)$ is $\partial S = \bar{S} \cap \left(\overline{\mathbb{R}^n \backslash S}\right)$.

Example 4.1.1.

$S = \cup_{n=1}^{\infty} I_n$

$$\begin{aligned} I_n &= \left[\frac{1}{2n+1}, \frac{1}{2n}\right] \\ S &= \left\{ \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{5}, \frac{1}{4}\right], \ldots. \right\} \\ \partial S &= \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots. \right\} \\ &= \{0\} \cup \left\{\frac{1}{n}, n \geq 2\right\} \\ \bar{S} &= S \cup \{0\} \end{aligned}$$

Example 4.1.2.

$$\begin{aligned} P &= [2,3] \subseteq \mathbb{R} \\ \partial P &= \bar{P} \cap \overline{\mathbb{R}\backslash P} \\ \bar{P} &= [2,3] \\ \mathbb{R}\backslash P &= (-\infty, 2) \cup (3, \infty) \\ \overline{\mathbb{R}\backslash P} &= (-\infty, 2] \cup [3, \infty) \\ \partial P &= \bar{P} \cap \overline{\mathbb{R}\backslash P} = \{2, 3\} \end{aligned}$$

Example 4.1.3.

$$\begin{aligned}
\partial Q &= \bar{Q} \cap \overline{\mathbb{R}\backslash Q} \\
\bar{Q} &= \mathbb{R} \\
\mathbb{R}\backslash Q &= \text{ Set of irrationals} \\
\overline{\mathbb{R}\backslash Q} &= \mathbb{R} \\
\therefore \quad \partial Q &= \mathbb{R} \cap \mathbb{R} = \mathbb{R}
\end{aligned}$$

Definition 4.1.2: Bounded set in $\mathbb{R}^n$

A set $S \subseteq \mathbb{R}^n$ in said to be bounded if it lies entirely within an n-ball $B(a, r)$ for some $r > 0$ and some $a \in \mathbb{R}^n$.

Example 4.1.4.

$S = [0, 1]$. S is bounded.

Example 4.1.5.

$P = (2, 3)$. P in bounded.

The notion of closed and bounded sets leads to the definition of compact set. We first need the definition of *covering of a set.* A collection F of sets is said to be covering of a given set S if $S = \cup_{A \in F} A$. The collection F is also said to cover S. If F is a collection of open sets, then F is called an open covering of S.

Example 4.1.6.

Consider $S = (0, 1)$, then the set $F = \{(\frac{1}{n}, 1 - \frac{1}{n}) : n \in Z^+\}$ is an open covering for S. But no finite subcover covers S. Hence, the open interval $(0, 1)$ is not a compact set. the non-compactness can also be deduced from the fact that $(0, 1)$ is an open set.

Definition 4.1.3: Heine-Borel

A set S in $\mathbb{R}^n$ is said to be compact if and only if every open covering of S contains a finite subcover that is, a finite subcollection which also covers S.

As an application of this theorem, we have that every closed and bounded set in $\mathbb{R}^n$ is compact.

Example 4.1.7.

$S = [2, 3]$. Is S compact. Yes.

Example 4.1.8.

$P = (1, 2)$. The subset P in not compact since it is not closed.

In $\mathbb{R}^2$, the closed-disk are compact. $\mathbb{R}$ in closed but not bounded.

Example 4.1.9. *The collection,* $\{(0, 2 - \frac{1}{k}) : k \in Z^+\}$ *is an open cover of* $(1, 2)$ *because* $(1, 2) \subset \cup_{k=1}^{\infty}(0, 2 - \frac{1}{k})$. *Again, this open cover does not have a finite subcover.*

It happens that in $\mathbb{R}^n$, closed and bounded subsets are compact.

Example 4.1.10. *$A = [2, 5]$ is a closed and bounded subset in $\mathbb{R}$. Hence, it is compact.*

Example 4.1.11. *Let $f(x) = ||x||$. Then the set $\Omega \triangleq \{x \in R^n : f(x) \leq c, c > 0\}$ is compact. The set Ω is called the sub-level set of the function f.*

LECTURE-5

5.1 Dense and Connected sets

Definition 5.1.1: Dense set

Let $S \subset \mathbb{R}^n$. Then S is dense in $\mathbb{R}^n$ if $\bar{S} = \mathbb{R}^n$.

Example 5.1.1.

The set of rationals Q are dense in R, that is $\bar{Q} = \mathbb{R}$. In other words, given $x \in \mathbb{R}$, then for every $r > 0$ we have $B(x, r) \cap Q \neq \emptyset$. A more practical example of a dense set is the following: This example uses the space of continuous functions on the closed interval $[a, b]$, denoted by $C([a, b])$. Let $f \in C([a, b])$. Then for every $\epsilon > 0$ there exists a polynomial p such that $||f - p||_\infty < \epsilon$.

Example 5.1.2. *Dyadic rationals*

Numbers of the form $\frac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}$ are dense in $\mathbb{R}$.

Example 5.1.3.

The set $\{\sin(n) : n \in \mathbb{N}\}$ is dense in $[-1, 1]$

Definition 5.1.2: Separated set

Let $A, B \subset \mathbb{R}^n$ be non-empty sets. Then, A and $B \subset \mathbb{R}^n$ are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. That is,

$$
\begin{aligned}
A \cap \bar{B} &= \emptyset \\
\bar{A} \cap B &= \emptyset.
\end{aligned}
$$

Example 5.1.4. *Consider the following three cases involving open and closed sets.*

It is clear from the above Table 5.1 a necessary condition for two sets to be separated is that they are disjoint. For sufficiency, the two sets should not have in common their closure points.

S.No	A	B	$A \cap B$	$\bar{A} \cap B$	$A \cap \bar{B}$	Remark
1.	$(0, 1)$	$(2, 3)$	$\emptyset$	$\emptyset$	$\emptyset$	Separated
2.	$[0, 1]$	$(1, 2)$	$\emptyset$	$\emptyset$	$\{1\}$	Not separated
3.	$[0, 1]$	$[1, 2)$	$\{1\}$	$\{1\}$	$\{1\}$	Not separated

Table 5.1: Separable and not separable sets.

Definition 5.1.3: Connected set

A set $S \subset \mathbb{R}^n$ is said to be connected if it cannot be written as the union of two non-empty separated sets. That is, $S \neq A \cup B$, where A, B are separated.

Example 5.1.5.

The intervals $(0, 1), (-\infty, \infty), [2, 3]$ are all connected sets.

Example 5.1.6.

Consider the continuous function $f : [a, b] \to \mathbb{R}$. The graph of the function $G = \{(x, f(x)) \in \mathbb{R}^2 : x \in [a, b]\}$ is a connected set.

Example 5.1.7.

If subset A and B are not disjoint and connected, then their union is connected.

LECTURE-6

6.1 Supremum and Infimum of a set

Let $A \subseteq \mathbb{R}$ be a non-empty.

Definition 6.1.1: Upper Bound

Let $S \subseteq \mathbb{R}$ be non-empty. If there is a real number b such that $x \leq b \; \forall \; x \in S$, then b is called an upper bound for S. We say S is bounded above by b.

Certainly any number $c \geq b$ is an upper bound for S, but we are interested in the least among the upper bounds.

Definition 6.1.2: Completeness Axiom for a Real line

Every non-empty subset of S of $\mathbb{R}$ that is bounded above has a least upper bound. Similarly, every non-empty subset of S of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Definition 6.1.3: Supremum

Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. A real number b is called a least upper bound for S, denoted by $b = \sup(S)$, if it has the following properties:

- b is an upper bound for S.
- No number less than b is an upper bound for S.

Supremum of S is the least upper bound.

Definition 6.1.4: Infimum

Let $S \subseteq \mathbb{R}$ be non-empty and bounded below. A real number a is called a greatest lower bound for S, denoted by $a = \inf(S)$, if it has the following properties

- a is a lower bound for S.
- No number greater than a is lower bound for S.

Infimum of S is the greatest lower bound.

Using the completeness axiom, it can proved that if a nonempty set is bounded above, then its supremum exists. Similarly, if it is bounded below, its infimum exists.

Example 6.1.1.

Let $A \subset B$, then $\sup(B)$ is an upper bound for A. But $\sup(A)$ is the least upper bound for A. Hence $\sup(A) \leq \sup(B)$. Similarly, $\inf(A) \geq \inf(B)$.

Example 6.1.2.

Consider $A = [0, 1]$, 1 is an upper bound for A. Infact any number greater than or equal to 1 is an upper bound for A. But, we are interested in the "least" such bound. Thus, $\sup(A) = 1 = \max(A)$. Similarly, $\inf(A) = 0 = \min(A)$.

Example 6.1.3.

Let $A = (0, 1)$. $\sup(A) = 1 \notin A$. $\inf(A) = 0 \notin A$

The infimum and supremum of a set need not belong to the set.

Example 6.1.4.

$A = (-\infty, 2)$. We have $\sup(A) = 2$. $\inf(A) = -\infty$

The given set has an upper bound but no lower bound.

Example 6.1.5.

$B = (5, \infty)$. We have $\sup(B) = +\infty$, $\inf(B) = 5$

The given set has a lower bound but not an upper bound.

Example 6.1.6.

$A = \{1, 2, 3\}$. $\sup(A) = 3$, $\inf(A) = 1$.

Example 6.1.7.

$A = \{x : -1 \leq x < 3\}$. We have $\sup(A) = 3$, $\inf(A) = -1$.

Example 6.1.8.

$A = \mathbb{N}$. We have $\sup(A) = \infty$, $\inf(A) = 1$.

Example 6.1.9.

$A = \{x : (x^2 + 1)^{(-1)} > \frac{1}{2}\}$. We have $\sup(A) = 1$, $\inf(A) = -1$.

Example 6.1.10.

$A = \{x : x^2 > 1\}$. We have $\sup(A) = \infty$, $\inf(A) = -\infty$.

Example 6.1.11.

Let A, B be non-empty bounded subsets of $\mathbb{R}$. Then

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$$

Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. This implies $x \leq \sup(A)$ or $x \leq \sup(B)$, that is $x \leq \max\{\sup(A), \sup(B)\}$. Conversely, $A \subseteq A \cup B$, and thus $\sup(A) \leq \sup(A \cup B)$. Similarly, $\sup(B) \leq \sup(A \cup B)$, that is, $\max\{\sup(A), \sup(B)\} \leq \sup(A \cup B)$.

Example 6.1.12. *What is supremum and infimum of an empty set?*

We have $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$. Suppose $B = \emptyset$. Then $\sup(A) = \max\{\sup(A), \sup(\emptyset)\}$. This requires $\sup(\emptyset) = -\infty$. Similarly, $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$. Suppose $B = \emptyset$. Then $\inf(A) = \min\{\inf(A), \inf(\emptyset)\}$. This requires $\inf(\emptyset) = +\infty$.

Definition 6.1.5: ϵ-interpretation of supremum and infimum of a set

Let $b = \sup(S), a = \inf(S)$. Then for every $\epsilon > 0$. There exists $x \in S$ such that $x > b - \epsilon$. Similarly, there exists $y \in S$ such that $y < a + \epsilon$.

LECTURE-7

7.1 Properties of Supremum and Infimum

Proposition 7.1.1

The supremum or infimum of a set $A \subset \mathbb{R}$ is unique if it exists. Moreover, if both exists, then $\inf(A) \leq \sup(A)$.

Proof. Let m and m' be the $\sup(A)$. Then, $m \leq m'$, since m' is the $\sup(A)$. Similarly, $m' \leq m$, since m is the $\sup(A)$. Thus $m = m'$. The proof for infimum is along similar lines.

If both $\inf(A)$ and $\sup(A)$ exists $\implies$ the set is non-empty. Choose $x \in A$. Then, $\inf(A) \leq x \leq \sup(A)$. Since, $x \in A$ is arbitrary, the result follows, i.e, $\inf(A) \leq \sup(A)$. □

7.1.1 Scaling of a set

Let $A \subset \mathbb{R}$ and $c \in \mathbb{R}$. Then, $cA = \{cx : x \in A\}$. For example, $A = \{1, 2, 3\}$, $c = 5$. Then, $cA = \{5, 10, 15\}$.

Proposition 7.1.2

If $c \geq 0$, then $\sup(cA) = c\sup(A)$ and $\inf(cA) = c\inf(A)$. If $c < 0$, then $\sup(cA) = c\inf(A)$ and $\inf(cA) = c\sup(A)$.

Proof. The result is trivial for $c = 0$. Consider, $c > 0$, let $b_1 = \sup(cA)$ and $b_2 = \sup(A)$. $b_1 = \sup(cA) \implies cx \leq b_1 \ \forall\ x \in A \implies x \leq \frac{b_1}{c}, c > 0$. But, b_2 is the $\sup(A)$. Therefore, $b_2 = \frac{b_1}{c}$ or $cb_2 = b_1$. Thus $\sup(cA) = c\sup(A)$. For $c < 0$, $a_2 = \sup(cA)$. Since, $a_2 = \sup(cA) \implies cx \leq a_2 \ \forall\, x \in A \implies x \geq \frac{a_2}{c} \ \forall\, x \in A$. But $\inf(A) = \frac{a_2}{c}$ or $a_2 = c\inf(A)$. Therefore, $\sup(cA) = c\inf(A)$. □

Example 7.1.1.

$A = \{10, 7, 2, 3\}, A_1 = \{7, 2, 3\}, A_2 = \{2, 3\}, A_3 = \{3\}$. Observe that, $A_3 \subset A_2 \subset A_1 \subset A$. We have, $\sup(A) = 10, \sup(A_1) = 7, \sup(A_2) = 3, \sup(A_3) = 3$ and $\inf(A) = 2, \inf(A_1) = 2, \inf(A_2) = 2, \inf(A_3) = 3$. We can thus conclude that:

Proposition 7.1.3

Suppose A and B are non-empty subsets of $\mathbb{R}$ such that $A \subset B$. If $\sup(A)$ and $\sup(B)$ exists then, $\sup(A) \leq \sup(B)$ and $\inf(A) \geq \inf(B)$.

Proof. We will show $\sup(A) \leq \sup(B)$. Since, $A \subset B$, the supremum of B is also an upper bound for the set A. Therefore, $\forall\ x \in A,\ x \leq \sup(B)$. Implies, $\sup(A) \leq \sup(B)$. Next, we show $\inf(A) \geq \inf(B)$. If $A \subset B$. Then, $-A \subset -B$ where, $-A = \{-x : x \in A\}$. From Proposition **??** we have $\sup(-A) \leq \sup(-B) \implies -\inf(A) \leq -\inf(B)$ or $\inf(A) \geq \inf(B)$. ☐

Proposition 7.1.4

Suppose A, B are non-empty sets of $\mathbb{R}$ such that $x \leq y\ \forall\ x \in A, y \in B$. Then, $\sup(A) \leq \inf(B)$.

Proof. Fix $y \in B$. Since, $x \leq y\ \forall\ x \in A$. This implies, y is an upper bound for A. Therefore, $y \geq \sup(A)$. $\sup(A)$ is a lower bound for B. But $\inf(B)$ is the greatest lower bound for B. Hence, $\sup(A) \leq \inf(B)$. ☐

Example 7.1.2.

$A = \{1, 3, 4\}$, $B = \{5, 7\}$. Then, $\sup(A) = 4$ and $\sup(B) = 7$. Therefore, $\sup(A) \leq \sup(B)$.

7.1.2 Set Addition and Set Subtraction

Let $A, B \subset \mathbb{R}$ be non-empty. Then, define set addition and set subtraction as follows.

$$A + B = \{x + y \in \mathbb{R} : x \in A, y \in B\}$$
$$A - B = \{x - y \in \mathbb{R} : x \in A, y \in B\}$$

Proposition 7.1.5

If A, B are nonempty sets, then

$$\sup(A + B) = \sup A + \sup B \qquad \inf(A + B) = \inf A + \inf B$$
$$\sup(A - B) = \sup A - \inf B \qquad \inf(A - B) = \inf A - \sup B$$

Proof. The set $A + B$ is bounded from above if and only if A and B are bounded from above, so $\sup(A + B)$ exists if and only if both $\sup A$ and $\sup B$ exist. In that case, if $x \in A$ and $y \in B$, then

$$x + y \leq \sup A + \sup B.$$

so $\sup A + \sup B$ is an upper bound of $A + B$ and therefore

$$\sup(A + B) \leq \sup A + \sup B$$

Conversely, let $\epsilon > 0$. Then there exists $x \in A$ and $y \in B$ such that

$$x > \sup A - \frac{\epsilon}{2} \qquad y > \sup B - \frac{\epsilon}{2}$$

It follows that

$$x + y > \sup A + \sup B - \epsilon$$

for every $\epsilon > 0$, which implies that $\sup(A + B) \geq \sup A + \sup B$. Thus, $\sup\ (A + B) = \sup A + \sup B$. ☐

We now move to the supremum and infimum of real-valued functions.

Definition 7.1.6: If $f : A \to \mathbb{R}$ is a function, then

$$\sup_A f = \sup\{f(x) : x \in A\}, \quad \inf_A f = \inf\{f(x) : x \in A\}.$$

A function f is bounded from above on A if $\sup_A f$ is finite, bounded from below on A if $\inf_A f$ is finite, and bounded on A if both are finite. Inequalities and operations on functions are defined pointwise as usual; for example, if $f, g : A \to \mathbb{R}$, then $f \leq g$ means that $f(x) \leq g(x)$ for every $x \in A$, and $f + g : A \to \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x)$.

Proposition 7.1.7

Suppose that $f, g : A \to \mathbb{R}$ and $f \leq g$. If g is bounded from above, then

$$\sup_A f \leq \sup_A g,$$

and if f is bounded from below, then

$$\inf_A f \leq \inf_A g.$$

Proof. If $f \leq g$ and g is bounded from above, then for every $x \in A$

$$f(x) \leq g(x) \leq \sup_A g.$$

Thus f is bounded from above by $\sup_A g$. So, $\sup_A f \leq \sup_A g$. Similarly, g is bounded from below by $\inf_A f$, so $\inf_A g \geq \inf_A f$. Note that $f \leq g$ does not imply that $\sup_A f \leq \inf_A g$ as the following example shows. □

Example 7.1.3.

Define $f, g : [0,1] :\to \mathbb{R}$ by $f(x) = 2x, g(x) = 2x + 1$. Then $f < g$ and

$$\sup_{[0,1]} f = 2, \quad \inf_{[0,1]} f = 0, \quad \sup_{[0,1]} g = 3, \quad \inf_{[0,1]} g = 1.$$

Thus, $\sup_{[0,1]} f > \inf_{[0.1]} g$.

Proposition 7.1.8

If $f(x) \leq g(y)$ for all $x, y \in A$, then $\sup_A(f) \leq \inf_A(g)$.

Proof. We will prove by contradiction. Suppose $\inf_A(g) < \sup_A(f)$. The there exists $x \in A$ s.t $\inf_A(g) < f(x) \leq \sup_A(f)$. There also exits $y \in A$ s.t $\inf_A(g) \leq g(y) < f(x) \leq \sup_A(f)$. We are lead to $g(y) < f(x)$, a contradiction. □

Like limits, the supremum and infimum do not preserve strict inequalities in general.

Example 7.1.4.

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Then $f < 1$ on $[0,1]$ but $\sup_{[0,1]} f = 1$. Next, we consider the supremum and infimum of linear combinations of functions. Scalar multiplication by a positive constant multiplies the inf or sup, while multiplication by a negative constant switches the inf and sup.

Proposition 7.1.9

Suppose that $f : A \to R$ is a bounded function and $c \in R$. If $c \geq 0$, then

$$\sup_A cf = c \sup_A f, \quad \inf cf = c \inf_A f$$

If $c < 0$, then

$$\sup_A cf = c \inf_A f, \quad \inf_A cf = c \sup_A f$$

Proof. Apply Proposition 7.1.1 to the set $\{cf(x) : x \in A\} = c\{f(x) : x \in A\}$. For sums of functions, we get an inequality. □

Proposition 7.1.10

If $f, g : A \to \mathbb{R}$ are bounded functions, then

$$\sup_A (f+g) \leq \sup_A f + \sup_A g, \quad \inf_A (f+g) \geq \inf_A f + \inf_A g$$

Proof. Since $f(x) \leq \sup_A f$ and $g(x) \leq \sup_A g$ for every $x \in [a,b]$, we have

$$f(x) + g(x) \leq \sup_A f + \sup_A g$$

Thus, $f+g$ is bounded from above by $\sup_A f + \sup_A g$, so $\sup_A (f+g) \leq \sup_A f + \sup_A g$. The proof for the infimum is on similar lines. $\square$

We may have strict inequality in Proposition 7 because f and g may take values close to their suprema (or infima) at different points.

Example 7.1.5.

Define $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x, g(x) = 1 - x$. Then

$$\sup_{[0,1]} f = \sup_{[0,1]} g = \sup_{[0,1]} (f+g) = 1,$$

Finally, we prove some inequalities that involve the absolute value.

Proposition 7.1.11

If $f, g : A \to \mathbb{R}$ are bounded functions, then

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|, \quad \left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|.$$

Proof. Since $f = f - g + g$ and $f - g \leq |f - g|$, we get

$$\sup_A f \leq \sup_A (f - g) + \sup_A g \leq \sup_A |f - g| + \sup_A g,$$

so

$$\sup_A f - \sup_A g \leq \sup_A |f - g|.$$

Exchanging f and g in this inequality, we get

$$\sup_A g - \sup_A f \leq \sup_A |f - g|$$

which implies that

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|.$$

Replacing f by $-f$ and g by $-g$ in this inequality and using the identity $\sup(-f) = -\inf f$, we get

$$\left|\inf_A f - \inf_A g\right| \leq \sup_A |f - g|.$$

□

Proposition 7.1.12

If $f, g : A \to \mathbb{R}$ are bounded functions such that

$$|f(x) - f(y)| \leq |g(x) - g(y)| \quad \text{for all } x, y \in A,$$

then

$$\sup_A f - \inf_A f \leq \sup_A g - \inf g.$$

Proof. The condition implies that for all $x, y \in A$, we have

$$f(x) - f(y) \leq |g(x) - g(y)| = \max[g(x), g(y)] - \min[g(x), g(y)] \leq \sup_A g - \inf_A g$$

which implies that

$$\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A g - \inf_A g.$$

From earlier Proposition

$$\sup\{f(x) - f(y) : x, y \in A\} = \sup_A f - \inf_A f$$

So the result follows. □

LECTURE-8

8.1 Pre-image of a subset and inverse of a function

Definition 8.1.1: Pre-image of a set

Let $f : X \to Y$ be a function. Then the pre-image of a subset $B \subset Y$ is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

The pre-image of a set is subset of the domain. If the function is surjective, then it is non-empty; if the function is injective, then the pre-image of a singleton is a singleton or empty. We next define the inverse of a function. It is again a map between sets.

Definition 8.1.2: Inverse of a function

It is a map $f^{-1} : Y \to X$, defined as

$$f^{-1}(y) = \{\text{Unique} \quad \text{element} \ \ x \in X \ \ \text{s.t} \ \ f(x) = y\}.$$

The inverse of a function exists if and only if f is bijective. Equivalently, let f, g be two injective functions. If $(f \circ g)(x) = x$ and $(g \circ f)(y) = y$, for all $x \in X$ and $y \in Y$, then we say f and g are inverses of each other.

The preimage preserves set operations as seen through the following lemma

Lemma 8.1.3:

If $f : S \to T$ and V, W are subsets of T. Then

- $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$.
- $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$.
- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(T) = S$.

Example 8.1.1.

Consider $f : \mathbb{R} \to [0, \infty)$ defined as $f(x) = x^2$. Then f has no inverse. However, if the domain is restricted to $[0, \infty)$, then f^{-1} exists and is given by $f^{-1}(y) = \sqrt{y}$.

Example 8.1.2.

Consider $f : \mathbb{R} \to [0, \infty)$ defined as $f(x) = a^x, a \in (0, 1)$. It is easy to verify that f is bijective. The inverse is given by $f^{-1}(y) = \log_a y$

Definition 8.1.4: Composition of maps

Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the composotion of g with f, denoted by $g \circ f$, is the map $g \circ f : A \to C$, and defined as $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

It can be easily show that the composition of bijective functions is also bijective.

Proposition 8.1.5

The unit circle S^1 is compact.

Proof. The proof involves by starting with an open cover for S^1 and then showing it has a finite sub-cover. Consider the $f \colon [0, 1] \to S^1$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Note that f is continuous and surjective. Let $\mathcal{U}$ be an open cover for S^1. That is, $\mathcal{U}$ is a collection of open sets such that $S^1 \subset \bigcup_{Q \in \mathcal{U}} Q$.

For each $Q \in \mathcal{U}$, the inverse image of Q under f, $f^{-1}(Q)$, is open in $[0, 1]$. This follows from the continuity of f. The collection, $\bigcup_{Q \in \mathcal{U}} f^{-1}(Q)$ is an open cover for $[0, 1]$. Since, $[0, 1]$ is compact, it has a finite subcover, that is $[0, 1] \subset f^{-1} \bigcup_{Q \in \mathcal{V}} (Q)$, where $\mathcal{V}$ is a finite collection of sets from $\mathcal{U}$.

Next, to show that every point in S^1 is in some $Q \in \mathcal{V}$. Let, $x \in S^1$, then by the bijection of f there exists $y \in [0, 1]$ such that $f(y) = x$. As, $[0, 1]$ is covered by a finite number of open sets, there must exists some $Q \in \mathcal{V}$ such that $y \in f^{-1}(Q)$ or $f(y) \in Q$ which implies $x \in Q$. Thus $\mathcal{V}$ is a finite sub cover of S^1 and the claim that S^1 is compact holds. $\square$

LECTURE-9

9.1 Cardinality of sets

The first thing that we learn as children is how to count. We need counting for our daily survival and our everyday transactions depend on it. Let us now delve into investigating what we are doing at a fundamental level when we count.

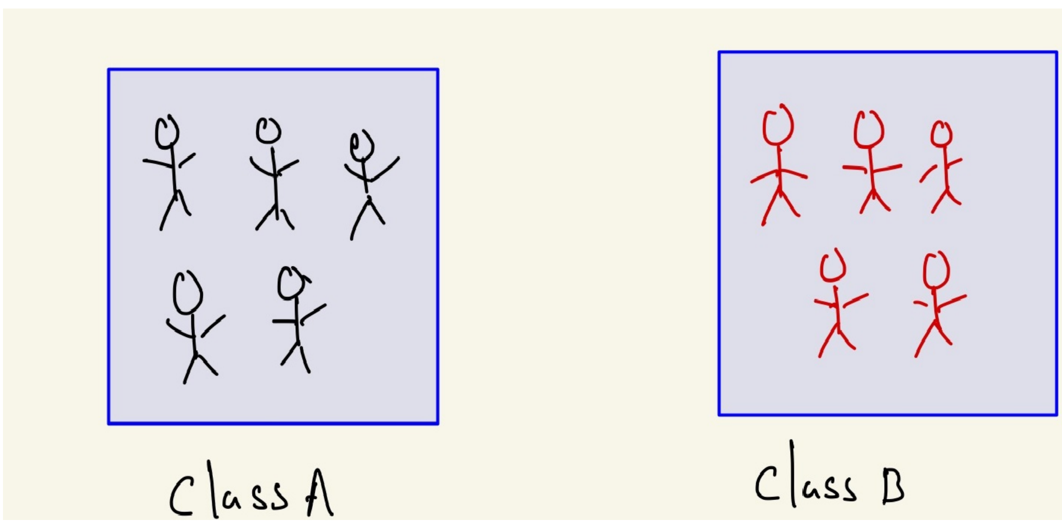


Figure 9.2: Class strength

Let us take a simple instance. At the basic level counting is comparing the amount of something between two different sets. For example, what do we mean when we say class A has the same strength as class B? Everyone will agree that both the classes A and B have the same strength. But how do we say that? We can see that every person in class A can be associated to an unique person in class B. In short, we can create a pairing between members of class A and class B so that every body in class A gets a unique pair from class B and everybody from B, gets a unique pair from A as shown in Figure 9.3. So this is the secret of saying that two sets of objects contain same amount of them. More precisely, what we are doing is that we are framing a bijective function from A to B, $f\colon A \to B$, $f^{-1}\colon B \to A$, f is a bijective funtion $\Rightarrow f$ and f^{-1} are functions.

f is a function $\Leftrightarrow$ ensures everyone from A gets a single unique pair from B. f^{-1} is a function $\Leftrightarrow$ ensures everyone from B gets a single unique pair from A.

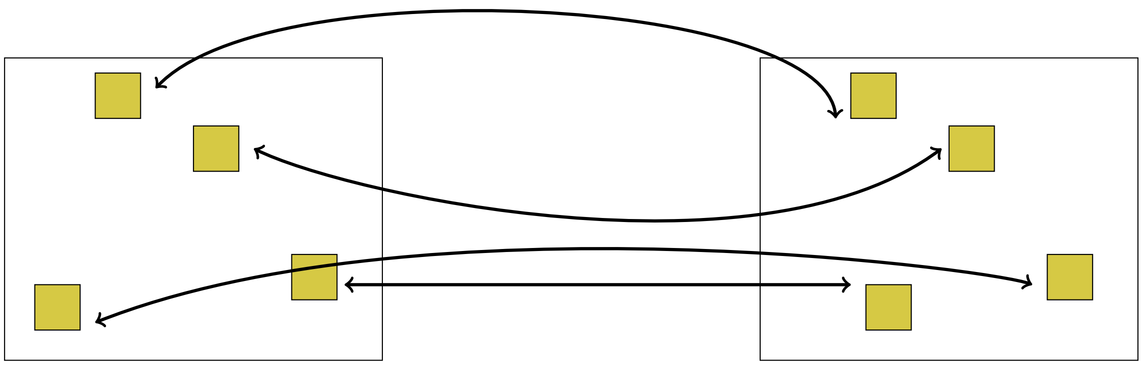


Figure 9.3: Counting scheme

So when we say two sets have the same number of objects, mathematically we have produced a bijective mapping between the two sets, a correspondence between every single member of A and every single member of B. Now in mathematics, the set of objects we deal with need not be finite in number. Instead of members from the class we may have the set of all natural numbers as a set. However, this logic extends to arbitrary sets A and B, even if you may not be physically able to count the members and compare them. With this we define the cardinality of a set.

Definition 9.1.1: [Cardinality] Two sets A and B are said to be equicardinal if there exists a bijective mapping $f : A \to B$.

Now let us define what we intutively do again most of the time. A set A is said to have cardinality of N for some natural number N, if it is equicardinal with the set $\{1, 2, 3, \dots, N\}$. Thus we say a set has six elements or cardinality 6 if it is equicardinal to the set $\{1, 2, 3, 4, 5, 6\}$. This is again something that we routinely do to compare two sets. To see if two sets A and B are equicardinal, check if they are both equicardinal to the same set of the form$\{1, 2, \dots, M\}$ for some M. This follows from the result that:

Theorem 9.1.2: If two sets are equicardinal to another set, then they are equicardinal to each other.

Proof. Given $f_{AC} : A \longrightarrow C, f_{BC} : B \longrightarrow C$ are bijective maps. Using the fact that the composition of two bijective functions is another bijective function, we have $f_{AB} = f_{BC}^{-1} \circ f_{AC}$ is a bijective map from A to B. □

Now, we define another notion. A set A is called finite if it is equicardinal to a set of the form $\{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. A set A is called infinite if it is not finite. An infinite set is called countably infinte if it is equicardinal to the set of natural numbers, $\mathbb{N}$. An infinite set is called uncountably infinite if it is not countably infinite. So this is how mathematically we define finite, countable and uncountable sets.

Example 9.1.1.

Natural numbers are countable. This is trivial as the identity function serves as a bijection from $\mathbb{N}$ to $\mathbb{N}$.

Example 9.1.2.

The set of even numbers are countable and hence equicardinal as $\mathbb{N}$.
This may sound initially shocking that there are as many even numbers as much as there are natural numbers but this is true if we accept the definition above of being equicardinal.
Note that this cannot happen in case of finite sets. A subset of a finite set cannot be equicardinal to itself! But with infinite sets we have to get used to the fact that a subset of an infinite set can be equicardinal to the given set itself as with the case of even numbers and natural numbers.

Example 9.1.3.

Integers are countable and equicardinal to $\mathbb{N}$.
Again this may sound surprising but it is true even if it appears that the integers are twice in number as natural numbers as we have negative integers also! But again in infinite sets, we have to let go of our intution and accept that a subset of a set can be of same cardinality as the set itself. We use odd numbers to count $0, 1, 2, 3, \ldots$ using the map $f(2n+1) = n$. We

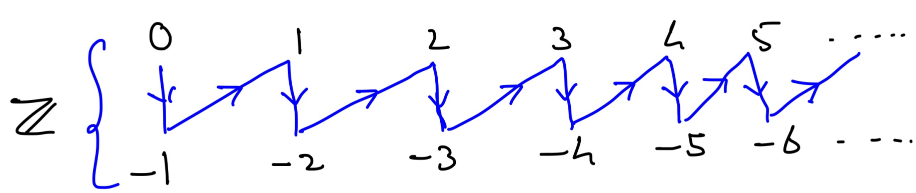


Figure 9.4: Counting scheme for the set of integers

use even numbers to count $-1, -2, -3, \ldots$ using the map $f(2n) = -n$. The inverse is given by $f^{-1}(n) = 2n+1, n = 0, 1, 2, \ldots$ and $f^{-1}(-n) = 2n, n = -1, -2, -3, \ldots$. Hence, $f \colon \mathbb{N} \to \mathbb{Z}$ is bijection.

We cannot get uncountable sets by taking countable union of a family of countable sets. Let, $A_i, i = 1, 2, 3, \ldots$ be a family of disjoint sets. Then we claim that $\bigcup_{i=1}^{\infty} A_i$ is still countable.

Proposition 9.1.3

Countable union of disjoint countable sets is countable.

Proof. The proof uses Cantor's diagonal argument as the counting scheme. Define $S = \cup_{i=1}^{\infty} A_i$ and let a_{ij} be the jth element of the ith set. Arrange them in the infinite array as shown.

$$\begin{array}{l} A_1 \to a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \ldots \\ A_2 \to a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25} \ldots \\ A_3 \to a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35} \ldots \\ A_4 \to a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45} \ldots \\ \qquad\qquad\qquad\qquad \vdots \end{array}$$

If we cross out the first diagonal, we have mapped element 1. If we crossed out the second diagonal, we have mapped elements 2 and 3. If we cross out the third diagonal, we have mapped elements $4, 5$ and 6. At the nth diagonal, we have cross out n elements. Thus we have, $f(1) = a_{11}, f(2) = a_{12}, f(3) = a_{21}, f(4) = a_{13}, f(5) = a_{22}, f(6) = a_{31}, \ldots$. The counting scheme is given by the map $f\colon \mathbb{N} \to S$ defined as follwos: $f(1) = a_{11}$, if $f(n) = a_{ij}$, then

$$f(n+1) = \begin{cases} a_{i+1,j-1} & \text{if } \ i, j \in \mathbb{N} \\ a_{1,i+j} & \text{if } \quad j - 1 \notin \mathbb{N}. \end{cases} \tag{9.1}$$

Since we never run out of elements in $\mathbb{N}$, eventually given any diagonal we have created a map to every element in it, thus establishing the surjectivity. Since every element in S is one of the diagonals, we have a one-to-one map. □

Example 9.1.4.

Rational numbers are countable: Now using this we can prove that rational numbers are countable. It also may sound surprising as there appear much more rational numbers than natural numbers as between any two rationals however small, there is always another rational number and hence dense in the real line. Compared to the natural numbers that are scattered here and there in the number line, rationals are dense and look like they fill the line completely. But still cardinality says that there are as many rationals as there are natural numbers. This is again due to a diagonal argument. Any rational number is of the form given by p/q where p and q are integers and hence they can also be arranged in an infinite array with some gaps corresponding to leaving out fractions with common factors. We will illustrate Cantor diagonalization method to show that the rationals are countable. Consider the infinite array

$$\begin{array}{llllll} 0 \\ \pm 1/1 & \pm 2/1 & \pm 3/1 & \pm 4/1 & \pm 5/1 & \cdots \\ \pm 1/2 & \pm 2/2 & \pm 3/2 & \pm 4/2 & \pm 5/2 & \cdots \\ \pm 1/3 & \pm 2/3 & \pm 3/3 & \pm 4/3 & \pm 5/3 & \cdots \\ \pm 1/4 & \pm 2/4 & \pm 3/4 & \pm 4/4 & \pm 5/4 & \cdots \end{array}$$

The infinite S array contains all the rationals. The first diagonal is just the element 0. Any element a/b along the nth the diagonal satisfies $a+b=n$. For example take $n=4$, the possible way one can add up to 4 is $(1,3),(2,2),(3,1)$. Every rational number is in some diagonal. For example, the rational $\frac{1002}{1000}$ is in the 2002th digonal. The length of the elements in any diagonal is finite. The counting mechanism will involve going along diagonals $1,2,3,\dots$ and count all $\pm\frac{a}{b}$ that are rational (skip any $\pm\frac{a}{b}$ with common factors). We then have the following bijection $f:\mathbb{N}\to S$ as follows:

$$
\begin{aligned}
1 &\to 0\\
2 &\to 1\\
3 &\to -1\\
4 &\to \frac{1}{2}\\
5 &\to -\frac{1}{2}\\
6 &\to 2\\
7 &\to -2\\
8 &\to \frac{1}{3}\\
9 &\to -\frac{1}{3}\\
&\text{skip } \pm\frac{2}{2}\\
10 &\to 3\\
&\vdots
\end{aligned}
$$

So the rationals are densely packed in the line but they can be counted as well. Looks like every set is countable. We cannot get uncountable sets by taking countable union of countable sets. What can we do to generate an uncountable set from a countable set? More generally given a set of some cardinality, how do we generate another set of higher cardinality? Looks like there is an easy way. The power set of a given set cannot be equicardinal to the given set. No set can use itself to enumerate its own power set. Recall that power set 2^X of a set X is the set of all subsets of X.

Theorem 9.1.4: There is no surjective map between X and 2^X.

Proof. The proof by contradiction. We will assume $\exists\, f\colon X\to 2^X$ is surjective and arrive at a contradiction. Take an element $x\in X$ and return a subset of X containing some elements of X. Define $D=\{a\in X : a\notin f(X)\}$.

For example, say $X=\{1,2\}$, then $P(X)=\{\{1\},\{2\},\emptyset,X\}$. Then suppose $x=1$, and $f(1)=\{2\}$. Then $1\notin f(1)=\{2\}$. Thus the set D is well-defined. Clearly, $D\in P(X)$. There are two cases for $f(x)$ given $x\in X$. We will show that $f(x)\notin D$ for all $a\in X$.

Case 1 : $x \in D \implies x \notin f(x) \implies f(x) \notin D$.

Case 2 : $x \notin D \implies x \in f(x) \implies f(x) \notin D$.

Thus, for all $x \in X$, $f(x) \notin D \implies f$is not surjective. □

Theorem 9.1.5: Power set of Natural numbers is uncountable.

We think of rational numbers as normal and irrational numbers as weird, hence the respective names. At first glance, there seems to be not much of a difference between rationals and reals as any real number can be approximated as much precisely as we like by a rational number. But the fact is still the irrationals are uncountable, because if they were countable then the real numbers which is the union of rationals and irrationals would be countable as a countable union of countable set is countable and we have assumed irrationals are countable and have already established rationals as countable. So this says that irrational numbers are much larger in number than rational numbers. It is time we get to know them better. Now having proved that the power set of a set is not equicardinal to the original set and has to be bigger in size we can produce a series of sets whose cardinalities are bigger and bigger as follows: finite, $\mathbb{N}, 2^{\mathbb{N}}, 2^{\mathbb{R}}, 2^{2^{\mathbb{R}}}, \ldots$. In other words, there are infinitely many infinities.

LECTURE-10

10.1 Sequences in $\mathbb{R}$

An infinite sequence x is a mapping $X\colon \mathbb{N} \to \mathbb{R}$ denoted by $x(n)$ or x_n, consisting of a non-terminating collection of real numbers. We will focus on the sequence of real numbers. We will denote a sequence by $\{x_n\}_{n=1}^{\infty}$. A sequence consists of elements $\{x_1, x_2, \ldots, x_n, x_{n+1,\ldots}\}$.

Example 10.1.1. $A = \{1, 2, 3, 4\}$. *This is not a sequence but a finite set.*

Example 10.1.2. $\{x_n\}_1^{\infty}, x_n = 3n$. *Then* $x_n = \{3, 6, 9, \ldots\}$.

Example 10.1.3. $\{x_n\}_5^{\infty} = \{x_5, x_6, \ldots\}$.

Example 10.1.4. $\{x_n\}$, *where* $x_n = (-2)^{2n}, n = 1, 2, \ldots$. *Here,* $x_n = \{4, 16, 64, 256, 1024, \ldots\}$.

Sequences can also be defined inductively or recursively.

Example 10.1.5. *Fibonacci sequence:* $x_1 = x_2 = 1$, $x_{n+1} = x_{n-1} + x_n, n \geq 2$. $x_n = \{1, 1, 2, 3, 5, 8, \ldots\}$.

Objective

We are given a sequence, and we need to find what happens to the sequence as n becomes very large. The limit of the sequence captures the notion.

Example 10.1.6. $\{x_n\}, x_n = \frac{n+1}{n}, n = 1, 2, 3, \ldots$
$x_n = \{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\}$.

The sequence looks like it is approaching 1 as n becomes very large. In fact, $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$.

Definition 10.1.1: Convergence of sequence

Let $\{x_n\}$ be a sequence of real numbers. We say that the sequence $\{x_n\}$ converges to a real number x, or tends to x, and we denote by $x = \lim\limits_{n\to\infty} x_n$ or $x_n \to x$ if for every $\epsilon > 0$, there is an $N_1 \in \mathbb{N}$ s.t $|x_n - x| < \epsilon \ \ \forall\, n \geq N_1$.

We say x is the limit of the sequence and say that the sequence $\{x_n\}$ is convergent if the limit exists.

Definition 10.1.2: Divergent Sequence

A sequence $\{x_n\}$ diverges or is divergent if it does not converge to any number.

Example 10.1.7. $\{x_n\} = n$. *The sequence x_n is divergent.*

Definition 10.1.3: Archimedian Property

Given $x \in \mathbb{R}, \exists\, n \in \mathbb{N}$ such that $n > x$.

Proposition 10.1.4

Uniqueness of limit: A sequence $\{x_n\} \in \mathbb{R}$ can have atmost one limit.

Proof. Let $x_n \to x$ and also $x_n \to x'$. We need to show $x = x'$. Given $\epsilon > 0$, since $x_n \to x$, $\exists\, N_1 \in \mathbb{N}$ s.t $|x_n - x| < \frac{\epsilon}{2}\ \forall\, n \geq N_1$. Also, since $x_n \to x'$, $\exists\, N_2 \in \mathbb{N}$ s.t $|x_n - x'| < \frac{\epsilon}{2}\ \forall\, n \geq N_2$. Now, $|x - x'| = |x_n - x' - x_n + x| \leq |x_n - x'| + |x_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall\, n \geq N$, where $N \triangleq \max\{N_1, N_2\}$. Since, $\epsilon > 0$ is arbitrary, $x = x'$. □

Example 10.1.8. $\{x_n\} = \frac{1}{n}$.
$\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. *Claim $x = 0$ is the limit point. Let $\epsilon > 0$ be given. We need to come-up with $N_1 \in \mathbb{N}$ s.t $|x_n - x| = |x_n| < \epsilon$, $\forall\, n \geq N_1$. This implies $\frac{1}{n} < \epsilon\ \forall\, n \geq N_1$ or $n > \frac{1}{\epsilon}\ \forall\, n \geq N_1$. Choose $N_1 > \frac{1}{\epsilon}$.*
Thus, $x_n \to 0$.

Example 10.1.9. $\{x_n\}, x_n = \frac{1}{\sqrt{n}}$
$\{x_n\} = \left\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \ldots\right\}$
Claim $x_n \to 0$.
Given $\epsilon > 0$ we need to show the existence of an integer N_1 s.t $|x_n - 0| < \epsilon\ \forall\, n \geq N_1$. Now, $\frac{1}{\sqrt{n}} < \epsilon \implies \sqrt{n} > \frac{1}{\epsilon} \implies n > \frac{1}{\epsilon^2}$. By Archimedian property such a number exists.
Take $N_1 = \left\lceil \frac{1}{\epsilon^2} \right\rceil + 1$, where $\lceil x \rceil$ is the greatest integer function of x.

Example 10.1.10. $\{x_n\}$, $x_n = \frac{1}{2^n}$
$\{x_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$. *Claim $x_n \to 0$. Given $\epsilon > 0$. $\left|\frac{1}{2^n} - 0\right| < \epsilon\ \forall\, n \geq N_1$. What is this integer N_1 ?*
$\frac{1}{2^n} < \epsilon \implies 2^n > \frac{1}{\epsilon} \implies n \ln 2 > -\ln \epsilon \implies n > -\frac{\ln \epsilon}{\ln 2}$. *Choose* $N_1 = \max\left\{1 + \left\lceil \frac{-\ln \epsilon}{\ln 2} \right\rceil, 1\right\}$

Definition 10.1.5

A sequence $\{x_n\}$ that converges to zero is called a null sequence.

We have seen several equivalent definitions of the convergence of a sequence. They are summarized in the following proposition.

Proposition 10.1.6

Let $\{x_n\}$ be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. The following statements are equivalent.

1. $x_n \to x$.
2. $\forall\, \epsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ s.t $|x_n - x| < \epsilon\ \forall\, n \geq N_1$.
3. $\forall\, \epsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ s.t $\forall\, n \geq N_1 \quad x - \epsilon < x_n < x + \epsilon$.
4. $\forall\, \epsilon$-ngbd of x, denoted by $B(x, \epsilon)$, $\exists\, N_1 \in \mathbb{N}$ s.t $\forall\, n \geq N_1$, $x_n \in B(x, \epsilon)$.

Proposition 10.1.7

If $\{x_n\}$ converges, then $\{x_n\}$ is bounded. A sequence is unbounded if it is not bounded.

Proof. A sequence $\{x_n\}$ is said to be bounded if there exists a number $M > 0$ such that $|x_n| \leq M\ \forall\, n \in \mathbb{N}$. We are given $x_n \longrightarrow x$. Take $\epsilon = 1$. Then, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq N_1$. Consider

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$$

for all $n \geq N_1$. We need to take care of the first $N_1 - 1$ terms $x_1, x_2, \ldots, x_{N_1-1}$ of the sequence, which is done by defining $M = \max\{|x_1|, |x_2|, \ldots, |x_{N_1-1}|, 1 + |x|\}$. Thus $|x_n| < M$ for all $n \in \mathbb{N}$. □

Example 10.1.11. *$\{x_n\}$, where $x_n = 2^n, n = 1, 2, \ldots$, $x_n = \{2, 4, 8, 16, \ldots\}$.*
The sequence is divergent. Proof by contradiction. Assume the sequence is convergent. Then by Proposition 10.1 we have,

$$\begin{aligned} |2^n| &\leq M \\ 2^n &\leq M \\ n \log 2 &\leq M \\ n &\leq \frac{M}{\log 2} \end{aligned}$$

Contradicting Archimedian property. Hence, the given sequence $\{x_n\}$, where $x_n = 2^n, n = 1, 2, \ldots$ is divergent.

LECTURE-11

11.1 Limit Theorems

Proposition 11.1.1

(Linearity Property) Suppose $x_n \to x$ and $y_n \to y$. Then, $x_n + y_n \to x + y$.

Proof. Since, $x_n \to x$. Given $\epsilon > 0, \exists\ N_1 \in \mathbb{N}$ s.t $|x_n - x| < \frac{\epsilon}{2}\ \forall\ n \geq N_1$. Similarly, $|y_n - y| < \frac{\epsilon}{2}\ \forall\ n \geq N_2$ where $N_2 \in \mathbb{N}$. Consider

$$\begin{aligned} |x_n + y_n - x + y| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ \forall\ n \geq \max\{N_1, N_2\}. \end{aligned}$$

□

Note: If $x_n \to x$ and $y_n \to y$. Then, $\alpha x_n + \beta y_n \to \alpha x + \beta y\ \forall\ \alpha, \beta \in \mathbb{R}$.

Proposition 11.1.2

Let $x_n \to x$ and $y_n \to y$. Then, $(x_n y_n) \to (xy)$.

Proof. Note that $\{x_n y_n\}$ is a new sequence formed using x_n and y_n. Since $x_n \to x$, fix $\epsilon_1 > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $|y_n - y| < \epsilon_1$. Consider

$$\begin{aligned} |x_n y_n| &= |x_n y_n - x_n y + x_n y - xy| \quad \text{(smuggling)} \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &\leq |x_n|\,|y_n - y| + |y|\,|x_n - x| \end{aligned}$$

Since $\{x_n\}$ is convergent $\implies \exists\ M > 0$ s.t $|x_n| \leq M\ \forall\ n \in \mathbb{N}$ and $y \in \mathbb{R}$.

$$\begin{aligned} \therefore |x_n y_n - xy| &< M|y_n - y| + |y||x_n - x| \\ &\leq \underbrace{\frac{M\epsilon_1}{2}}_{\forall\ n \geq N_1} + \underbrace{\frac{|y|\epsilon_1}{2}}_{\forall\ n \geq N_2} = \frac{(M + |y|)}{2}\epsilon_1\ \forall\ n \geq \max\{N_1, N_2\} \end{aligned}$$

Choose $M_1 = \max\{M, |y|\}$. Then,

$$|x_n y_n - xy| \leq \frac{M_1\epsilon_1}{2} + \frac{M_1\epsilon_1}{2} = M_1\epsilon_1 \;\forall\; n \geq \max\{N_1, N_2\}$$

Define $\epsilon_1 < \frac{\epsilon}{M_1}$. Then, $|x_n y_n - xy| < \epsilon \;\forall\; n \geq \max\{N_1, N_2\}$. □

Proposition 11.1.3

(Quotient Rule) If $x_n \to x$ & $y_n \to y$ s.t $y_n \neq 0 \;\forall\; n$ & $y \neq 0$. Then, $(\frac{x_n}{y_n}) \to (\frac{x}{y})$.

Note: We will prove an equivalent result. If $y_n \to y, y_n \neq 0 \;\forall\; n$ & $y \neq 0$. Then, $(\frac{1}{y_n}) \to (\frac{1}{y})$.

Proof. Given $\epsilon > 0$. To show that there exists $N_1 \in \mathbb{N}$ such that $|\frac{1}{y_n} - \frac{1}{y}| < \epsilon \;\forall\; n \geq N_1$. Consider $|\frac{1}{y_n} - \frac{1}{y}| = |\frac{y - y_n}{y y_n}| = \frac{|(y_n - y)|}{|y||y_n|}$. We first need to show $\frac{1}{|y_n|}$ is bounded. We know $y_n \to y$, so we can fix $\epsilon_1 = \frac{|y|}{2}$ such that there $\exists\; N_2 \in \mathbb{N}$ s.t $|y_n - y| \leq \frac{|y|}{2}$ for all $n \geq N_2$. Using the reverse triangle inequality $|y_n - y| \geq |y| - |y_n|$ or

$$\begin{aligned} |y_n| &\geq |y| - |y_n - y| \\ &\geq |y| - \frac{|y|}{2} = \frac{|y|}{2} \\ \implies \frac{1}{|y_n|} &\leq \frac{2}{|y|}, y \neq 0 \;\forall\; n \geq N_2 \end{aligned}$$

Thus $\frac{1}{|y_n|}$ is bounded. Again, since $y_n \to y$, so we can fix ϵ_2 such that there $\exists\; N_3 \in \mathbb{N}$ s.t $|y_n - y| < \epsilon_2$ for all $n \geq N_3$.

Consider

$$|\frac{1}{y_n} - \frac{1}{y}| = \frac{|(y_n - y)|}{|y||y_n|} < \epsilon_2 \frac{2}{|y|^2}.$$

Choose $\epsilon_2 = \epsilon |y|^2/2$. Then $\frac{|(y_n - y)|}{|y||y_n|} < \epsilon \;\forall\; n \geq \max\{N_2, N_3\}$. □

Example 11.1.1. *Find* $\lim\limits_{n\to\infty} \frac{3n^2-2}{n^2+n}$

It is of the form $\{\frac{x_n}{y_n}\}$ *where* $x_n = 3n^2 - 2 \& y_n = n^2 + n$.

$$\begin{aligned} \lim_{n\to\infty} \frac{3n^2 - 2}{n^2 + n} &= \lim_{n\to\infty} \frac{3 - \frac{2}{n^2}}{1 + \frac{1}{n}} \\ &= \frac{3}{1} \end{aligned}$$

Proposition 11.1.4

If $x_n \to x$ *and* $x_n \geq 0 \;\forall\; n \in \mathbb{N}$, then $x \geq 0$.

Proof. By Contradiction.
Suppose $x < 0 \implies -x > 0$. Given $-x > 0$ (treat it as epsilon), $\exists\ N_1 \in \mathbb{N}$ s.t

$$|x_n - x| < \epsilon = -x \ \forall \ n \geq N_1$$
$$\implies x_n - x < -x$$
$$\implies x_n < -x + x = 0$$
$$\text{or } \ x_n < 0$$

A contradiction. Hence, proved. □

Proposition 11.1.5

If $\{x_n\}$ *and* $\{y_n\}$ are convergent sequences and if $x_n \leq y_n \ \forall \ n$, then $\lim x_n \leq \lim y_n$.

Proof. Define $z_n = y_n - x_n$. $\forall \ n \in \mathbb{N}$ we have $z_n \geq 0$

$$\implies \lim_{n\to\infty} z_n \geq 0$$
$$\implies \lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n \geq 0$$
$$\implies \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n$$

□

LECTURE-12

12.1 Monotonic sequences

Proposition 12.1.1

Suppose $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} |x_n| = |x|$.

Proof. We will use the inequality $||a| - |b|| \leq |a - b|$. Given $\epsilon > 0$, since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ s.t $|x_n - x| \leq \epsilon$ for all $n \geq N_1$. Consider

$$||x_n| - |x|| < |x_n - x| < \epsilon \;\; \forall \, n \geq N_1.$$

□

Proposition 12.1.2

Suppose $x_n \to x$ and $x_n \geq 0$ then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof. We have two cases:
Case 1: $x = 0$. Given that $x_n \to 0$. Let $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ s.t. $|x_n - 0| \leq \epsilon$ for all $n \geq N_1$. Consider

$$\begin{aligned}
|x_n| &< \epsilon^2 \\
x_n &\geq 0 \\
x_n &< \epsilon^2 \\
\sqrt{x_n} &< \epsilon \\
|\sqrt{x_n} - 0| &< \epsilon \, \forall \, n \geq N_1 \implies \lim_{n\to\infty} \sqrt{x_n} = \sqrt{0} = 0.
\end{aligned}$$

Case 2: $x > 0$. Since, $x_n \to x$, Fix $\epsilon_1 < \epsilon\sqrt{x}$. Then there exists $N_2 \in \mathbb{N}$ such that

$|x_n - x| < \epsilon_1, \forall n \geq N_2$. Consider,

$$|\sqrt{x_n} - \sqrt{x}| = |\frac{(\sqrt{x_n - \sqrt{x}})(\sqrt{x_n + \sqrt{x}})}{(\sqrt{x_n + \sqrt{x}})}|.$$
$$= |\frac{(x_n - x)}{\sqrt{x_n} + \sqrt{x}}|$$
$$< \frac{|x_n - x|}{\sqrt{x}}$$
$$< \epsilon_1/\sqrt{x}$$
$$< \epsilon, \forall\, n \geq N_2.$$

□

Monotonic sequences

Let $\{x_n\} \subseteq \mathbb{R}$ is a sequence. Then we say, $\{x_n\}$ is

1. Bounded above if there exists $M \in \mathbb{R}$ such that $x_n \leq M \;\forall\, n \in \mathbb{N}$.
2. Bounded below, if there exists $m \in \mathbb{R}$ such that $m \leq x_n \;\forall\, n \in \mathbb{N}$.
3. Bounded if it is bounded above and below.
4. Monotonically increasing if $x_n \leq x_{n+1} \leq x_{n+2} \ldots \;\forall\, n \in \mathbb{N}$.
5. Strictly increasing if $x_n < x_{n+1} < x_{n+2} \ldots \;\forall\, n \in \mathbb{N}$.
6. Monotonically decreasing if $x_n \geq x_{n+1} \geq x_{n+2} \ldots \;\forall\, n \in \mathbb{N}$.
7. Strictly decreasing if $x_n > x_{n+1} > x_{n+2} \ldots \;\forall\, n \in \mathbb{N}$.
8. Strictly monotonic if it is strictly increasing or strictly decreasing.
9. Alternating, if it changes it sign alternatively.

Example 12.1.1. $\{1/n\} = \{1, 1/2, 1/3 \ldots\}$. *Strictly decreasing and bounded.*

Example 12.1.2. $\{n\} = \{1, 2, \ldots\}$. *Strictly increasing and bounded below.*

Example 12.1.3. $\{(-1)^{n-1}_n\} = \{1, -2, 3, -4 \ldots\}$. *Neither increasing, nor decreasing. It is unbounded.*

Example 12.1.4. $\{(-1)^n/n\} = \{-1, 1/2, -1/3, \ldots\}$. *Neither increasing nor decreasing. It is bounded.*

Example 12.1.5. $\{n^{1/n}\} = \{1, 2^{1/2}, 3^{1/3} \ldots\}$. *Not monotonic. Tail is strictly decreasing. Look after $n \geq 4$. We say the sequence is eventually decreasing. Bounded above and below. In fact, the sequence is converging to 1.*

Theorem 12.1.3: (Monotone Convergence Theorem) Every increasing sequence that is bounded above converges. Also, every decreasing sequence that is bounded below converges.

Proof. Given $\{a_n\}$

1. $\{a_n\}$ is bounded above.
2. It is increasing.

$A = \{a_n \in \mathbb{R} : n \in \mathbb{N}\}$. A is non-empty and bounded above. By Completeness Axiom of the real line, sup(A) exists. Let $a = \sup(A)$. Given $\epsilon > 0$ $\exists$ $N_1 \in \mathbb{N}$ s.t $a_{N_1} > a - \epsilon$. We know $\{a_n\}$ is an increasing sequence

$$\begin{aligned} a - \epsilon < a_{N_1} \leq & a_n \leq a < a + \epsilon \;\forall\; n \geq N_1 \\ \implies a - \epsilon < & a_n < a + \epsilon \\ \implies |a_n - & a| < \epsilon \;\forall\; n \geq N_1. \end{aligned}$$

The proof for decreasing sequence is along similar lines. □

Example 12.1.6. $\{a_n\}$, $a_n = \frac{n}{n+1}$. $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$. $\{a_n\}$ *is an increasing sequence and bounded above by* 1. $\therefore a_n \to a$ *for some* $a \in \mathbb{R}$ *by MCT (Monotone Convergence Theorem).*

Example 12.1.7. $\{a_n\}$, $a_n = 1 + \frac{1}{n}$. $\{a_n\} = \{2, \frac{3}{2}, \frac{4}{3}, \ldots\}$. $\{a_n\}$ *is a decreasing sequence and bounded below by* 1. $\therefore a_n \to a$ *for some* $a \in \mathbb{R}$ *by MCT (Monotone Convergence Theorem).*

Example 12.1.8. $\{a_n\}$, $a_n = 1 \;\forall\; n \in \mathbb{N}$. *It is both increasing and decreasing sequence* $a_1 = a_2 = a_3 = \ldots$. *It is both bounded above and below by* 1. $\therefore a_n \to a$ *for some* $a \in \mathbb{R}$ *by MCT (Monotone Convergence Theorem). Here* $a = 1$.

Equivalent result

A monotonic sequence converges if and only if it is bounded.

Question: Does a bounded sequence converge?
Answer: No, a counterexample is as follows.

Example 12.1.9. $\{a_n\} = \{1, -1, 1, -1, \ldots\}$
It is bounded but oscillating.

Example 12.1.10. *Suppose the sequence* $\{a_n\}$ *satisfies* $0 < a_n < 1$, $a_n(1 - a_{n+1}) > \frac{1}{4}$ *for all* $n \in \mathbb{N}$. *Establish the convergence of the same and find the limit. Using AM-GM inequality, we have*

$$\frac{a_n + (1 - a_{n+1})}{2} \geq \sqrt{a_n + (1 - a_{n+1})} > 1/2$$

$$a_n - a_{n+1} > 0$$

This implies that the sequence is decreasing and it is bounded. By MCT, the sequence converges to a limit a. The limit is obtained by solving the quadratic $4a^2 - 4a + 1 = 0$ which yields $a = \frac{1}{2}$.

Example 12.1.11. *Show that the sequence $\{a_n\} = \frac{n}{2^n}$, $n \geq 1$ is strictly decreasing and find the limit.*

Proof. $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, a_3 = \frac{3}{8}, \ldots$. We have $a_3 \leq a_2$. Assume $a_n \leq a_{n-1}$ which implies $\frac{n}{2^n} \leq \frac{2(n-1)}{2^n}$. Consider $a_{n+1} = \frac{n+1}{2^{n+1}} = \frac{a_n}{2} + \frac{1}{2^{n+1}} \leq \frac{a_{n-1}}{2} + \frac{1}{2^{n+1}} = a_n - \frac{1}{2^{n-1}} < a_n$. Thus, by induction we have established that $\{a_n\}$ is a strictly decreasing sequence. Since the sequence is decreasing and bounded below, by MCT, it has a limit point, say a.
Next, $a_{n+1} = \frac{(n+1)a_n}{2n}$. For large n, we have $a = \frac{(n+1)a}{2n}$ which implies $na = a$ for all n. Hence, $a = 0$. □

LECTURE-13

13.1 The Squeeze/Sandwhich Rule

Theorem 13.1.1: The Sandwhich/Squeeze Theorem

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be the three sequences in $\mathbb{R}$ such that $a_n \leq b_n \leq c_n \ \forall \ n \geq N_1$ and for some $N_1 \in \mathbb{N}$.

1. If $a_n \to L \ \& \ c_n \to L$, then $b_n \to L$.
2. If $b_n \to \infty$, then $c_n \to \infty$
3. IF $c_n \to -\infty$, then $b_n \to -\infty$.

Proof. Let $\epsilon > 0$ be given. Since $a_n \to L \ \exists \ N_2 \in \mathbb{N}$ s.t $|a_n - L| < \epsilon \ \forall \ n \geq N_2$. Since $c_n \to L \ \exists \ N_3 \in \mathbb{N}$ s.t $|c_n - L| < \epsilon \ \forall \ n \geq N_3$. Since $a_n \leq b_n \leq c_n \ \forall \ n \geq N_1$

$$
\begin{aligned}
& L - \epsilon \leq a_n \leq b_n \leq c_n \leq L + \epsilon \ \forall \ n \geq \max\{N_1, N_2, N_3\} \\
\implies & L - \epsilon \leq b_n \leq L + \epsilon \\
& \implies b_n \to L.
\end{aligned}
$$

□

Corollary 13.1.2

If $\{c_n\}$ is a null sequence of nonnegative real numbers, and $|b_n| \leq c_n \ \forall \ n \geq N$. Then $\{b_n\}$ is a null sequence.

Example 13.1.1. $c_n = \{\frac{1}{\sqrt{n}}\}$
c_n *is*

1. *non-negative.*
2. *null sequence i.e,* $c_n \to 0$.

Consider $\{b_n\} = \frac{1}{\sqrt{n+1}}$.
Now $\left|\frac{1}{\sqrt{n+1}}\right| < \frac{1}{\sqrt{n}}$
$c_n = \{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{5}}, \ldots\}$
$b_n = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{5}}, \ldots\}$

$$b_1 \leq c_1$$
$$b_2 \leq c_2$$
$$\vdots$$

By Corollary 13.1, $b_n \to 0$.

If $a_n \to 0$ & $|b_n - L| \leq a_n \ \forall\ n \geq N$. Then, $b_n \to L$.

Example 13.1.2. $\{a_n\} = \frac{\cos(n^2)}{n}$
$-\frac{1}{n} \leq \frac{\cos(n^2)}{n} \leq \frac{1}{n}$.
Both $-\frac{1}{n}$ *and* $\frac{1}{n}$ *are null sequences. By Squeeze Theorem 13.1,* $a_n \to 0$.

13.2 Subsequences

It is an infinite ordered subset of a sequence. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Let $\{n_k\}_{k\geq 1}$ be a strictly increasing sequence of positive integers. The integers $\{n_k\}$ are called the subsequence indices. Then the sequence $\{a_{n_k}\}_{k\geq 1}$ is called a subsequence of $\{a_n\}_{n\geq 1}$.

Example 13.2.1. $\{a_n\} = \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\}$
$\{b_m\} = \{a_2, a_4, a_7, a_{100}, \ldots\}$. *Then* $\{b_m\}$ *is a subsequence of* $\{a_n\}$. *It can also be written as* $\{a_{n_k}\}$, *where* $n_1 = 2, n_2 = 4, n_3 = 7, n_4 = 100$ *and note* $n_1 < n_2 < n_3 < n_4 < \ldots$ *is strictly increasing.*

Example 13.2.2. *Let* $\{a_n\}$ *sequence. The sequence* $\{a_{7k+1}\}_{k\geq 1} = \{a_8, a_{15}, a_{22}, \ldots\}$ *is subsequence. Here* $n_1 = 8, n_2 = 15, n_3 = 22$ *which is an increasing sequence of positive integers. Thus* $\{a_{7k+1}\}$ *is a subsequence.*

Example 13.2.3. *Let* $\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots\}$. *Define* $\{b_m\} = \{a_2, a_5, \mathbf{a_1}, a_7, \ldots\}$, *not a subsequence.*

Observations

1. Can we say that every subsequences of a convergent sequence must converge and converge to the same limit?

2. Every sequence is a subsequence of itself?

3. If a sequence has convergent subsequences whose limits are different, what can we say about the sequence $\{a_n\}$?

Define $\{a_n\} = (-1)^n \to$ not a convergent sequence. i.e, $\{a_n\} = \{-1, 1, -1, 1, -1, \ldots\}$. Define subsequences

$$a_{2k} = \{1, 1, 1, 1, \ldots\} \to 1$$
$$a_{2k-1} = \{-1, -1, -1, -1, \ldots\} \to -1$$

Both the subsequences converge, but to different limits. Subsequences can be convergent but need not converge to the same limit.

Subsequential limits

Subseqential limits: Let $\{a_n\}$ be a sequence. A subsequential limit is any real number or symbol $+\infty$ & $-\infty$ that is the limit of some subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Example 13.2.4. *Consider $a_n = 1 + (-1)^n$. The subsequence $\{a_{2k}\}_{k\geq 1} = 1 + (-1)^{2k} = \{2, 2, 2, 2, \ldots\} \to 2$ and the subsequence $\{a_{2k-1}\}_{k\geq 1} = \{0, 0, 0, 0, \ldots\} \to 0$. The subsequential limits are $\{0, 2\}$.*

Example 13.2.5. *x Consider $a_n = \{n(-1)^n\}$. It is clear that $+\infty$ & $-\infty$ are the subsequential limits of the sequence $\{a_n\}$.*

Question

If a sequence $\{a_n\}$ converges, what can we say about all its subsequences?

Invariance property of subsequences.

Proposition 13.2.1

If $\{a_n\}$ converges, then every subsequence $\{a_{n_k}\}$ of it converges. Also, if $a_n \to +\infty$, then $a_{nk} \to +\infty$.

Proof. We prove the first part. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Given $a_n \to a$. Given $\epsilon > 0$, $\exists\ N_1 \in \mathbb{N}$ s.t $|a_n - a| < \epsilon\ \forall\ n \geq N_1$. To show $|a_{n_k} - a| < \epsilon$. Choose $k \geq N_1 \implies n_k \geq N_1$ and

$$|a_{n_k} - a| < \epsilon\ \forall\ n_k \geq N_1.$$

□

Corollary 13.2.2

The sequence $\{a_n\}$ is divergent if it has convergent subsequence with different subsequential limits. Also, $\{a_n\}$ is divergent if it has a subsequence that tends to $+\infty$ or has a subsequence that tends to $-\infty$.

Example 13.2.6.

$$\begin{aligned}
a_n &= \sum_{k=1}^{n} \frac{1}{k} \\
a_1 &= 1 \\
a_2 &= 1 + \frac{1}{2} \\
a_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
a_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\vdots \\
a_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n}
\end{aligned}$$

Clearly

1. $a_n > 0 \ \forall \ n$
2. *strictly increasing*

We will show that the sequence diverges by showing that a subsequence $\{a_{n_k}\}$ *of* $\{a_n\}$ *is unbounded.*

$$\begin{aligned}
a_{2^n} &= \sum_{k=1}^{2^n} \frac{1}{k} \\
a_2 &= 1 + \frac{1}{2} && \text{(2nd element of } \{a_n\}\text{)} \\
a_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} && \text{(4th element of } \{a_n\}\text{)} \\
a_8 &= 1 + \frac{1}{2} + \ldots + \frac{1}{8} && \text{(8th element of } \{a_n\}\text{)}
\end{aligned}$$

$$\begin{aligned}
\{a_n\} &= a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots \\
\{a_{n_k}\} = \{a_{2^n}\} &= \underbrace{b_1}_{a_2}, \underbrace{b_2}_{a_4}, \underbrace{b_3}_{a_8}, \ldots \\
\therefore a_{2^n} &= \underbrace{\underbrace{\underbrace{1 + \frac{1}{2}}_{b_1} + \frac{1}{3} + \frac{1}{4}}_{b_2} + \frac{1}{5} + \ldots + \frac{1}{8}}_{b_3} + \ldots + \frac{1}{1 + 2^{n-1}} + \ldots + \frac{1}{2^n}
\end{aligned}$$

$$
\begin{aligned}
a_{2^n} &> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \ldots + \frac{1}{8}) + \ldots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \\
&= 1 + \frac{1}{2}\underbrace{(1 + 1 + 1 \ldots)}_{n\ terms} \\
&= 1 + \frac{n}{2} \quad \longleftarrow \textit{unbounded}
\end{aligned}
$$

$\therefore \{a_{2^n}\}$ *is unbounded which further implies* $\{a_{2^n}\}$ *diverges. This implies* $\{a_n\}$ *also diverges.*

LECTURE-14

14.1 Bolzano-Weierstrass Theorem

Theorem 14.1.1: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence. In other words, if $\{a_n\}$ is a bounded sequence, i.e $|a_n| \leq M \, \forall n \in \mathbb{N}$ and some $M > 0$, then there exists a number, $a \in [-M, M]$ and a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to a$.

The outline of the proof is:

i. It uses the fact that every sequence in $\mathbb{R}$ has a monotonic subsequence.

ii. Invoke monotone convergence theorem (MCT) to the subsequence.

The following Theorem is useful in finding the limit point.

Theorem 14.1.2: If a sequence converges, then the limit must be the least upper bound of the sequence. As a consequence, if the given sequence converges to a, then the limit must satisfy $a_{n+1} = f(a_n)$ and $a = f(a)$.

Example 14.1.1. *Consider, $\{a_n\}$ defined by: $a_1 = 1, a_{n+1} = \sqrt{2 + a_n}, n \geq 1$. $a_n = \{1, \sqrt{3}, \sqrt{2+\sqrt{3}}, \dots\}$. Note $a_n > 0$. We next claim, $\{a_n\}$ is increasing by induction. True for $a_2 > a_1$. Assume it is true for $a_n < a_{n+1}$. To show that $a_{n+2} > a_{n+1} \; \forall \; n \in \mathbb{N}$. Since, $a_{n+1} > a_n \; \forall \; n \in \mathbb{N}$. $a_{n+2} = \sqrt{2 + a_{n+1}} > a_{n+1} \; \forall \; n \in \mathbb{N}$. Therefore $\{a_n\}$ is monotonically increasing sequence. Let a be the limit of the sequence, then, $a = \sqrt{2 + a}, a^2 = 2 + a \implies a = 2, -1$. The root -1 is ruled out. Hence, $a = 2$.*

Let, $\{a_n\}$ be a convergent, say $a_n \to a$. The difference between a_n and a is small for very large $N \in \mathbb{N}$. This also implies that the difference between a_n and a_m for very large $m, n \in \mathbb{N}$ is also small. Give, $\epsilon > 0$, then there exists $m, n \in \mathbb{N}$ such that, $|a_n - a| < \epsilon/2$, $|a_m - a| < \epsilon/2$ which implies $\implies |a_m - a_n| = |a_m - a + a - a_n|$
$\leq |a_m - a| + |a - a_n| < \epsilon/2 + \epsilon/2 = \epsilon$. This leads to the notion of Cauchy sequence.

Definition 14.1.3: Cauchy sequence

A sequence, $\{a_n\}$ is called a Cauchy sequence, if for every $\epsilon > 0$, there is a positive integer $N \in \mathbb{N}$ such that for all $m, n \geq N$, it follows that $|x_n - x_m| < \epsilon$.

Example 14.1.2. $a_n = \{1/n\}$ *is Cauchy.*
$|1/m - 1/n| < 1/m < \epsilon$ *for all* $m > \left\lceil \frac{1}{\epsilon} \right\rceil + 1$.

Example 14.1.3. $a_n = 1/n$
$a_n = \{1, 1/2, 1/3, 1/4 \dots\} \rightarrow$ *decreasing.*
As we go closer to the limit point, the terms of the sequence are close to each other.

What can we say if a sequence is convergent?

Proposition 14.1.4

Every convergent sequence is Cauchy.

Proof. Given, $\epsilon > 0$, $\exists\ N \in \mathbb{N}$, such that $|a_n - a| < \epsilon/2\ \forall\ n \geq N$. For $m, n \geq N$, $|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \epsilon/2 + \epsilon/2 = \epsilon$. □

Proposition 14.1.5

Every Cauchy sequence is bounded.

Proof. Let, $\{x_n\}$ be Cauchy. Since $\{x_n\}$ is Cauchy, fix $\epsilon = 1$. Then there exists $N_1 \in \mathbb{N}$ such that, $\forall\, n, m \geq N_1$, $|x_n - x_m| < \epsilon = 1$. Consider, $|x_n| = |x_n - x_m + x_m| \leq |x_n - x_m| + |x_m| < 1 + |x_m|$. Take $m = N_1$. Then, $|x_n| < 1 + |x_{N_1}|$. □

Completeness criterion for a sequence

Theorem 14.1.6: A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. Assume, $\{a_n\} \subset \mathbb{R}$ is Cauchy. To show, $\{a_n\}$ is convergent. Let $\epsilon > 0$. Since $\{a_n\}$ is Cauchy, there $\exists\ N \in \mathbb{N}$ such that $\forall\, n, m \geq N, |a_n - a_m| < \epsilon/2$. Also $\{a_n\}$ is bounded. By BWT, $\{a_n\}$ has a convergent subsequence. That is, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that, $\{a_{n_k}\} \rightarrow a \in \mathbb{R}$. Then, $\exists\ K \in \mathbb{N}$ such that, $|a_{n_k} - a| < \epsilon/2\ \forall\ k > K$. Let $m = n_k$. Then, $|a_n - a_{n_k}| < \epsilon/2\ \forall\ n > n_k \geq N$. Choose, k large such that $k > K$ and $n_k > N$. Then, $|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon/2 + \epsilon/2 = \epsilon\ \forall\ n \geq N$. □

Caution: This result is not necessarily true if the space is non-Euclidean.

LECTURE-15

15.1 Limit superior and limit inferior

Consider a sequence in $\mathbb{R}$, say $\{a_n\}$.Then for each $k \in \mathbb{N}$, let

$$\begin{aligned} M_1 &= \sup\{a_1, a_2, \dots\} \\ M_k &= \sup\{a_k, a_{k+1}, \dots\} \\ &= \sup\{a_n : n \geq k\}. \end{aligned}$$

Observe that, $M_k \geq M_{k+1}$, that is, $\{M_k\}$ is a decreasing sequence. $\{M_k\} = +\infty$ if the sequence is not bounded above. Define,

$$\begin{aligned} m_1 &= \inf\{a_1, a_2, \dots, \} \\ m_k &= \inf\{a_k, a_{k+1}, \dots\} \\ &= \inf\{a_n : n \geq k\}. \end{aligned}$$

Since $m_k \leq m_{k+1}$ i.e $\{m_k\}$ is an increasing sequence, the sequence $\{m_k\} = -\infty$ if the sequence is not bounded below.

Example 15.1.1. *Consider the sequence* $A = \{-1, 5, 2, -3, 4, 1, \dots\}$. *Then,* $M_1 = 5, M_2 = 5, M_3 = 4, M_4 = 4, M_5 = 4, M_6 = 1 = M_7 = \dots$. *Thus* $\{M_k\} = \{5, 5, 4, 4, 4, 1, 1, \dots\}$ *is a decreasing sequence.*
Similarly, $m_1 = \inf\{-1, 5, 2, -3, 4, 1, \dots\} = -1, m_2 = -3, m_3 = -3, m_4 = 1, m_5 = 4$. *Thus* $\{m_k\} = \{-1, -3, -3, 1, 1, 1 \dots\}$, *an increasing sequence. Thus,* $m_1 \leq m_2 \leq \dots m_{k+1} \leq M_{k+1} \leq M_k \dots \leq M_2 \leq M_1$. *We have,* $\{M_k\}$ *is a decreasing sequence in* $\mathbb{R}$ *and* $\{m_k\}$ *as an increasing sequence in* $\mathbb{R}$. *Since,* $\{M_k\}$ *and* $\{m_k\}$ *are monotonic,*

$$\left.\begin{aligned} M &= \lim_{k \to \infty} M_k \\ m &= \lim_{k \to \infty} m_k \end{aligned}\right\} \textit{exists (proper and improper limit).}$$

A Proper limit means that the limit is in $\mathbb{R}$. An Improper limit means that the limit is $+\infty$ or $-\infty$.

M is called the limit superior of $\{a_n\}$ and m is called the limit inferior of $\{a_n\}$.

$$\begin{aligned}
M &= \limsup_{n\to\infty} a_n \\
m &= \liminf_{n\to\infty} a_n \\
M &= +\infty \text{ if } \{a_n\} \text{ is not bounded above.} \\
M &= -\infty \text{ if } \lim_{n\to\infty} a_n = -\infty \\
m &= -\infty \text{ if } \{a_n\} \text{ is not bounded below.} \\
m &= +\infty \text{ if } \lim_{n\to\infty} a_n = +\infty.
\end{aligned}$$

Example 15.1.2. *Let* $\{a_n\} = \left\{\frac{1}{n}\right\}$

$$\begin{aligned}
M_1 &= \sup\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = 1 \\
M_2 &= \sup\{\frac{1}{2}, \frac{1}{3}, \ldots\} = \frac{1}{2} \\
&\vdots \\
M_k &= \frac{1}{k}
\end{aligned}$$

$\therefore M = \lim_{k\to\infty} M_k = 0$. *Similarly,*

$$\begin{aligned}
m_1 &= \inf\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = 0 \\
m_2 &= \inf\{\frac{1}{2}, \frac{1}{3}, \ldots\} = 0 \\
&\vdots \\
m_k &= \inf\{1/k, 1/k+1, \ldots\} = 0
\end{aligned}$$

$\therefore m = \lim_{k\to\infty} m_k = 0$. *In this case,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 0$.

Theorem 15.1.1

For any sequence of real numbers, $\{a_n\}$, we have $\lim_{n\to\infty} a_n = a$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a.$$

Example 15.1.3. *Let*

$$a_n = \begin{cases} n, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$$

$$a_n = \{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots\}$$

It is easy to see that $M_k = \infty$ *and thus* $M = \infty$*, while* $m = 0$*. Therefore* $\limsup_{n\to\infty} a_n = \infty$ *and* $\liminf_{n\to\infty} a_n = 0$*. Therefore,* $\{a_n\}$ *does not converge (the sequence diverges).*

Equivalent interpretation of Limsup and Liminf

$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} M_k = \inf_{k\geq 1} \sup_{n\geq k} a_n$
$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} m_k = \sup_{k\geq 1} \inf_{n\geq k} a_n.$

Example 15.1.4. $\{x_n\}$, $x_n = \dfrac{(-1)^n}{n}$

$$\begin{aligned} M_k &= \sup\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \frac{(-1)^{k+2}}{k+2}, \ldots\right\} \\ &= \sup\left\{\frac{(-1)^k}{k}, \frac{-(-1)^k}{k+1}, \ldots\right\} \\ &= \begin{cases} \dfrac{1}{k+1}, & k \text{ is odd} \\ \dfrac{1}{k} & k \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} m_k &= \inf\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \ldots\right\} \\ &= \inf\left\{\frac{(-1)^k}{k}, \frac{-(-1)^k}{k+1}, \ldots\right\} \\ &= \begin{cases} \dfrac{-1}{k}, & k \text{ is odd} \\ \dfrac{-1}{k+1}, & k \text{ is odd} \end{cases} \end{aligned}$$

$\lim_{k\to\infty} M_k = \lim_{k\to\infty} m_k = 0 \implies \{x_n\}$ *converges with limit equal to zero.*

Example 15.1.5. *Consider* $\{a_n\}$, $a_n = (-1)^n \frac{n+1}{n}$

$$
\begin{aligned}
M_1 &= \sup\left\{-2, \frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \ldots\right\} = \frac{3}{2} \\
M_2 &= \sup\left\{\frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \ldots\right\} = \frac{3}{2} \\
M_3 &= \sup\left\{\frac{-4}{3}, \frac{5}{4}, \ldots\right\} = \frac{5}{4} \\
M_4 &= \sup\left\{\frac{5}{4}, \frac{-6}{5}, \ldots\right\} = \frac{5}{4} \\
M_k &= \frac{k+1}{k} = 1 + \frac{1}{k} \\
m_1 &= \inf\left\{-2, \frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \frac{-6}{5}, \ldots\right\} = -2 \\
m_2 &= \inf\left\{\frac{3}{2}, \frac{-4}{3}, \frac{5}{4} \ldots\right\} = \frac{-4}{3} \\
&\vdots \\
m_k &= -\frac{k+1}{k}
\end{aligned}
$$

$M = \lim_{k\to\infty} M_k = 1$, $m = \lim_{k\to\infty} m_k = -1$
Let us see what is the supremum and infinum of $\{a_n\}$.

$$
\begin{aligned}
\sup_{n\geq 1} a_n &= \frac{3}{2} \\
\inf_{n\geq 1} a_n &= -2.
\end{aligned}
$$

Here, $\lim_{n\to\infty} \sup a_n$ *and* $\sup(a_n)$ *are not be confused as same. Similarly* $\lim_{n\to\infty} \inf a_n$ *and* $\inf(a_n)$.
The plot of a_n *is shown in Figure 15.5.*

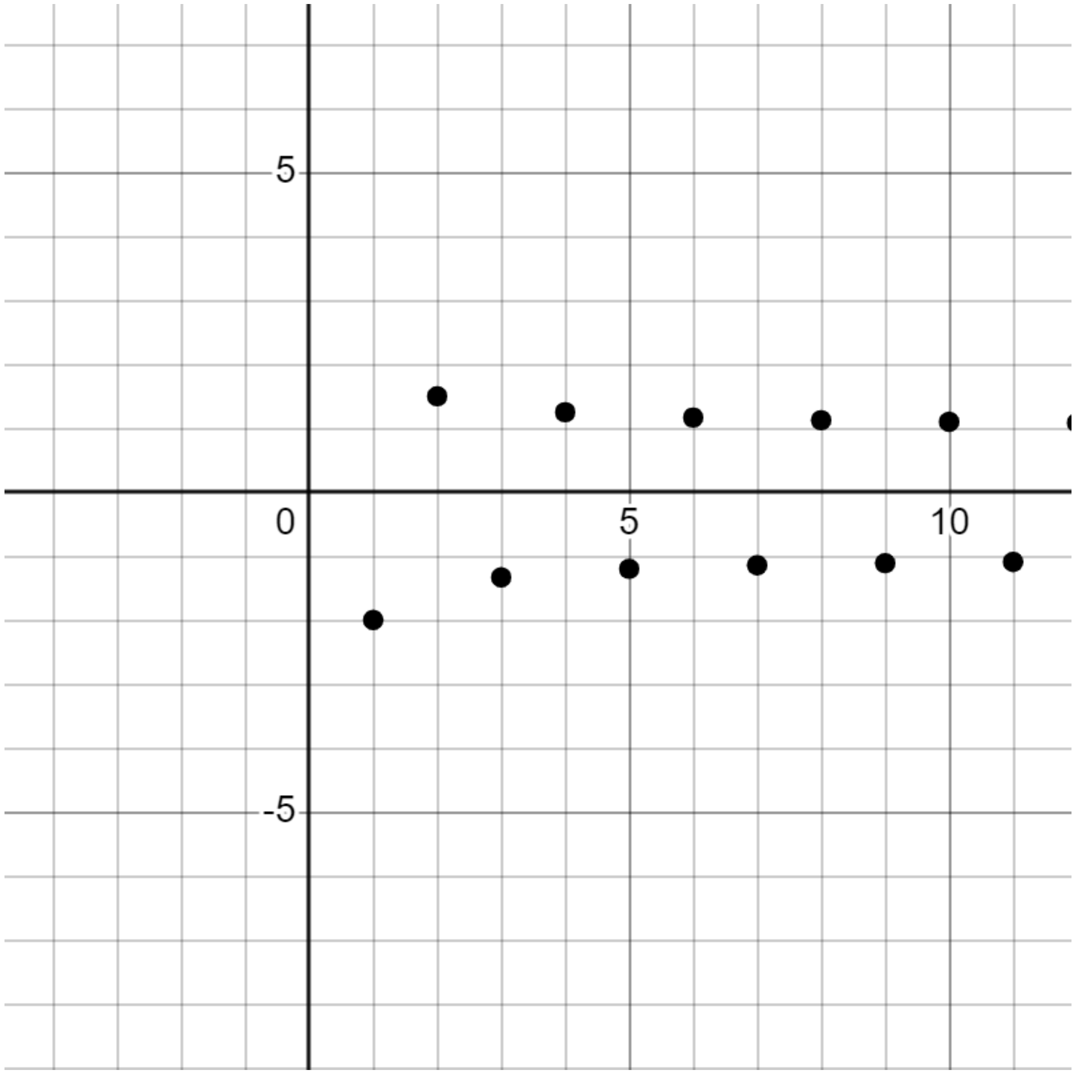


Figure 15.5: Ex. $a_n = (-1)^n \frac{n+1}{n}$: Even though the sequence is not converging, the limsup and liminf exists

We end this lecture with a very useful interpretation of limsup and liminf.

Proposition 15.1.2

Suppose $\{a_n\} \subset \mathbb{R}$ with $L = \limsup_{n\to\infty} a_n$ and $l = \liminf_{n\to\infty} a_n$. Then for every $\epsilon > 0$ there exists integers N_1 and N_2 such that

$$\begin{aligned} a_n - L &< \epsilon \text{ for all } \quad n \geq N_1 \\ a_n - L &> -\epsilon \text{ for infintely many } \quad n \geq N_1 \end{aligned}$$

and

$$\begin{aligned} a_n - l &> -\epsilon \text{ for all } \quad n \geq N_2 \\ a_n - l &< \epsilon \text{ for infintely many } \quad n \geq N_2 \end{aligned}$$

Proof. We have $L = \lim_{k\to\infty} M_k$. Given $\epsilon > 0$ there exists integer N_1 s.t $|M_k - L| < \epsilon$ for all $k \geq N_1$. In other words $L - \epsilon < M_k < L + \epsilon$ or

$$a_k \leq \sup\{a_k, a_{k+1}, \ldots\} = M_k < L + \epsilon$$

for all $k \geq N_1$. Since $\{M_k\}$ is decreasing sequence, $L \leq \sup_{k\geq 1} M_k$. In particular, $L \leq \sup\{a_1, a_2, \ldots\}$. By definition of sup, there exists n_1 such that $a_{n_1} > M_1 - \epsilon > L - \epsilon$. Taking $k = n_1$ we obtain that $L \leq M_{n_1}$. So, there exists n_2 such that $a_{n_2} > M_{n_1} - \epsilon > L - \epsilon$. Proceeding indefinitely, we obtain integers $n_1 < n_2 < n_2 \cdots n_k < \cdots$ such that $a_{n_k} > L - \epsilon$ for all $k \in \mathbb{N}$. We have $l = \lim_{k\to\infty} m_k$. Given $\epsilon > 0$ there exists integer N_2 s.t $|m_k - l| < \epsilon$ for all $k \geq N_2$. In other words $l - \epsilon < m_k < l + \epsilon$ or

$$a_k \geq \inf\{a_k, a_{k+1}, \ldots\} = m_k < l + \epsilon$$

for all $k \geq N_2$. Since $\{m_k\}$ is increasing sequence, $l \geq \inf_{k\geq 1} m_k$. In particular, $l \geq \inf\{a_1, a_2, \ldots\}$. By definition of inf, there exists n_1 such that $a_{n_1} < m_1 + \epsilon < l + \epsilon$. Taking $k = n_1$ we obtain that $l \geq m_{n_1}$. So, there exists n_2 such that $a_{n_2} < m_{n_1} + \epsilon < l + \epsilon$. Proceeding indefinitely, we obtain integers $n_1 < n_2 < n_2 \cdots n_k < \cdots$ such that $a_{n_k} < l - \epsilon$ for all $k \in \mathbb{N}$. $\square$

LECTURE-16

16.1 Arithmetic on the Extended Real Line

We work with the extended real line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Basic conventions

$$a + (+\infty) = +\infty, \qquad a + (-\infty) = -\infty \quad \text{for all } a \in \overline{\mathbb{R}},\ a \neq -\infty, +\infty,$$
$$(+\infty) + (+\infty) = +\infty, \qquad (-\infty) + (-\infty) = -\infty.$$

Undefined sums:

$$(+\infty) + (-\infty) \quad \text{is left undefined.}$$

Multiplication

For $a \in \mathbb{R}, a \neq 0$:

$$a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0, \end{cases} \qquad a \cdot (-\infty) = \begin{cases} -\infty & \text{if } a > 0, \\ +\infty & \text{if } a < 0. \end{cases}$$

$$(+\infty) \cdot (+\infty) = +\infty, \quad (-\infty) \cdot (-\infty) = +\infty, \quad (+\infty) \cdot (-\infty) = -\infty.$$

Undefined products:

$$0 \cdot (\pm\infty) \quad \text{are left undefined.}$$

Order

The order extends naturally:

$$-\infty < a < +\infty \quad \text{for all } a \in \mathbb{R}.$$

16.2 Some more results on limsup and liminf

Example 16.2.1.

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Then

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + + \limsup_{n\to\infty} b_n$$

provided the RHS is not of the form $\infty - \infty$.

Proof. Define

$$\begin{aligned} g_k &= \sup\{a_k, a_{k+1}, \ldots\} \\ h_k &= \sup\{b_k, b_{k+1}, \ldots\} \\ p_k &= \sup\{(a_k+b_k), (a_{k+1}+b_{k+1}), \ldots\}. \end{aligned}$$

By definition $a_n \leq g_k$ and $b_n \leq h_k$ for all $n \geq k$. Thus $a_n + b_n \leq g_k + h_k \;\; \forall\, n \geq k$. Taking sup on the LHS gives

$$\sup_{n\geq k}(a_n+b_n) = p_k \leq g_k + h_k \;\; \forall\, n \geq k.$$

As this holds for all k, letting $k \to \infty$ gives

$$\begin{aligned} \lim_{k\to\infty} p_k \leq \lim_{k\to\infty}(g_k+h_k) &= \lim_{k\to\infty} g_k + \lim_{k\to\infty} h_k \\ \limsup_{n\to\infty}(a_n+b_n) &\leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \end{aligned}$$

□

Example 16.2.2. *Give an example where* $\limsup_{n\to\infty}(a_n+b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

Proof. Let $a_n = (-1)^n, b_n = (-1)^{n+1}$. Then $\sup(a_n) = 1, \sup(b_n) = 1$, while $\sup(a_n+b_n) = 0$. □

Example 16.2.3. *Show that* $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$

Proof. We have $\liminf_{n\to\infty}(a_n) = \sup_{n\geq 1}\inf_{k\geq n}(a_k)$. Consider

$$\liminf_{n\to\infty}(-a_n) = \sup_{n\geq 1}\inf_{k\geq n}(-a_k) = \sup_{n\geq 1}(-\sup_{k\geq n}(a_k)) = -\inf_{n\geq 1}\sup_{k\geq n}(a_k) = -\limsup_{n\to\infty}(a_n)$$

□

Example 16.2.4. *Let a_n be a sequence s.t $\liminf_{n\to\infty}(a_n) = L$. Then for any convergent subsequence p_n of a_n, we have $\lim_{n\to\infty} p_n \geq L$.*

Proof. We have $\liminf_{n\to\infty}(a_n) = L = \sup_{n\geq 1}\inf_{k\geq n}(a_k) = \sup_{n\geq 1} z_n$, where $z_n = \inf_{k\geq n}(a_k)$. Given $\epsilon > 0, \exists\ N \in \mathbb{N}$ s.t $z_n \geq L - \epsilon\ \ \forall\, n \geq N$. This implies $\inf_{k\geq n}(a_k) > L - \epsilon\ \ \forall\, n \geq N$, which further implies $a_n > L - \epsilon\ \ \forall\, n \geq N$. Now $p_k = s_{n_k} > L - \epsilon\ \ \forall\, k$ s.t $n_k \geq N$. Thus

$$L - \epsilon < p_k$$
$$L - \epsilon < \liminf_{k\to\infty} p_k = \lim_{k\to\infty} p_k = p.$$

Since $\epsilon > 0$ is arbitrary, $L \leq p$. □

Example 16.2.5. *Let a_n be a sequence and S be the set of all subsequential limit points of a_n. Then $\liminf_{n\to\infty}(a_n) = \inf(S)$.*

Proof. Let $\liminf_{n\to\infty}(a_n) = L$. Then, by earlier example, for any convergent subsequence $\{c_n\}$ of $\{a_n\}$, we have $\lim_{n\to\infty} c_n \geq L$. Taking infimum over all subsequential limits we get

$$\inf(S) \geq \liminf_{n\to\infty}(a_n). \tag{16.2}$$

Define $p_k = \inf\{a_n : n \geq k\}$. Then, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ s.t $p_k \leq a_{n_k} \leq p_k + \frac{1}{k}$. By squeeze theorem, the subsequence a_{n_k} converges to $\lim_{k\to\infty} p_k$. This means the limit infimum of our sequence a_n is the limit for some convergent subsequence. That is, the limit infimum is a part of the set S. This gives

$$\liminf_{n\to\infty}(a_n) \geq \inf(S) \tag{16.3}$$

From (16.2)-(16.3), we get the equality. □

Example 16.2.6. *Show that a sequence s_n converges to a limit L if and only if $\liminf_{n\to\infty}(s_n) = \limsup_{n\to\infty}(a_n) = L$.*

Proof. Let $\liminf_{n\to\infty}(s_n) = \limsup_{n\to\infty}(s_n) = L$. Let $a_n = \inf_{k\geq n} s_k, b_n = \sup_{k\geq n} s_k$. Then for any n we have $a_n \leq s_n \leq b_n$. Letting $n \to \infty$, $L \leq s_n \leq L$. By squeeze Theorem, $s_n \to L$.
Conversely, let $s_n \to L$. This implies that every subsequence of s_n also converges to L. That is the set of all subsequential limits of s_n is L, that is $\{S\} = L$. From previous examples, $\limsup_{n\to\infty} s_n = \sup(S) = L = \inf(S) = \liminf_{n\to\infty} s_n$. □

Proposition 16.2.1

Least upperbound property

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Let $A = \sup\{a_n : n \geq 1\} \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Let $M \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ be an upper bound for $\{a_n\}$. Further, let $B \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ such that $B < A$. Then

- $a_n \leq A$ for all $n \geq 1$.

- $A \leq M$
- There exists atleast one $n \geq 1$ for which $B < a_n \leq A$.

Proposition 16.2.2

Greatest lower bound property

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Let $A = \inf\{a_n : n \geq 1\} \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Let $M \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ be a lower bound for $\{a_n\}$. Further, let $B \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ such that $B > A$. Then

- $a_n \geq A$ for all $n \geq 1$.
- $A \geq M$
- There exists atleast one $n \geq 1$ for which $B > a_n \geq A$.

Theorem 16.2.3

Let $\{a_n\}$ be a sequence of real numbers.

- For every $\limsup_{n\to\infty}(a_n) < a$, there exists an $N \geq 1$ such that $a_n < a$ for all $n \geq N$. In other words, for every $\limsup_{n\to\infty}(a_n) < a$, the elements of the sequence $\{a_n\}$ are eventually less than a. Similarly, for every $\liminf_{n\to\infty}(a_n) > l$, there exists an $N \geq 1$ such that $a_n > L$ for all $n \geq N$.
- For every $\limsup_{n\to\infty}(a_n) > a$, and every $N \geq 1$, there exists an $n \geq N$ such that $a_n > a$. In other words for every $\limsup_{n\to\infty}(a_n) > a$, the elements of the sequence $\{a_n\}$ exceed a infintely often. Similarly, for every $\liminf_{n\to\infty}(a_n) < l$ and every $N \geq 1$, there exists an $n \geq N$ such that $a_n < l$.

Proof. i. $\limsup_{n\to\infty}(a_n) < a$.

We have

$$\limsup_{n\to\infty}(a_n) < a$$
$$\inf_{n\geq 1}\sup_{k\geq n} a_k < a$$
$$\inf_{N\geq 1} z_N < a, \quad \text{where} \quad z_N = \sup_{k\geq N} a_k.$$

$\{z_N\}$ is a non-increasing sequence. We have $\inf_{N\geq 1} z_N < a$. By GLB property, there exists atleast one $N \geq 1$ s.t $z_N < a$. This means $\sup_{k\geq N} a_k < a$, which implies by LUB property $a_n < a$ for all $n \geq N$ (eventually).

ii. $\limsup_{n\to\infty}(a_n) > a$.
We have

$$\begin{aligned}\limsup_{n\to\infty}(a_n) &> a\\ \inf_{n\geq 1}\sup_{k\geq n} a_k &> a\\ \inf_{N\geq 1} z_N &> a, \quad \text{where } \; z_N = \sup_{k\geq N} a_k.\end{aligned}$$

$\{z_N\}$ is a non-increasing sequence. Now $\inf_{N\geq 1} z_N > a$ by GUB property implies $z_N > a \;\; \forall\, N$. This further implies $\sup_{k\geq N} a_k > a$. Then by LUB property imples that there exists atleast one $n \geq N$ s.t $a_n > a$ (infintely often).

iii. $\liminf_{n\to\infty}(a_n) < l$.
We have

$$\begin{aligned}\liminf_{n\to\infty}(a_n) &< l\\ \sup_{n\geq 1}\inf_{k\geq n} a_k &< l\\ \sup_{N\geq 1} z_N &< l, \quad \text{where } \; z_N = \inf_{k\geq N} a_k.\end{aligned}$$

$\{z_N\}$ is a non-decreasing sequence. Now $\sup_{N\geq 1} z_N < l$ implies by LUB property $z_N < l \;\forall\, N$. This further implies $\inf_{k\geq N} a_k < l$. Then by GLB property there exists atleast one $n \geq N$ s.t $a_n < l$

iv. $\liminf_{n\to\infty}(a_n) > l$.
We have

$$\begin{aligned}\liminf_{n\to\infty}(a_n) &> l\\ \sup_{n\geq 1}\inf_{k\geq n} a_k &> l\\ \sup_{N\geq 1} z_N &> l, \quad \text{where } \; z_N = \inf_{k\geq N} a_k.\end{aligned}$$

$\{z_N\}$ is a non-decreasing sequence. Now $\sup_{N\geq 1} z_N > l$. Then by LUB property, there exists atleast one $N \geq 1$ s.t $z_N < l$. This means $\inf_{k\geq N} a_k > l$. This further implies by GUB property that $a_n > l$ for all $n \geq N$ (eventually).

□

Suppose $a_n \leq b_n$ and $a_n \to a$. Then $\lim_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$. If $a_n \geq b_n$, then use $\lim_{n\to\infty} A_n \geq \limsup_{n\to\infty} b_n$

LECTURE-17

17.1 Series

Definition 17.1.1: An infinite series is any expression of the form $\sum_{i=1}^{\infty} a_i$, where $a_i \in \mathbb{R}$.

17.1.1 Conditional Convergent Series

Definition 17.1.2: Consider the sequence $\sum_{i=1}^{N} a_i$. For any $N \geq 1$, define the N^{th} partial sum S_N of this series to be $S_N = \sum_{i=1}^{N} a_i$.

- **Note S_N is a real number.**
- **If the sequence $\{S_N\}_{N=1}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that this infinite series $\sum_{i=1}^{\infty} a_i$ is convergent and converges to L. We also say $L = \sum_{i=1}^{\infty} a_i$ and L is the sum of the infinite series.**
- **If $\{S_N\}_{N=1}^{\infty}$ diverges, then we say this series $\sum_{i=1}^{\infty} a_i$ is divergent.**

Example 17.1.1. *Consider*

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n} &= 2^{-1} + 2^{-2} + 2^{-3} + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\end{aligned}$$

The partial sum is

$$S_N = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{N}$$
$$= 1 - 2^{-N}.$$

Now consider the sequence $\{S_N\}_{N=1}^{\infty}$. The sequence converges to 1. Hence the series $\sum_{n=1}^{\infty} 2^{-n}$ is convergent with sum equal to 1. On the other hand, consider the series

$$\sum_{n=1}^{\infty} 2^n = 2 + 2^2 + 2^3 + \cdots$$

The partial sum is

$$\begin{aligned} S_N &= 2 + 2^2 + \cdots + 2^N \\ 2S_N &= 2^2 + 2^3 + \cdots + 2^{N+1} \\ &= \underbrace{2^2 + 2^3 + \cdots + 2^N}_{(S_N - 2)} + 2^{N+1} \\ 2S_N &= (S_N - 2) + 2^{N+1} \\ S_N &= 2^{N+1} - 2. \end{aligned}$$

The sequence $\{S_N\}_{N=1}^{\infty}$ diverges since it is unbounded. Thus the series $\sum_{n=1}^{\infty} 2^n$ is divergent.

The following proposition shows that a series converges if and only if the 'tail' of the series is eventually less that ϵ for any $\epsilon > 0$.

Proposition 17.1.3

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Then, $\sum_{n=1}^{\infty} a_n$ converges if and only if, for every real number $\epsilon > 0$, there exists an integer $N \geq 1$ s.t $|S_{n+p} - S_n| = |\sum_{k=1}^{p} a_{n+k}| < \epsilon \ \forall \ n \geq N$ and $p \geq 1$.

Tests for Convergence:

Zero test:

Let, $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then we must have $\lim\limits_{n \to \infty} a_n = 0$.
In other words, if $\lim\limits_{n \to \infty} a_n$ is non-zero or divergent, then the series,$\sum_{n=1}^{\infty} a_n$ is divergent.

Example 17.1.2.

$$\{a_n\}, a_n = \{1, 1, 1 \cdots\}$$
$$a_n \to 1$$

Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 17.1.3. $\{a_n\}, a_n = (-1)^n$
The sequence diverges, hence the series $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 17.1.4. $\{a_n\}, a_n = \frac{1}{n}$
The sequence, $\{a_n\}$ *converges to zero. But,* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*
Note: If the sequence converges to zero, the series $\sum a_n$ *may or may not be convergent.*

Example 17.1.5. $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a.r^n, r \geq 0$ *is a Geometric sequence.*
$a\rightarrow$ *First term.*
$r\rightarrow$ *Common ratio.*
$\frac{a_{n+1}}{a_n} = \frac{a.r^{n+1}}{a.r^n} = r$ *(Independent of n).*
Partial Sum

$$\begin{aligned} S_n &= a(1 + r + r^2 + \cdots + r^n) \\ r.S_n &= a(r + r^2 + r^3 + \cdots + r^{n+1}) \\ &= a(r + r^2 + \cdots + r^n) + a.r^{n+1} \\ r.S_n &= (S_n - a) + a.r^{n+1} \\ S_n(r-1) &= -a + a.r^{n+1} \\ S_n &= \frac{a.r^{n+1} - a}{r-1} = \frac{a(r^{n+1} - 1)}{r-1} \end{aligned}$$

If $0 \leq r < 1$*, then* $\lim_{n\to+\infty} r^{n+1} = 0$ *and* $S_n \rightarrow \frac{a}{1-r}$*. The series converges with sum* $\frac{a}{1-r}$*. If* $r > 1, r^{n+1}$ *is unbounded and the series is divergent.*

Definition 17.1.4: Absolute Convergence: Let $\sum a_n$ be a series of real numbers. We say this series is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Proposition 17.1.5

(Absolute Convergence Test): Let, $\sum a_n$ be a series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent . Furthermore, in this case we have the triangle inequality, $|\sum_{n=1}^{\infty} a_n| < \sum_{n=1}^{\infty} |a_n|$.

NOTE: The converse of this proposition is not true.

LECTURE-18

Series Laws

1. Let $\sum a_n$ and $\sum b_n$ be convergent series such that $\sum a_n = a$ and $\sum b_n = b$. Then the $\sum(a_n + b_n) = \sum a_n + \sum b_n$ and converges to $a + b$.
2. Let $\sum a_n$ be a convergent series such that $\sum a_n = a$. Let $c \in \mathbb{R}$. Then, $\sum c a_n = c \sum a_n = ca$.
3. Let $\sum a_n$ be a convergent series. Let $k \in \mathbb{N}$. If one of the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=k}^{\infty} a_n$ are convergent, then the other is also convergent and we have the identity $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{\infty} a_n$.
4. Let $\sum a_n$ be a convergent series converging to a and let $k \in \mathbb{N}$. Then $\sum_{n=k}^{\infty} a_{n-k}$ also converges to a.

Proposition 18.0.6

Telescopic Series: Let $\{a_n\}$ be a sequence of real numbers which converge to zero. Then the series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ converges to a_0.

Example 18.0.6. *Let* $\frac{5}{1*3} + \frac{5}{2*4} + \frac{5}{3*5} + \ldots = \sum_{n=1}^{\infty} \frac{5}{n(n+2)}$

$$\frac{5}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

Substituting $n = 0$ *and* $n = 1$ *we get* $A = \frac{5}{2}$ *and* $B = \frac{-5}{2}$

.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5}{n(n+2)} &= \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} \\
&= \frac{5}{2}\{1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \ldots\} \\
&= \frac{5}{2}\{1 + \frac{1}{2}\} = \frac{15}{4}.
\end{aligned}$$

Proposition 18.0.7

Alternating Series Test: Let $\{a_n\}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}\ \forall\ n \geq 1$. Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent if and only if the sequence $\{a_n\}$ converges to zero.

Example 18.0.7. $\{a_n\} = \frac{1}{n}$. *The sequence* $\{a_n\}$ *is non-negative and decreasing with* $\lim_{n\to\infty} a_n \to 0$. *Consider the series* $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$. *By alternating series test, this series is convergent.*

18.1 Sums of non-negative numbers

Consider $\sum a_n$, where $a_n \geq 0\ \forall\ n \geq 1$. Then, the partial sums S_N are increasing. i.e, $S_{N+1} \geq S_N\ \forall\ N \geq 1$.Therefore, $\{S_N\}$ is an increasing sequence. If it is upper bounded then, the sequence $\{S_N\}$ is convergent.

Proposition 18.1.1

Let $\sum a_n$ be a series of non-negative real numbers. Then, this series is convergent if and only if there is a real number M s.t $\sum_{n=1}^{N} a_n \leq M\ \forall\ N \geq 1$.

Corollary 18.1.2

Comparison Test: Let $\sum a_n$ and $\sum b_n$ be two series of real numbers with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

- $\sum a_n$ is convergent if $\sum b_n$ is convergent.
- If $\sum a_n$ is divergent then so is $\sum b_n$.

Example 18.1.1. *Prove that the series* $\sum \frac{n}{n+3}x^n$ *converges for each* $x \in [0,1)$. *Fix* $x \in [0,1)$. *Set* $a_k(x) = \frac{kx^k}{k+3}$. *Note*

- $a_k(x) \geq 0$, *for each fixed* $x \in [0,1)$.
- $\frac{k}{k+3} < 1\ \forall\ k \geq 1$.

Hence, $a_k(x) \leq x^k\ \forall\ x \in [0,1)$.

Thus, for $x \in [0,1)$

$$\begin{aligned} 0 \le S_n = \sum_{k=1}^{n} a_k(x) &\le \sum_{k=1}^{n} x^k \\ = \frac{x(1-x^n)}{1-x} &\le \frac{x}{1-x} := M \end{aligned}$$

Thus, $\{S_n\}$ *is increasing and bounded above. This implies* $\{S_n\}$ *converges* $\implies \sum \frac{n}{n+3} x^n$ *converges for* $x \in [0,1)$.

LECTURE-19

19.1 Tests for convergence

19.1.1 Ratio Test (D'Alembert)

Theorem 19.1.1: Consider $\sum a_n$, $a_n > 0\ \forall\ n$. Consider the ratio $\dfrac{a_{n+1}}{a_n} = r_n$ and let $L = \lim\limits_{n\to\infty} |r_n|$. Then,

1. **$\lim\limits_{n\to\infty} r_n$ does not exist. Then this test is inconclusive.**
2. **If the $\lim_{n\to\infty} r_n$ exists and if $L < 1$, then the series $\sum a_n$ converges.**
3. **If $L > 1$, then the series diverges.**
4. **If $L = 1$, the test is inconclusive.**

Proof. Assume $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r_n = L < 1$ (Limit exists).

- Note $L \geq 0$ since, $a_n \geq 0\ \forall\ n$. Choose an interval (R_1, R_2) s.t $L \in (R_1, R_2)$ and $L < R_2 < 1$, $0 < R_1 < L$.
- $r_n = \dfrac{a_{n+1}}{a_n} \to L$. By definition of convergence of sequence, $\exists\ N \in \mathbb{N}$ s.t $\forall\ n \geq N$, $r_n \in (R_1, R_2)$
- Now, consider the sequence $\{r_n\}$ after N. That is consider the series $\sum_{n=N}^{\infty} a_n$.

Now $\dfrac{a_{N+1}}{a_N} < R_2 < 1$

i.e,

$$\begin{aligned}
a_{N+1} &< R_2 a_N \\
a_{N+2} &< R_2 a_{N+1} = R_2^2 a_N \\
a_{N+3} &< R_2^3 a_N \\
&\vdots \vdots \\
\text{say } a_N &= a \\
a_{N+1} &= R_2 a \\
a_{N+2} &= R_2^2 a \\
&\vdots \\
\therefore \sum_{i=1}^{\infty} a_{N+i} &< \sum_{n=1}^{\infty} R_2^n a
\end{aligned}$$

Now $\sum_{n=1}^{\infty} R_2^n a$ is a convergent geometric series since $R_2 < 1$.
By Comparison test the series $\sum_{i=1}^{\infty} a_{N+i}$ converges.
Now

$$\sum_{i=1}^{\infty} a_i = \underbrace{\sum_{i=1}^{N} a_i}_{\textbf{finite sum and hence convergent}} + \underbrace{\sum_{i=N+1}^{\infty} a_i}_{\textbf{convergent}}$$

$\therefore \sum_{i=1}^{\infty} a_i$ is also convergent. $\square$

Example 19.1.1. *Let $x \in \mathbb{R}$. Consider $s_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$. i.e, $\lim_{n\to\infty} s_n = e^x$.*

$$\begin{aligned}
a_n &= \frac{x^n}{n!} \\
a_{n+1} &= \frac{x^{n+1}}{(n+1)!} \\
\frac{a_{n+1}}{a_n} &= \frac{x^{n+1}}{(n+1)!}.\frac{n!}{x^n} = \frac{x}{n+1} \\
\lim_{n\to\infty} \frac{x}{n+1} &= 0 \ \forall x
\end{aligned}$$

$\therefore$ The series $\sum_{n=1}^{\infty} s_n$ converge, infact to e.

Example 19.1.2. *Consider the harmonic series* $\sum a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \cdots$.

$$\begin{aligned}
a_n &= \frac{1}{n} \\
a_{n+1} &= \frac{1}{n+1} \\
\frac{a_{n+1}}{a_n} &= \frac{n}{n+1} \\
\lim_{n\to\infty} \frac{n}{n+1} &= 1
\end{aligned}$$

Test is inconclusive. (This series is divergent, we have already seen it).

Example 19.1.3. *Consider the series* $\sum a_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \ldots$

$$\begin{aligned}
a_n &= \left(\frac{1}{n}\right)^2 \\
a_{n+1} &= \left(\frac{1}{n+1}\right)^2 \\
\therefore \frac{a_{n+1}}{a_n} &= \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2+1+2n} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 1
\end{aligned}$$

This test is inconclusive. (This series is actually convergent).

Example 19.1.4. $a_k = \dfrac{k!}{1.4.7.\ldots(3k+1)}$.

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{\dfrac{(k+1)!}{1.4.7.\ldots(3(k+1)+1)}}{\dfrac{k!}{1.4.7.\ldots(3k+1)}} \\
&= \lim_{k\to\infty} \frac{(k+1)!}{1.4.7.\ldots(3k+4)}.\frac{1.4.7.\ldots(3k+1)}{k!} \\
&= \lim_{k\to\infty} \frac{k+1}{3k+4} = \lim_{k\to\infty} \frac{1+\dfrac{1}{k}}{3+\dfrac{4}{k}} \\
&= \frac{1}{3} < 1
\end{aligned}$$

$\therefore$ *By Ratio test, the series is convergent.*

Example 19.1.5. $\sum_{n=1}^{\infty} \dfrac{3^{1-2n}}{n^2+1}$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{1-2(n+1)}}{(n+1)^2+1} \cdot \frac{n^2+1}{3^{1-2n}} \\ &= \lim_{n\to\infty} \frac{n^2+1}{n^2+1+2n+1} 3^{-2} \\ &= \lim_{n\to\infty} \frac{3^{-2}(n^2+1)}{n^2+2n+2} = \frac{1}{9} < 1\end{aligned}$$

The series converges.

Example 19.1.6. $\sum_{n=3}^{\infty} \dfrac{e^{4n}}{(n-2)!}$
Consider

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{e^{4(n+1)}}{(n+1-2)!} \cdot \frac{(n-2)!}{e^{4n}} \\ &= \lim_{n\to\infty} e^4 \frac{(n-2)!}{(n-1)(n-2)!} = \lim_{n\to\infty} \frac{e^4}{n-1} = 0\end{aligned}$$

The series converges.

Proposition 19.1.2

Consider an increasing sequence $\{a_n\}$ of real numbers. If a subsequence of $\{a_n\}$ is bounded, then the sequence $\{a_n\}$ is also bounded.

Proof. We have $a_1 \leq a_2 \leq \ldots$. Suppose $\{a_n\}$ is unbounded, then there exists an $M > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > M$. Consider a subsequence $\{a_{n_k}\}$ of $\{a_n\}$. It is increasing and bounded (say M). Choose $n_k > N$. Then, $a_N \leq a_{n_k} \leq M$, a contradiction. Hence, $\{a_n\}$ is bounded. □

Example 19.1.7. *The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges for* $p > 1$. *The sequence* $a_n = \frac{1}{n^p}$ *is non-negative. The partial sum*

$$\begin{aligned}S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2^p} \\ S_3 &= 1 + \frac{1}{2^p} + \frac{1}{3^p} \\ &\vdots\end{aligned}$$

The sequence $\{S_n\}$ *is increasing. We show it is bounded. It is enough to show a subsequence of* $\{S_n\}$ *is bounded. Consider a subsequence* $\{S_{k_j}\}$ *of* $\{S_n\}$. *Choose* $k_1 = 2^1 - 1 = 1, k_2 =$

$2^2 - 1 = 3, k_3 = 2^3 - 1 = 7$ *and so on. Next,*

$$\begin{aligned}
S_{k_1} &= 1 \\
S_{k_2} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} < 1 + \frac{1}{2^p} + \frac{1}{2^p} = S_{k_1} + \frac{1}{2^{p-1}} \\
S_{k_3} &= S_{k_2} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \\
&< S_{k_2} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} \\
&= S_{k_2} + \frac{1}{4^{p-1}} = S_{k_2} + \frac{1}{2^{2(p-1)}} \\
&\vdots \\
S_{k_j} &< 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \cdots + \frac{1}{(2^{(p-1)})^{(j-1)}}
\end{aligned}$$

The RHS is a geometric series with ratio $r = \frac{1}{2^{p-1}}$ *with sum* $\frac{1}{1-\frac{1}{2^{p-1}}}$. *Hence* $\{S_{k_j}\}$ *is monotonically increasing and bounded. Hence, the claim.*

19.1.2 Integral test

Suppose f is a nonnegative, continuous, and decreasing function for $x \geq n_0$. Then the infinite series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges if and only if the integral

$$\int_1^{\infty} f(x)\, dx$$

converges or diverges. Moreover, if f is positive, continuous, and decreasing, then

$$\int_1^{\infty} f(x)\, dx \leq \sum_{k=1}^{\infty} f(k) \leq \int_1^{\infty} f(x)\, dx + f(1)$$

Example 19.1.8. *Harmonic Series*

For $p > 1$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Consider the function $f(x) = \frac{1}{x^p}$, $p > 1$. Then $f(x)$ is positive, continuous, and decreasing for $x > 0$. The integral test gives:

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^{\infty} = \frac{1}{p-1}$$

which converges when $p > 1$. Thus, the series $\sum_{k=2}^{\infty} \frac{1}{k^p}$ converges for $p > 1$. When $p = 2$, we have $1 \leq \sum \frac{1}{n^2} \leq 2$, while it is well-known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

LECTURE-21

21.1 Conditionally convergent

LECTURE-21

21.1 Limits

1. Neighbourhood of a point.
2. Deleted neighbourhood $B(x_0, \epsilon) \setminus \{x_0\}$.

Definition 21.1.1: Limit Point: Let $A \subset \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is a limit point of A if every $\epsilon - ngbd\ B(x_0, \epsilon)$ of x_0 contains a point of A other than x_0.

Theorem 21.1.2: Let $A \subset \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is a limit point of A if and only if there exists a sequence $\{x_n\} \subseteq A$ with $x_n \neq x_0\ \forall\ n$ such that $x_n \to x_0$ as $n \to \infty$.

Proof. Choose $x_n \in B(x_0, \frac{1}{n}) \setminus \left(\overline{B(x_0, \frac{1}{n+1})} \cap A\right)$ as shown in Figure 21.6. Clearly, $x_n \in A$, $x_n \neq x_0$ for every n and $x_n \to x_0$ as $n \to \infty$. □

We next see if we can characterize the limit of a function by the limit of sequences?

21.1.1 Sequential characterization of limits

Let $A \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of A. Suppose that f is a function defined on A except possibly at x_0. Then f is said to have limit l as $x \to x_0$, we write

$$\lim_{x \to x_0} f(x) = l \ or \quad f(x) \to l \quad \text{as} \quad x \to x_0$$

If $f(x_n) \to l$ for each sequence $\{x_n\}$ in A with $x_n \neq x_0\ \forall\ n$ and $x_n \to x_0$ as $n \to \infty$.

- If $\lim_{x \to x_0} f(x) \to l \Leftrightarrow \lim_{x \to x_0} |f(x) - l| \to 0$.

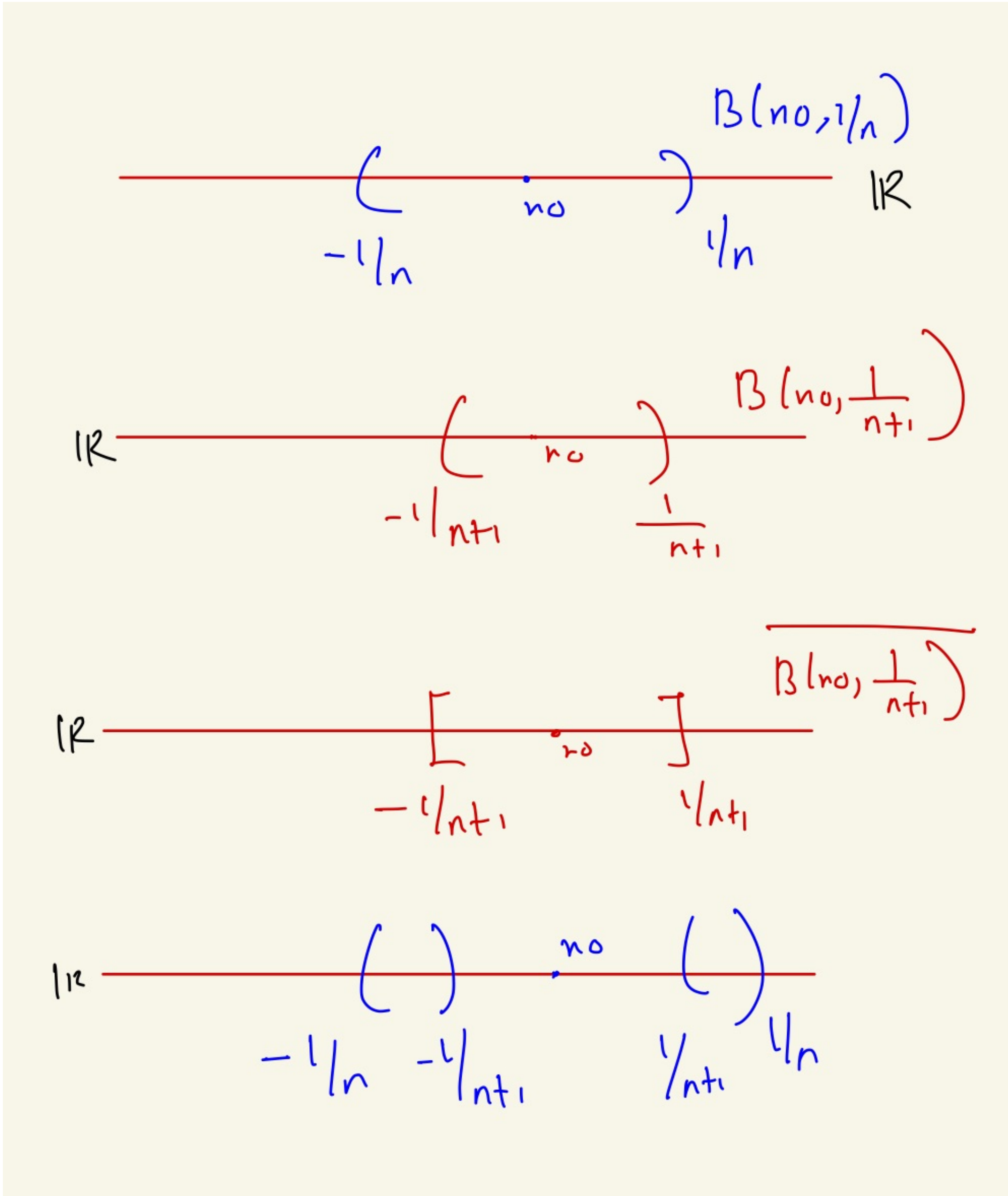


Figure 21.6: Various sets

As x gets close to x_0, $f(x)$ gets close to l.
Definition for the limit of a function through the following theorem:

Theorem 21.1.3: Let $A \subset \mathbb{R}$ and x_0 be a limit point of A, and $f : A \to \mathbb{R}$. Then the following are equivalent.

a. $\lim_{x \to x_0} f(x) = l$

b. For every $\epsilon > 0$, there exists a $\delta > 0$ s.t $|f(x) - l| < \epsilon$ when $x \in A$ and $x \in B(x_0, \delta) \setminus \{x_0\}$.

Proof. $(a) \implies (b)$
Assume $\lim_{x \to x_0} f(x) = l$. We will prove by contradiction. Assume (b) is not true (negation). Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there corresponds a point $x \in A$ such that

$$0 < |x - x_0| < \delta \text{ and } |f(x) - l| \geq \epsilon_0.$$

Then for each $n \in \mathbb{N}, \exists\, x \in A \cap B(x_0, \frac{1}{n})$, denoted by x_n such that

$$0 < |x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - l| \geq \epsilon_0.$$

So, $x_n \to x_0$, but $f(x_n) \not\to l$. We had assumed that $\lim_{x \to x_0} f(x) = l$. Hence, a contradiction. Hence (b) holds.
Conversely, we will show that (b) $\implies$ (a). Assume (b) holds. Consider a sequence $\{x_n\} \subset A, x_n \to x_0, x_n \neq x_0$ for every n. Given $\epsilon > 0$, $|f(x_n) - l| < \epsilon$ when $x_n \in A$ and $0 < |x_n - x_0| < \delta$. Now choose $N \in \mathbb{N}$ such that

$$\begin{aligned} & 0 < |x_n - x_0| < \delta \quad \forall \quad n \geq N \\ & \implies |f(x_n) - l| < \epsilon \quad \forall \quad n \geq N \\ & \implies f(x_n) \to l. \end{aligned}$$

□

Observations

1. f need not be defined at x_0 in order to have a limit. So $\lim_{x \to x_0} f(x) = l$ does not depend on $f(x_0)$ even if f is defined at x_0.

   Example 21.1.1.

$$\begin{aligned} & f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \\ & f(x) = \frac{x^2 - 1}{x - 1} \quad , f(\cdot) \textit{ is undefined at } x = 1 \\ & \lim_{x \to 1} f(x) = \left. \frac{(x+1)(x-1)}{(x-1)} \right|_{x=1} = 2 \end{aligned}$$

   Even though the function is undefined at $x = 1$, the limit exists.

2. It is only a deleted neighbourhood of $B(x_0, \delta) \setminus \{x_0\}$ of x_0 that is involved. So x_0 need not belong to A.

3. Even if $x_0 \in A$ holds, we may have $f(x_0) \neq l$. For example

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Different ways in which x approaches x_0

1. $x > 0$, x approaching x_0 from the right.

2. $x < 0$, x approaching x_0 from the left.

3. x can approach x_0 in an oscillating manner (from both the left and the right).

- If $f(x)$ has a limit l as $x \to x_0$, then we say $f(x)$ approaches l as x approaches x_0.

- If $f(x)$ does not get close to any value as $x \to x_0$, then we say $f(.)$ does not have a limit at x_0.

Example 21.1.2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$
$f(x) = \dfrac{|x|}{x}$
$\lim\limits_{x\to 0} f(x)$ *does not exist.*

Example 21.1.3. $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Claim: $f(x)$ has no limit point at $x = 0$ in $\mathbb{R}$. Suppose $\lim\limits_{x\to 0} f(x) = L$. Then for every sequence of $\{a_n\}$ of non-zero numbers converging to zero, we have $\lim\limits_{n\to\infty} f(a_n) = L$. Since, $\{\frac{1}{n}\}$ is one such sequence, $L = \lim\limits_{n\to\infty} f(\frac{1}{n}) = 1$, since $\dfrac{1}{n} \in \mathbb{Q}$. Likewise, $\left\{\dfrac{\sqrt{2}}{n}\right\}$ is another sequence and $L = \lim\limits_{n\to\infty} f(\frac{\sqrt{2}}{n}) = 0$, since $\frac{\sqrt{2}}{n} \notin \mathbb{Q}$. We have $L = 1$ & $L = 0$. But the limit is unique. Therefore, $f(.)$ has no limit at $x = 0$.

LECTURE-22

22.1 Divergence Criterion:

Let $A \subset \mathbb{R}$, Let $x_0 \in \mathbb{R}$ be a limit point of A, and suppose $f : A \to \mathbb{R}$. Let $l \in \mathbb{R}$ be given. Then, $f(x) \not\to l$ as $x \to x_0$ if and only if there exists a sequence $\{x_n\} \subseteq A$ with $x_n \neq x_0$ for all n such that $x_n \to x_0$ as $n \to \infty$ but, $f(x_n) \not\to l$ as $n \to \infty$.

$\epsilon - \delta$ definition of limit:

Definition 22.1.1: Let f be defined in some neighborhood of $x_0 \in \mathbb{R}$, except possibly at x_0. We say that $\lim_{x \to x_0} f(x)$ exists if there exists a real number l satisfying the following: For every $\epsilon > 0, \exists\ \delta > 0$ such that, $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$.

To show the divergence, if there exists an $\epsilon > 0$, such that for every $\delta > 0$, there exists an $x_\delta \in A$ such that $0 < |x_\delta - x_0| < \delta$, but $|f(x_\delta) - f(x_0)| \geq \epsilon$.

Example 22.1.1. *Find $\lim_{x \to 3} x^2$. We have $f(x) = x^2, x_0 = 3$. Consider $|f(x) - 9| = |x^2 - 3^2| = |x+3||x-3|$. Take $\delta \leq 1$, then $|x - 3| < \delta \leq 1 \implies x \in (2, 4)$ and hence $x + 3 \in (5, 7)$. Therefore, $|x^2 - 9| < 7|x - 3|$. Choose, $\delta = \min\{1, \frac{\epsilon}{7}\}$, then $|f(x) - 9| < \epsilon$.*

Example 22.1.2. *Show that*

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right),$$

does not exist.

Sandwich rule for Functions

Let, f, g, h be defined in a deleted neighborhood of x_0 such that:

- $g(x) \leq f(x) \leq h(x)\ \forall\ x$ in a neighborhood of x_0 , $x \neq x_0$.
- $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = l$ then, $\lim_{x\to x_0} f(x) = l$.

One-sided limits:

Let f be defined on $A = (x_0, x_0 + \delta)$ for some $\delta > 0$. We say $f(x)$ approaches the limit l as x approach x_0 from the right,

$$\lim_{x\to x_0^+} f(x) = l.$$

We denote

$$\lim_{x\to x_0^+} f(x) = l = f\left(x_0^+\right).$$

Similarly f is defined on

$$A = (x_0 - \delta, x_0).$$

we say, $f(x)$ approaches x_0 from left if

$$\lim_{x\to x_0^-} f(x) = l.$$

We denote,

$$\lim_{x\to x_0^-} f(x) = f(x_0^-).$$

Theorem 22.1.2: Let f be defined in a deleted neighborhood of x_0. Then, $\lim_{x\to x_0} f(x) = l$ if and only if $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ both exists and both are equal to l.

Example 22.1.3. *Consider $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined as $f(x) = \frac{|x|}{x}$. Then $f(0^+) = 1$ and $f(0^-) = -1$. Note that $f(0^+) \neq f(0^-)$, we say f has a jump discontinuity at $x_0 = 0$.*

Example 22.1.4. *Consider*

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$f(0^+) = f(0^-) = 0$ while $f(0) = 1$. Then, $x_0 = 0$ is called as removable discontinuity.

Properties of limits of functions

Suppose, f and g are functions defined on $I \subset \mathbb{R}$. Then,

- $(f+g)(x) = f(x) + g(x)$, (Point-wise addition).
- $(fg)(x) = f(x)g(x)$, (Point-wise multiplication).
- $(f/g)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ on I.

Combination rule

Theorem 22.1.3: Let, $A \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of A. Suppose f and g are defined on A with

$$\lim_{x \to x_0} f(x) = l_1 \textbf{ and } \lim_{x \to x_0} g(x) = l_2.$$

Then

1. $\lim_{x \to x_0} [f(x) + g(x)] = l_1 + l_2$
2. $\lim_{x \to x_0} [f(x).g(x)] = l_1 l_2$.
3. $\lim_{x \to x_0} [f(x)/g(x)] = \frac{l_1}{l_2}$.

Lemma 22.1.4: Sign preserving property

Let $\lim_{x \to x_0} f(x) = l \neq 0$, Then there exists a deleted neighborhood $B(x_0, \delta) \setminus \{x_0\}$ on which $f(x) \neq 0$. Moreover, $f(x)$ has the same sign as l on $B(x_0, \delta) \setminus \{x_0\}$.

Proof. Fix $\epsilon = \frac{|l|}{2}$, then there $\exists\, \delta > 0$, such that (where l could be either positive or negative). $|f(x) - l| < \epsilon = \frac{|l|}{2}$ whenever $0 < |x - x_0| < \delta$. This implies $l - \frac{|l|}{2} < f(x) < \frac{|l|}{2} + l$ $\forall\, x \in B(x_0, \delta) \setminus \{x_0\}$. Suppose $l > 0$, then

$$\frac{l}{2} < f(x) < \frac{3l}{2} \Rightarrow f(x) > 0.$$

Suppose $l < 0$, then

$$-\frac{3|l|}{2} < f(x) < -\frac{|l|}{2} \Rightarrow \quad f(x) < 0.$$

In either case $f(x) \neq 0$ and has the same sign as l on $B(x_0, \delta) \setminus \{x_0\}$. □

LECTURE-23

23.1 Continuity

Let $f : I \to \mathbb{R}$, where I is an open interval containing x_0. Then f is said to be continuous at $x_0 \in I$ if $\lim\limits_{x \to x_0} f(x) = f(x_0)$. If f is continuous at each point of I, then f is said to be continuous on I. The function f is said to be continuous if it is continuous on the domain of f.

Note

If $x_0 \in \partial I$ (boundary of I), then we can talk only of left and right continuity.

Theorem 23.1.1: Let $f : I \to \mathbb{R}$ and let $x_0 \in I$. Then the following are equivalent:

1. **f is continuous at x_0.**
2. **For a given $\epsilon > 0$, there exists a $\delta = \delta(x_0, \epsilon) > 0$ s.t $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.**
3. **The following three conditions hold:**
   $f(x_0)$ is defined, $\lim\limits_{x \to x_0} f(x)$ exists and $\lim\limits_{x \to x_0} f(x) = f(x_0)$.
4. **f is continuous at x_0 if and only if, for every sequence $\{x_n\} \subset I$ s.t $x_n \to x_0$, then $f(x_n) \to f(x_0)$.**

Example 23.1.1. *Let* $f : \mathbb{R} \to \mathbb{R},\ f(x) = c$
$f(\cdot)$ *is continuous on* $\mathbb{R}$. *Since given* $\epsilon > 0$, $|f(x) - f(y)| = |c - c| = 0 < \epsilon \ \forall\ y \in B(x, \delta)$.

Example 23.1.2. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x$
$\lim\limits_{x \to x_0} f(x) = x_0$

1. $f(x_0)$ *is defined.*
2. *limit exists.*

3. $\lim_{x \to x_0} f(x) = f(x_0)$

$\therefore$ *$f(\cdot)$ is continuous at every $x \in \mathbb{R}$.*

Example 23.1.3. *$f(x) = x^k$, $k \in \mathbb{N}$. We will prove f is continuous by induction. For $k = 1, f(x) = x$, which is continuous. Assume the result is true for k, that is $\lim_{x \to x_0} x^k$ exists. That is $\lim_{x \to x_0} x^k = x_0^k$. Next, consider x^{k+1}. We have*

$$|x^{k+1} - x_0^{k+1}| = |x - x_0||x^k + x^{k-1}x_0 + \ldots + xx_0^{k-1} + x_0^k|.$$

Take $\delta = 1$, then $|x - x_0| < \delta \implies x \in (x_0 - \delta, x_0 + \delta)$.

For $x \in (x_0 - \delta, x_0 + \delta)$, the term $|x^k + x^{k-1}x_0 + \ldots + xx_0^{k-1} + x_0^k|$ is bounded, by say, M. Then $|x^{k+1} - x_0^{k+1}| < \delta M$ Choose $\delta = \min\{1, \frac{\epsilon}{M}\}$ from which it follows $|x^{k+1} - x_0^{k+1}| < \epsilon$.

Arithmetic operations on functions

Given two functions, $f : I \to \mathbb{R}, g : I \to \mathbb{R}$. We can define

1. $\underbrace{(f+g)(x)}_{\text{sum}} = \underbrace{f(x) + g(x)}_{\text{point-wise addition}}$
2. $\underbrace{(f-g)(x)}_{\text{difference}} = \underbrace{f(x) - g(x)}_{\text{point-wise subtraction}}$.
3. $\max\{f, g\}(x) = \max\{f(x), g(x)\}$
4. $\min\{f, g\}(x) = \min\{f(x), g(x)\}$
5. $\underbrace{(fg)(x)}_{\text{product}} = \underbrace{f(x)g(x)}_{\text{point-wise multiplication}}$
6. $(\frac{f}{g})(x) = \dfrac{f(x)}{g(x)}$, provided $g(x) \neq 0 \; \forall \; x \in I$
7. $(cf)(x) = cf(x) \; \forall \; c \in \mathbb{R}$.

Composition of functions

Suppose $f : I \to J$, $g : J \to \mathbb{R}$. Then the composition of f with g, denotes $g \circ f$, is a map $g \circ f : I \to \mathbb{R}$ defined as $x \mapsto g(f(x)), x \in I$.

Note

$g \circ f \neq f \circ g$, composition is not commutative.

Theorem 23.1.2: (Composition rule): If $\lim_{x \to x_0} f(x) = y_0$ and g is a function that is continuous at y_0, then $\lim_{x \to x_0} (g \circ f)(x) = g(y_0)$.

Proof. Since, g is continuous at $y_0 \implies \forall\, \epsilon > 0, \exists\, \delta g > 0$ s.t

$$|g(y) - g(y_0)| < \epsilon \quad \text{whenever} \quad |y - y_0| < \delta g \tag{23.4}$$

Treat δg as the epsilon for f. That is for $\delta g > 0$, $\exists\ \delta > 0$ s.t $|f(x) - y_0| < \delta y$ where $0 < |x - x_0| < \delta$.
In (23.4), put $y = f(x)$ then $|(g \circ f)(x) - g(y_0)| = |g(f(x)) - g(y_0)| < \epsilon$ where $0 < |x - x_0| < \delta$. □

Corollary 23.1.3

Composition of two continuous functions is continuous.

Corollary 23.1.4

(Squeeze Rule) Let f, g and h be defined in a ngbd of x_0 s.t

1. $g(x) \leq f(x) \leq h(x)\ \forall\ x$ in a ngbd of x_0.
2. g and h are continuous at x_0 and $g(x_0) = f(x_0) = h(x_0)$. Then f is continuous at x_0.

Example 23.1.4. *Dirichlet's function on $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(z) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then f is discontinuous at each point of $\mathbb{R}$.

Proof. Suppose that $a \in \mathbb{R}$ and $\lim_{x \to a} f(x) = l$. Then given $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t $\{x_n\} \subseteq \mathbb{Q}$ and $\{y_n\} \subseteq \mathbb{R} \setminus \mathbb{Q}$ s.t $x_n \to a$ and $y_n \to a$.

$$|f(x_n) - l| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x_n - a| < \delta$$
$$\&$$
$$|f(y_n) - l| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |y_n - a| < \delta$$

Hence, $\exists\, N \in \mathbb{N}$ s.t $\forall\, n \geq N$,

$$\begin{aligned} 1 = |f(x_n) - f(y_n)| &= |f(x_n) - l + l - f(y_n)| \\ &\leq |f(x_n) - l| + |f(y_n) - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$1 \not< \epsilon\ \forall\ \epsilon > 0$. Thus, $f(x)$ does not approach a limit as $x \to a$ whether a is rational or irrational. Therefore, f is discontinuous at each point in $\mathbb{R}$. □

Proposition 23.1.5

Pre-image of a an open set under a continuous map

Let $f : X \subseteq \mathbb{R} \to Y \subseteq R$ be continous. Let $V \subset Y$ be open. Then, $f^{-1}(V)$ is open in X.

Proof. □

23.1.1 Uniform Continuity

Definition 23.1.6: (Uniform Continuity): Consider $f : I \to \mathbb{R}$. The function f is said to be uniformly continuous on I if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ (depending only on ϵ), such that for all $x, y \in I$

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Example 23.1.5. *Every Linear funciton $f(x) = ax + b$ is uniformly continuous on $\mathbb{R}$.*

Example 23.1.6. $f(x) = \sin x$
Since, $|f(x) - f(y)| = |\sin x - \sin y| < |x - y|\ \forall\ x, y \in \mathbb{R}$
Take $\delta = \epsilon$.

Example 23.1.7. $f(x) = x^2$ *is uniformly continuous on* $[a, b]$
Since, $|x^2 - y^2| = |x + y||x - y| \leq 2b|x - y|\ \forall\ x, y \in [a, b]$.
Take $\delta < \dfrac{\epsilon}{2b}$.

Example 23.1.8. $f(x) = \dfrac{1}{x}$ *is U.C on* $[b, \infty)$
Since, $\left|\dfrac{1}{x} - \dfrac{1}{y}\right| = \left|\dfrac{y - x}{xy}\right| \leq \dfrac{1}{b^2}|y - x|$
Take $\delta < b^2\epsilon$.

Example 23.1.9. *$f(x) = x^2$ on $\mathbb{R}$ is not U.C on $\mathbb{R}$.*

Proof. It suffices to show that f is not a UC on a smaller subet $[0, \infty)$. Let $\epsilon > 0$. We will show that $|f(x) - f(y)| \geq \epsilon$ for any $\delta > 0$ and for all x, y satisfying $|x - y| < \delta$. WLG, assume $x < y$ and let $y - x = a \geq 0$. We have $a < \delta$. Now

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \geq \epsilon \\ &= |x - y|a \geq \epsilon \\ &= a(2x + a) \geq \epsilon \\ \implies x &\geq \frac{\epsilon - a^2}{2a} \end{aligned}$$

Since $x \geq 0$, we need $a \leq \sqrt{\epsilon}$. Therefore $\delta = \min\{\delta, \sqrt{\epsilon}\}$. Then, $x = \frac{\epsilon - a^2}{2a}$, $y = \frac{\epsilon + a^2}{2a}$ satisfy $|x - y| < \delta$ while $|f(x) - f(y)| = \epsilon$. Hence, the claim. Conversely, assume f is uniformly continuous. Fix $x_0 \in [a, b]$. Given $\varepsilon > 0$, by uniform continuity there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ for all $x, y \in [a, b]$. Setting $y = x_0$ gives: if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. Thus f is continuous at x_0. Since x_0 was arbitrary, f is continuous on $[a, b]$. □

Example 23.1.10. *$f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.*
For every $\delta > 0$, choose x s.t $0 < x < \min\{1, \delta\}$ & $y = \dfrac{x}{2}$. Then $|x - y| = \dfrac{x}{2} < \delta$ &

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \left|\frac{1}{x} - \frac{2}{x}\right| = \left|\frac{1}{x}\right| = \frac{1}{x} > 1 \end{aligned}$$

Theorem 23.1.7: Every continuous function f on a bounded closed interval $[a, b]$ is uniformly continuous.

Proof. Assume f is continuous on $[a, b]$. We will prove by contradiction. Suppose f is not uniformly continuous on $[a, b]$. Then $\exists\, \epsilon > 0$ s.t for every $\delta > 0, \exists\, x, y \in [a, b]$ s.t $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. In particular, we can define two sequences $\{x_n\}, \{y_n\} \in [a, b]$ s.t for every $n \geq 1$ have $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. Since, $\{x_n\}$ is bounded, by the Bolzano-Weierstrass theorem, it contains a convergent subsequence $\{x_{n_k}\}$ that converges to some point $c \in [a, b]$. Let $\{y_{n_k}\}$ be a subsequence in $[a, b]$. Then

$$\begin{aligned} |y_{n_k} - c| &= |y_{n_k} - x_{n_k} + x_{n_k} - c| \\ &\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \\ &< \frac{1}{k} + |x_{n_k} - c| \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have $\lim_{k\to\infty} |y_{n_k} - c| \to 0$
$\implies$ the sequence $\{y_{n_k}\} \to c$ as $k \to \infty$. Since f is continuous at c, we have $f(x_{n_k}) - f(y_{n_k}) \to f(c) - f(c) = 0$ as $k \to \infty$. This contradicts the assumption that $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon \ \forall\ k \in \mathbb{N}$. Thus f must be uniformly continuous. □

Proposition 23.1.8

Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. Then f is uniformly continuous on A if and only if for every pair of sequences $\{x_n\}, \{y_n\} \subset A$ with $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$.

Proof. ($\Rightarrow$) Assume f is uniformly continuous. Let $\{x_n\}, \{y_n\} \subset A$ satisfy $|x_n - y_n| \to 0$. Given $\varepsilon > 0$, uniform continuity gives $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in A$. Since $|x_n - y_n| \to 0$, there exists N with $|x_n - y_n| < \delta$ for all $n \geq N$, hence $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$. Thus $|f(x_n) - f(y_n)| \to 0$.

($\Leftarrow$) Suppose the sequential condition holds. If f were not uniformly continuous then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are $x, y \in A$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$. Choosing $\delta = 1/n$ for each n yields sequences $\{x_n\}, \{y_n\}$ with $|x_n - y_n| < 1/n \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n, contradicting the hypothesis. Hence f is uniformly continuous. □

Example 23.1.11. $f(x) = \dfrac{1}{x}$ *on* $(0, 1)$
$x_n = \dfrac{1}{n+1}, y_n = \dfrac{1}{n}$

$$\begin{aligned}|x_n - y_n| &= \left|\frac{1}{n+1} - \frac{1}{n}\right| \\ &= \left|\frac{n - n - 1}{n(n+1)}\right| \\ &= \frac{1}{n+1} \to 0 \quad as \quad n \to \infty\end{aligned}$$

However, $|f(x_n) - f(y_n)| = n + 1 - n = 1$ *as* $n \to \infty$.

Example 23.1.12. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$
Take $x_n = n + \dfrac{1}{n}, y_n = n$
Then $|x_n - y_n| = \left|\dfrac{1}{n}\right| \to 0$ *as* $n \to \infty$
However, $|f(x_n) - f(y_n)| = \left|n^2 + \dfrac{1}{n^2} + 2 - n^2\right| = 2$ *as* $n \to \infty$.

23.2 Lipschitz Functions

Definition 23.2.1: Lipschitz continuity

Let $D \subset \mathbb{R}$ be a dmonain (open and connected set). A function

$$f : D \to \mathbb{R}$$

is said to be **Lipschitz continuous** (or simply **Lipschitz**) if there exists a constant $L \geq 0$ such that

$$|f(x_1) - f(x_2)| \leq L|x_1, x_2| \quad \forall\, x_1, x_2 \in D.$$

The smallest such constant L is called the **Lipschitz constant** of f.

Examples

1. The identity function $f(x) = x$ on $\mathbb{R}$ is Lipschitz with constant $L = 1$.
2. Any linear function $f(x) = ax + b$ on $\mathbb{R}$ is Lipschitz with constant $L = |a|$.
3. The absolute value function $f(x) = |x|$ is Lipschitz with $L = 1$.
4. The function $f(x) = \sqrt{x}$ on $[0, \infty)$ is *not* Lipschitz near 0, because

$$\frac{|f(x) - f(0)|}{|x - 0|} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to \infty \text{ as } x \to 0^+.$$

Remarks and Properties

- Every Lipschitz function is **uniformly continuous**. Indeed, given $\varepsilon > 0$, choose $\delta = \varepsilon/L$. Then if $|x_1 - x_2| < \delta$, we have

$$|f(x_1) - f(x_2)| \leq L|x_1, x_2| < \varepsilon.$$

- If f and g are Lipschitz with constants L_f and L_g, respectively, then
  - $f + g$ is Lipschitz with constant $L_f + L_g$;
  - cf is Lipschitz with constant $|c|L_f$ for any scalar c.
- If f is differentiable on an interval $I \subset \mathbb{R}$ and $|f'(x)| \leq M$ for all $x \in I$, then f is Lipschitz with constant $L = M$ (by the Mean Value Theorem).
- A function can be uniformly continuous without being Lipschitz. Example: $f(x) = \sqrt{x}$ on $[0, 1]$ is uniformly continuous but not Lipschitz.

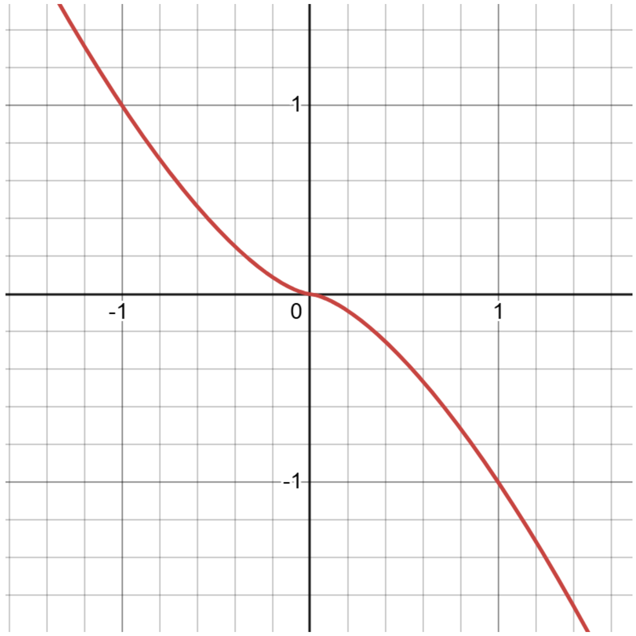


Figure 23.7: $f(x) = -|x|^{3/2}\text{sign}(x)$

Geometric Interpretation

The Lipschitz condition bounds how steeply f can change:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$$

means that the graph of f lies within a cone of slope L around any point, preventing arbitrarily sharp oscillations.

Example 23.2.1. *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -|x|^\beta \text{sign}(x), \beta \geq 1$ a shown in Figure **??**. Let us test the Lipschitz continuity at $x = 0$.*

$$|f(x) - f(0)| = |-|x|^\beta \text{sign}(x) - 0| = ||x|^\beta - 0| \leq |x - 0|$$

for all $x \in [-1, 1]$. Thus, f is Lipschitz continuous with Lipschitz constant $L = 1$ on the compact interval $[-1, 1]$.

Piece-wise continuous function

$f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous if it has at most a finite number of discontinuities on $[a, b]$ and the one-sided limit exists at each point of discontinuity.

Definition 23.2.2: **A function $f : [a.b] \to \mathbb{R}$ is said to be piece-wise continuous if there exists a partition, $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that,**

1. $x_0 = a; x_n = b$.

2. f is continuous on each sub-interval $(x_{k-1}, x_k), 1 \leq k \leq n-1$.

3. $f(x_k^+)$ for $0 \leq k \leq n-1$ and $f(x_k^-)$ for $1 \leq k \leq n$ exists.

Example 23.2.2. *$f(x_0^+) = f(x_0^-)$ but, $f(x_0^+) = f(x_0^-) \neq f(x_0)$.*

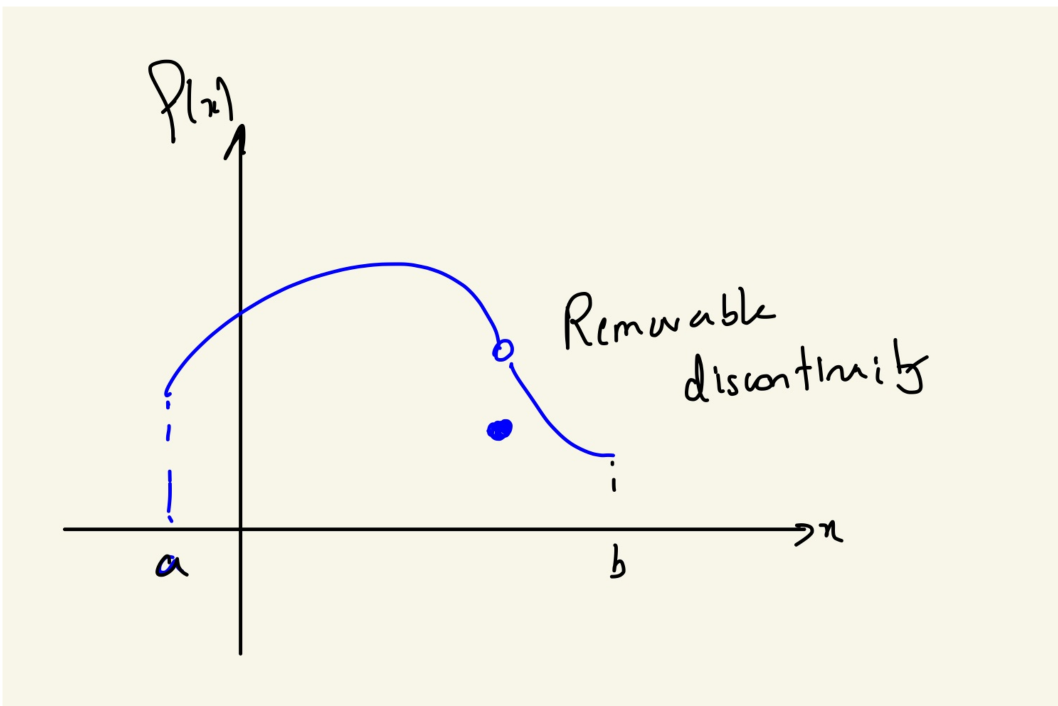


Figure 23.8: Piece-wise continuous function

Example 23.2.3. *The function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \operatorname{sign}(x)$ is peiece-wise continuous.*

LECTURE-24

24.1 Converse, Inverse, Contrapositive and Negation statements

Consider the statement *If the weather is nice, then I will wash the car.*

Let
p: the weather is nice
q: I will wash the car
If p, then q can be written $p \to q$.
For negations we use the notation: $\sim p$ & $\sim q$
$\sim p =$ the weather is not nice
$\sim q =$ I will not wash the car.
Three new statements can be formed using p, q and their negations.

1. Converse $q \to p$
   If I wash the car, then the weather is nice.

2. Inverse $\sim p \to \sim q$
   If the weather is not nice, then I will not wash the car.

3. Contrapositive: $\sim q \to \sim p$
   If I don't wash the car, then the weather will not be nice.

Note

- For a if then statement, the contrapositive is also true.
- The converse and inverse may or may not be true.
- If $p \to q$ & $q \to p$, then $p \leftrightarrow$ (if and only if) q

Example 24.1.1. *$p =$ I will walk to school.$q =$ I will be late. $p \to q$: If I will walk to school, then I will be late.*

1. *Converse $q \to p$: If I am late, then I walk to school.*

2. *Inverse $\sim p \to \sim q$: If I don't walk to school, then I won't be late.*

3. *Contrapositive* $\sim q \to \sim p$: *I am not late, then I don't walk to school.*

Example 24.1.2. *If* $n > 2$*, then* $n^2 > 4$*. Let* $p : n > 2, q : n^2 > 4$*.*

1. *Converse* $q \to p$: *If* $n^2 > 4$*, then* $n > 2$*.* *<u>False</u> Take* $n = -3$*.*

2. *Inverse* $\sim p \to \sim q$: *If* $n \not> 2$*, then* $n^2 \not> 4$*.* *<u>False</u> Take* $n = -3$*.*

3. *Contrapositive* $\sim q \to \sim p$: *If* $n^2 \not> 4$*, then* $n \not> 2$*.* *<u>True</u>.*

Example 24.1.3. p : *Two points are on the same line.* q : *They are collinear.*
$p \to q$: *If two points are on the same line, then they are collinear.*

1. *Converse* $q \to p$: *If two points are collinear, then they are on the same line.* *<u>True</u>.*

2. *Inverse* $\sim p \to \sim q$: *If two points are not on the same line, then they are not collinear.* *<u>True</u>.*

3. *Contrapositive* $\sim q \to \sim p$: *If two points are not collinear, then they are not on the same line.* *<u>True</u>.*

Negation

Negation p and not q, contradicts the implication

Example 24.1.4. p : *The weather is nice.*
q : *I will wash the car.*
$\sim (p \to q)$: *The weather is nice, but I will not wash the car.*

Negation with logical quantifiers

Suppose X is a set, A is a subset of X, and P is a statement about the general element of X. Consider the following statement:

$$\text{For every } x \in A \text{ , statement } P \text{ holds.} \tag{24.5}$$

The negation of this statement is: For atleast one $x \in A$, statement P does not hold. Therefore, for the negation of a statement as in (24.5), replace "for every" by "for atleast one" and replace the statement P by its negation. The process works in reverse just as well. For atleast one $x \in A$, statement Q holds.
Negation: For every $x \in A$, statement Q does not hold.

Definition 24.1.1: (Uniform Continuity): Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Then f is said to be uniformly continuous if for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t $\forall\ x, y \in A$, the following holds

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Negation of the above definition is as follows.
Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Then f is **not uniformly continuous** if $\exists$ atleast one $\epsilon > 0$ s.t $\forall\, \delta > 0,\ \exists\, x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

LECTURE-25

25.1 Differentiability

Let I be an open interval in $\mathbb{R}$. Let $a \in I$. Since I is open, for all h such that, $|h|$ is small, the point $a + h \in I$.
$f : I \to \mathbb{R}$. Define the slope of the graph of f at $(a, f(a))$ to be the limit $x \to a$ of the slope of the chord through the points $(a, f(a))$ and $(x, f(x))$.
Difference quotient for f at a is given by:

$$\frac{f(x) - f(a)}{x - a}$$

So, the slope of the graph at $(a, f(a))$ is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

provided the limit exists and is finite.
The limit denoted by $f'(a)$ is called the derivative of f at a. If f has a derivative at a, then the function is said to be differentiable at $x = a$. If f is differentiable at every point in I, then f is said to be differentiable on I.
If $f : [a, b) \to \mathbb{R}$, then we have right derivative of f defined as:

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

provided the limit exists and is finite. Similarly, if $f : (a, b] \to \mathbb{R}$, then we have left derivative of f defined as:

$$f'_-(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{b - x} = \lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$

provided the limit exists and is finite.

Theorem 25.1.1: Suppose, $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Then f is differentiable at c if both $f'_+(c)$ and $f'_-(c)$ exists and $f'_+(c) = f'_-(c)$.

Definition 25.1.2: $f : [a, b] \to \mathbb{R}$ is differentiable if $f'(x)$ exists on (a, b) and both $f'_+(a)$ and $f'_-(b)$ exists.

25.1.1 Basic Properties:

Theorem 25.1.3: Let f be defined on an open interval (a, b) and $x \in (a, b)$. If f is differentiable at x, then f is continuous at x.

Proof. We have,

$$f(x+h) - f(x) = \frac{f(x+h) - f(x)}{h} h$$

Taking limit as $h \to 0$ on both sides,

$$\begin{aligned} \lim_{h\to 0}(f(x+h) - f(x)) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} h \\ &= f'(x).0 = 0 \end{aligned}$$

$\therefore \lim_{h\to 0} f(x+h) = f(x)$
$\implies f$ is continuous. □

Example 25.1.1. $f : (-1, 1) \to \mathbb{R}, f(x) = |x|$
Then for $h \neq 0$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{f(h) - f(0)}{h} \\ \frac{f(h)}{h} &= \frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases} \end{aligned}$$

$\therefore f(x) = |x|$ *is not differentiable at* $x = 0$*. However,* f *is continuous on* $(-1, 1)$*.*

Example 25.1.2. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ *is differentiable everywhere.*

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{(x_0 + h)^2 - x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0^2 + h^2 + 2x_0 h - x_0^2}{h} \\ &= \lim_{h\to 0} h + 2x_0 = 2x_0. \end{aligned}$$

Example 25.1.3. *$f:\mathbb{R}\to\mathbb{R}, f(x)=\sin x$. Test differentiability at $x_0\in\mathbb{R}$*

$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h}\\
&= \lim_{h\to 0}\frac{\sin x_0\cos h+\cos x_0\sin h-\sin x_0}{h}\\
&= \lim_{h\to 0}\frac{\sin x_0(1-\cos h)}{h}+\cos x_0\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x_0\lim_{h\to 0}\left[\frac{(1-(1-\frac{h^2}{2!}+\frac{h^4}{4!}+\ldots))}{h}\right]+\cos x_0\\
&= \sin x_0\lim_{h\to 0}\frac{(\frac{h^2}{2!}-\frac{h^4}{4!}+\ldots)}{h}+\cos x_0\\
&= 0+\cos x_0=\cos x_0.
\end{aligned}$$

Since x_0 is arbitrary, $f'(x)=\cos x$.

Example 25.1.4.

$$f(x)=\begin{cases}1+x^2, & x\in[-1,0]\\ \cos x, & x\in[0,2\pi]\end{cases}$$

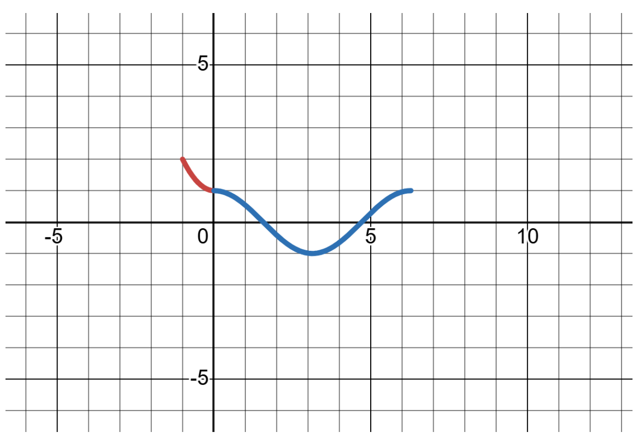


Figure 25.9: Piecewise function

Determine if f (see Figure 25.9) is differentiable at $x=0$. Determine if f has right and left derivative at $x=-1$ and 2π

At $c = -1$*, we compute the right-derivative*

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} \\ \lim_{h\to 0^+} \frac{f(-1+h)-2}{h} &= \lim_{h\to 0^+} \frac{1+(-1+h)^2-2}{h} \\ \lim_{h\to 0^+} \frac{1+1+h^2-2h-2}{h} &= \lim_{h\to 0^+} h-2 \\ &= -2. \end{aligned}$$

$\therefore f'_+(-1)$ *exists and equal* -2*.*
At $x = 2\pi$*, for* $h \to 0^-$

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(h+2\pi)-f(2\pi)}{h} &= \lim_{h\to 0^-} \frac{\cos(h+2\pi)-\cos 2\pi}{h} \\ &= \lim_{h\to 0^-} \frac{\cos h\cos 2\pi - \sin h \sin 2\pi - \cos 2\pi}{h} \\ &= \lim_{h\to 0^-} \frac{\cos h - 1}{h} = 0 \end{aligned}$$

Limits $f'_-(2\pi)$ *exists and equal* 0*.*
At $x = 0$*, for* $h \to 0^+$

$$\begin{aligned} f'_+(0) &= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\cos h - 1}{h} = 0 \\ f'_-(0) &= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{1+h^2-1}{h} = 0 \end{aligned}$$

Thus, both $f'_+(0)$ *and* $f'_-(0)$ *exists and equal* 0*.*
Further, f *is differentiable on* $(-1, 2\pi)$ *and differentiable from right at* -1 *and differentiable from left at* $x = 2\pi$*.*

LECTURE-26

26.1 $\epsilon - \delta$ interpretation of derivative

Suppose f is differentiable at x_0. Given$\epsilon > 0, \exists\, \delta(x_0, \epsilon) > 0$, such that:

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \epsilon$$

whenever

$$0 < |x - x_0| < \delta$$

Proposition 26.1.1

Let I be an open subset of $\mathbb{R}$. Suppose, $f : I \to \mathbb{R}$ is differentiable at x_0 if and only if there exists a function η that is continuous at x_0 and satisfies,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)\eta(x)\; \forall\, x \in B(x_0, \delta).$$

Equivalently, f is differentiable at x_0 if and only if

$$f(x) = f(x_0) + L(x) + E(x)$$

where, $L(x) \stackrel{\triangle}{=} f(x_0) + f'(x_0)(x - x_0)$ is a linear (affine) approximation of f at x_0 and $E(x)$ is a function defined in a neighbourhood of x_0 and satisfies $\lim\limits_{x \to x_0} \frac{E(x)}{x - x_0} = 0$, $|E(x)|$ is small in comparison with $|x - x_0|$ for x close to x_0. In other words, $|E(x)| = o(|x - x_0|)$.

Proof. ($\Rightarrow$) Suppose f is differentiable at x_0. Then by definition,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

For $x \neq x_0$, define

$$\eta(x) = \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0).$$

Then for all x near x_0,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)\eta(x).$$

Since $\dfrac{f(x) - f(x_0)}{x - x_0} \to f'(x_0)$ as $x \to x_0$, we have $\eta(x) \to 0$. Define $\eta(x_0) = 0$ to make η continuous at x_0. Thus the desired representation holds.

If we set $E(x) = (x - x_0)\eta(x)$, then

$$\frac{E(x)}{x - x_0} = \eta(x) \to 0 \quad \text{as } x \to x_0,$$

so

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + E(x)$$

with $\lim_{x \to x_0} \dfrac{E(x)}{x - x_0} = 0$.

($\Leftarrow$) Conversely, suppose there exists a function η continuous at x_0 such that

$$f(x) = f(x_0) + A(x - x_0) + (x - x_0)\eta(x)$$

for all x near x_0, where A is a constant. Then for $x \neq x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} = A + \eta(x).$$

Taking the limit as $x \to x_0$ and using the continuity of η,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = A + \eta(x_0).$$

Hence f is differentiable at x_0 with derivative $f'(x_0) = A + \eta(x_0)$.

If we take $A = f'(x_0)$ and $\eta(x_0) = 0$, this matches the forward direction.

Equivalently, if

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + E(x),$$

where $\lim_{x \to x_0} \dfrac{E(x)}{x - x_0} = 0$, then

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \frac{E(x)}{x - x_0} \to f'(x_0),$$

showing that f is differentiable at x_0.

$$\boxed{f(x) = L(x) + E(x), \quad L(x) = f(x_0) + f'(x_0)(x - x_0), \quad \lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0.}$$

□

26.1.1 Smooth and piecewise smooth functions

Suppose, f is differentiable on an interval I. Then, we obtain a new function f' (Whose domain may be a subset of I). Even if $f'(x)$ exists on I, the new function f' may not be continuous on I.

Definition 26.1.2: C^1

We say f is of class C^1 on I,denoted by $f \in C^1(I)$, if f is differentiable on I and f' is continuous on I. If $f \in C^1(I)$, then f is said to be continuously differentiable on I.

Definition 26.1.3: Piece-wise smooth

A function, $f : [a, b] \to \mathbb{R}$ is said to be piece-wise smooth if there exists a partition, $P = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ such that f is continuously differentiable on each sub-interval $(x_{k-1}, x_k), 1 \leq k \leq n$ where $x_0 = a, x_n = b$.

Example 26.1.1. *Define$f, g, \phi : \mathbb{R} \to \mathbb{R}$ by:*

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$g(x) = xf(x) \text{ and } \phi(x) = x^2 f(x).$$

1. *f is not continuous on $\mathbb{R}$.*

2. *g is continuous on $\mathbb{R}$ but not differentiable at 0.*

3. *ϕ is differentiable on $\mathbb{R}$ but not continuously differentiable.*

NOTE:f, g, ϕ are continuously differentiable on $\mathbb{R} \setminus \{0\}$.

- *$f(.)$ is not continuous at $x = 0$*

  $$|f(x) - f(0)| = |\sin\left(\frac{1}{x}\right) - 0| = |\sin\left(\frac{1}{x}\right)|$$

  There are points at which $|\sin(\frac{1}{x})| = 1$, no matter how small $\delta > 0$ is made.

- *To show that g is continuous at $x = 0$,*

  $$|g(x) - g(0)| = |x \sin\left(\frac{1}{x}\right) - 0| = |x \sin\left(\frac{1}{x}\right)| \leq |x| < \delta$$

  Choose, $\delta = \epsilon$.

- *To show, ϕ is continuous at $x = 0$*

  $$|\phi(x) - \phi(0)| = |x^2 \sin\left(\frac{1}{x}\right) - 0| \leq |x^2| < \delta^2$$

  $|x| < \delta$, choose $\delta = \sqrt{\epsilon}$.

- *To show that g is not differentiable at $x = 0$.*

$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{g(h)-0}{h}$$
$$\lim_{h\to 0}\frac{h.\sin(\frac{1}{h})}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

Therefore, g is not differentiable at $x = 0$.
Next, to show that ϕ is differentiable at $x = 0$.

$$\lim_{h\to 0}\frac{\phi(h)-\phi(0)}{h} = \lim_{h\to 0}\frac{h^2.\sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h.\sin\left(\frac{1}{h}\right) = 0$$
$$\phi'(x) = \begin{cases} 2x\sin(\frac{1}{x}) + x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
$$= \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Consider, $2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. Now $\lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) = 0$. However, $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ takes values, $+1, -1$ for x arbitrarily close to 0.
$\phi'(.)$ is not continuous. $\lim_{z\to 0}\phi'(z)$ does not exist even though $\phi'(0) = 0$.

Definition 26.1.4: L'Hôpital's Rule

We have seen

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

provided

1. $\lim_{x\to a} f(x)$ exists.
2. $\lim_{x\to a} g(x)$ exists, $g(x) \neq 0$.
3. $\lim_{x\to a} g(x) \neq 0$

When $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ such cases are called indeterminate forms. Other such indeterminate forms are $\infty.0, \infty - \infty, 1^\infty, 0^0$. The rule to evaluate such limits is called L'Hôpital's rule.

Theorem 26.1.5: L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable at x_0, with $f(x_0) = g(x_0) = 0$. If $g'(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof. Since $g'(x_0) \neq 0$. By sign preserving property $\exists$ a ngbd of x_0 s.t $g'(x) \neq 0$. Then, $\underbrace{g(x)}_{\neq 0} - \underbrace{g(x_0)}_{=0} \neq 0 \; \forall \; x$ in the ngbd of x_0. Then

$$\frac{f'(x_0)}{g'(x_0)} = \frac{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \to x_0} \frac{\underbrace{(g(x) - g(x_0))}_{\neq 0}}{x - x_0}} = \lim_{x \to x_0} \frac{\frac{f(x) - \underbrace{f(x_0)}_{=0}}{x - x_0}}{\frac{g(x) - \underbrace{g(x_0)}_{=0}}{x - x_0}}$$

$$= \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

□

The following examples illustrate the different types of indeterminate forms that can be solved using L'Hôpital's Rule.

Type	Example	Result
$\frac{0}{0}$	$\lim_{x \to 0} \frac{\sin x}{x}$	1
$\frac{\infty}{\infty}$	$\lim_{x \to \infty} \frac{2x^2 + 3}{5x^2 + 7x}$	$\frac{2}{5}$
$0 \cdot \infty$	$\lim_{x \to 0^+} x \ln x$	0
$\infty - \infty$	$\lim_{x \to \infty} (\sqrt{x^2 + x} - x)$	$\frac{1}{2}$
0^0	$\lim_{x \to 0^+} x^x$	1
1^∞	$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$	e
∞^0	$\lim_{x \to 0^+} \left(\frac{1}{x}\right)^x$	1

Table 26.2: Resolving inderminate forms using L'Hôpital's Rule

LECTURE-27

27.1 Generalization of the derivative

27.1.1 Derivative of a real-valued function

Let $A \subset \mathbb{R}, \phi : A \to \mathbb{R}$. Suppose A contains the neighbourhood of the point a. The derivative of ϕ at a is defined as:

$$\phi'(a) = \lim_{h \to 0} \frac{\phi(a+h) - \phi(a)}{h}$$

provided the limit exists. If the limit exists, we say that ϕ is differentiable at a.

Facts about differentiable functions

- Differentiable functions are continuous.
- Composition of differentiable functions is differentiable.

Attempt to generalize the functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition 27.1.1

Let $A \subset \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Suppose A contains the neighborhood of a, given $u \in \mathbb{R}^m$ with $u \neq 0$, define

$$f'(a; u) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

provided the limit exists. The limit depends on both a and u. $f'(a, u)$ if it exists is called the directional derivative of f at a with respect to the vector u.

Example 27.1.1. $f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = xy$, $a = (a_1, a_2)$ *and* $u = (1, 0)$

$$\begin{aligned} f'(a,u) =& \lim_{t\to 0} \frac{f(a_1 + tu_1, a_2 + tu_2) - f(a_1, a_2)}{t} \\ & \lim_{t\to 0} \frac{(a_1 + tu_1)(a_2 + tu_2) - a_1a_2}{t} \\ & \lim_{t\to 0} \frac{(a_1 + t)(a_2) - a_1a_2}{t} = a_2 \\ \therefore f'(a,u) =& a_2. \end{aligned}$$

$f : A \subset \mathbb{R}^m \to \mathbb{R}^n$. Suppose A contains the neighbourhood of a. We say f is differentiable at a if there exists a $n \times m$ matrix B such that

$$\frac{f(a+h) - f(a) - Bh}{||h||} \to 0 \text{ as } h \to 0.$$

Here, $h \in \mathbb{R}^m$.

1. Bh is a linear approximation of the increment $f(a+h) - f(a)$.
2. The matrix B which is unique is called as the derivative of f at a, denoted by $Df(a)$.

Proof that $Df(a)$ is unique.

Proof. Suppose C is another such matrix such that $C \neq B$ and satisfies

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Ch}{||h||} = 0$$
$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Bh}{||h||} = 0.$$

Subtracting and letting $h = tu, u \neq 0$

$$\begin{aligned} &\frac{(B-C)h}{||h||} = 0 \text{ as } h \to 0 \\ &\frac{(B-C)tu}{t||u||} = 0 \text{ as } t \to 0 \\ &\frac{(B-C)u}{||u||} = 0 \\ &(B-C)u = 0; u \neq 0 \\ &\implies B = C. \end{aligned}$$

□

With this definition, it can be shown that

1. Differentiable functions are continuous.

2. Composition of differentiable functions is differentiable.
3. Differentiability of f at a implies the existence of all the directional derivatives of f at a.

Theorem 27.1.2

Let $A \in \mathbb{R}^m$, and $f : A \to \mathbb{R}^n$. If f is differentiable at a, then f is continuous at a.

Proof. Let $B = Df(a)$. For h near 0 but different from 0,

$$f(a+h) - f(a) = ||h||\left[\frac{f(a+h) - f(a) - Bh}{||h||}\right] + Bh$$

$$||h||\frac{f(a+h) - f(a) - Bh}{||h||} \to 0 \text{ as } h \to 0$$

$$\therefore \lim_{h\to 0} f(a+h) - f(a) = 0.$$

Thus, f is continuous at a. □

Example to illustrate the existence of all directional derivatives of f at a does not imply that f is differentiable at a.

Example 27.1.2. $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let $u \neq 0, u = (u_1, u_2)$

$$\frac{f(0+tu) - f(0)}{t} = \frac{t^2u_1^2tu_2}{t(t^4u_1^4 + t^2u_2^2)} = \frac{u_1^2u_2}{t^2u_1^4 + u_2^2}$$

If $u_2 = 0$, *consider* $t = \frac{1}{n}$. *Then*

$$f'(0;u) = \lim_{n\to\infty} \left.\frac{u_1^2u_2n^2}{(u_1^4 + n^2u_2^2)}\right|_{u_2=0} = 0.$$

Thus, we have

$$f'(0,u) = \begin{cases} \dfrac{u_1^2}{u_2} & \text{if } u_2 \neq 0 \\ 0 & \text{if } u_2 = 0. \end{cases}$$

Thus $f'(0,u)$ exists for all $u \neq 0$. We will show that f is not differentiable at $(0,0)$. If $g : \mathbb{R} \to \mathbb{R}^2$ is a differentiable function at 0, the $Dg(0)$ is a 1×2 matrix of the form $[a\ b]$. Recall $B \in \mathbb{R}^{n\times m}$. Here, $n = 1$ and $m = 2$.
Then

$$\begin{aligned} g'(0;u) =& Df(0).u \\ =& [a\ \ b].[u_1\ \ u_2]' \\ =& au_1 + bu_2 \end{aligned}$$

which is a linear function of u. But $f'(0,u)$ is not a linear function of u. In fact, f is not even continuous at the origin. Consider $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n^2})$. Then, $\lim\limits_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \neq f(0,0)$.

We next show that the differentiability of f implies the existence of the directional derivatives in all directions.

Theorem 27.1.3

Let $A \subset \mathbb{R}^m \to \mathbb{R}^n$. If f is differentiable at a, then all the directional derivatives of f at a exists and $f'(a;u) = Df(a).u$

Proof. Let $Df(a) = B$. Set $h = ut, t \neq 0$. Then,

$$\frac{f(a+h) - f(a) - Bh}{||h||} \to 0 \text{ as } h \to 0$$
$$\frac{f(a+tu) - f(a) - Btu}{|t|||u||} \to 0 \text{ as } t \to 0$$

Since $\lim\limits_{t\to 0}$ involves both one-sided limits, they must exist and be equal. If t approaches from positive values,

$$\frac{f(a+tu) - f(a) - Btu}{t||u||} \to 0 \text{ as } t \to 0$$
$$\frac{f(a+tu) - f(a)}{t||u||} - \frac{Bu}{||u||} \to 0 \text{ as } t \to 0$$

Multiplying by $||u||$,

$$\frac{f(a+tu) - f(a)}{t} - Bu \to 0 \text{ as } t \to 0$$
$$f'(a;u) = Bu.$$

If it approaches from the negative side,

$$\frac{f(a+tu) - f(a) - Btu}{-t||u||} \to 0 \text{ as } t \to 0$$

Multiplying by $-||u||$,

$$\frac{f(a+tu)-f(a)}{t} - Bu \to 0 \text{ as } t \to 0$$
$$f'(a;u) = Bu$$

□

Utility of this theorem - If atleast one of the directional derivatives do not exist at a, then f is not differentiable at a.

Partial Derivatives

Let $f : A \subseteq \mathbb{R}^m \to \mathbb{R}$. The jth partial derivative of f at a is the directional derivative of f at a with respect to the vector e_j provided that the derivative exists and is denoted by $D_j f(a)$.

$$D_j f(a) = \lim_{t\to 0} \frac{f(a+te_j)-f(a)}{t}.$$

Theorem 27.1.4

Let $f : A \subset \mathbb{R}^m \to \mathbb{R}$. If f is differentiable at a, then $Df(a) = [D_1 f(a)\ D_2 f(a)\ \cdots D_m f(a)]$.

Theorem 27.1.5

Let $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$. Suppose A contains neighborhood of a. Let $f = [f_1\ f_2\ \cdots\ f_n]\top$. Then

1. The function f is differentiable at a if each $f_i, i = 1, 2, \ldots, n$ is differentiable at a.

2. If f is differentiable at a, then its derivative $Df(a)$ is given by

$$Df(a) = \begin{bmatrix} D_1 f_1(a) & \cdots & D_m f_1(a) \\ \vdots & \vdots & \vdots \\ D_1 f_n(a) & \cdots & D_m f_n(a) \end{bmatrix}$$

Example 27.1.3.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2y^2+(y-x)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a. For which vectors $u \neq 0$ does $f'(0,u)$ exist? Evaluate it when it exists.

b, Does $D_1 f$ and $D_2 f$ exists at 0.

c. Is f differentiable at 0.

d. Is f continuous at 0.

Solution:

a.

$$\begin{aligned} f'(u;0) &= \lim_{t\to 0} \frac{t^2u_1^2u_2^2}{(t^2u_1^2u_2^2 + (u_2 - u_1)^2)t} \\ &= \lim_{t\to 0} \frac{tu_1^2u_2^2}{(t^2u_1^2u_2^2 + (u_2 - u_1)^2)} \end{aligned}$$

If $(u_1 - u_2) = 0$, then $f'(0;u)$ does not exist.
If $u_1 \neq u_2$, then

$$\lim_{t\to 0} \frac{u_1^2u_2^2}{tu_1^2u_2^2 + \dfrac{(u_2 - u_1)^2}{t}} = 0$$

b. $D_1f(0) = 0 = D_2f(0)$

c. The function is not differentiable at 0.

d. Consider $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$, then $\lim\limits_{n\to\infty} f(x_n, y_n) = \lim\limits_{n\to\infty} n$ and so the limit does not exist. Hence f is not continuous at the origin.

Example 27.1.4.

$$f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Solution:

a.

$$\begin{aligned} f'(0,u) &= \lim_{t\to 0} \frac{tu_1|tu_2|}{t\sqrt{t^2u_1^2 + t^2u_2^2}} \\ &= \lim_{t\to 0} \frac{t|t|u_1|u_2|}{t|t|\sqrt{u_1^2 + u_2^2}} \\ &= \frac{u_1|u_2|}{\sqrt{u_1^2 + u_2^2}} \end{aligned}$$

$\therefore f'(0;u)$ exists for all $u \neq 0$.

b. $D_1 f(0) = D_2 f(0) = 0$

c. f is not differentiable at 0 since $f'(0, u)$ is not linear in u.

d. Given $\epsilon > 0$,$B(0, \delta) = \{(x, y) = \sqrt{x^2 + y^2} < \delta\}$.

$$\begin{aligned} &|f(x, y) - f(0, 0)| \\ =&\left|\frac{x|y|}{\sqrt{x^2 + y^2}}\right| \\ =&\frac{|x||y|}{\sqrt{x^2 + y^2}} \leq \frac{|x||y|}{\sqrt{x^2}} = |y| \end{aligned}$$

Now $(x, y) \in B(0, \delta) \implies \sqrt{0^2 + y^2} \leq \sqrt{x^2 + y^2} < \delta \implies |y| < \delta$. $\therefore \dfrac{|x||y|}{\sqrt{x^2 + y^2}} \leq |y| < \delta$.
Take $\delta = \epsilon$.

LECTURE-28

28.1 Mean value theorem

Theorem 28.1.1: Mean value theorem

If $f : [a, b] \to \mathbb{R}$ is continuous at each point on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that:

$$f(b) - f(a) = f'(c)(b - a).$$

Theorem 28.1.2

Let A be an open subset of $\mathbb{R}^n$. Suppose that the partial derivatives $D_i f_i(x)$ of the component functions of f exist at each point in A and are continuous on A. Then f is differentiable at each point of A.

A function satisfying the hypothesis of the above theorem is said to be continuously differentiable or of class C^1 on A. If the partial derivatives of f of all orders $\leq r$ are continuous on A, we say f is of class C^r on A. Thus a function f is of class C^r if each function $D_i f_i(x)$ is of class C^{r-1} on A.

We say f is of class C^∞ on A if all orders of partial derivatives exist and are continuous on A.

Theorem 28.1.3: Equality of mixed partials

Let A be an open set in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ be of class C^r. Then for each $a \in A$:

$$D_i D_j f(a) = D_j D_i f(a).$$

Example 28.1.1. *Show that the function $f(x, y) = |xy|$ is differentiable at 0, but is not of class C^1 in any neighborhood of 0.*

Solution:

Let us compute the partial derivatives at $(0, 0)$:

$$D_1 f(0) = \lim_{t \to 0} \frac{f(0 + te_1) - f(0)}{t} = 0$$

Similarly, $D_2f(0) = 0$*. Suppose* f *is differentiable at* 0*. Then,* $Df(0) = [0\ 0]$*. Consider*

$$\lim_{h\to 0} \frac{f(0+h) - f(0) - D(0)h}{||h||}$$

$$= \lim_{h\to 0} \frac{f(h)}{||h||}$$

Now consider $|x||y| = \min(|x|,|y|)\max(|x|,|y|) = \min(|x|,|y|)||x,y||_\infty$
Using equivalence of norm $||x,y||_\infty \ \le ||x,y||_2$

$$|x||y| \le \min(|x|,|y|)||(x,y)||$$

$$\frac{|x||y|}{||(x,y)||} \le \min(|x|,|y|)$$

$$\lim_{h\to 0} \frac{f(h)}{||h||} = 0$$

Therefore, f is differentiable at 0 with $Df(0) = [0,0]$*.*

Example 28.1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(t) = t^2 \sin\left(\frac{1}{t}\right)$ *for* $t \neq 0$ *and* $f(0) = 0$*.*

a. *Show that* f *is differentiable at* 0*.*

b. *Compute* $f'(t)$ *for* $t \neq 0$*.*

c. *Show that* f' *is not continuous at* 0*.*

d. *Conclude that* f *is differentiable on* $\mathbb{R}$*, but not of class* C^1*.*

Solution:

a.

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = 0$$

b. For $t \neq 0$:

$$f'(t) = 2t\sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)$$

c. Consider the sequence

$$t_n = \frac{1}{2\pi n}$$

$$\lim_{n\to\infty} f'(t_n) = \lim_{n\to\infty} \frac{1}{\pi n}[\sin(2\pi n) - \cos(2\pi n)] = -1$$

But $f'(0) = 0$ for all $n \in \mathbb{N}$.

d. Thus, f is differentiable on $\mathbb{R}$ but not differentiable at 0. Therefore, f is not of class C^1.

Example 28.1.3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$f(r, (\theta)) = (r\cos(\theta), r\sin(\theta))$$

It is polar coordinate transformation.

a. *Compute* $Df(r, \theta)$

b. *Sketch the images of f of the set* $[1, 2] \times [0, \pi]$

Solution:

a)

$$Df(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

b) The boundary of S is made up of four line segments:

- $B_1 = 1 \times [0, \pi] \subset \mathbb{R}^2$
- $B_2 = 2 \times [0, \pi] \subset \mathbb{R}^2$
- $B_3 = [1, 2] \times 0 \subset \mathbb{R}^2$
- $B_4 = [1, 2] \times \pi \subset \mathbb{R}^2$

Their images under the map f are

- $f(B_1) = \{(\cos(\theta), \sin(\theta)) : \theta \in [0, \pi]\}$
- $f(B_2) = \{2(\cos(\theta), \sin(\theta)) : \theta \in [0, \pi]\}$
- $f(B_3) = \{(r, 0) : r \in [1, 2]\}$
- $f(B_4) = \{(-r, 0) : r \in [1, 2]\}$

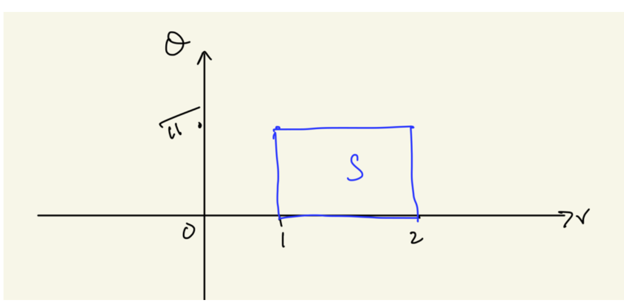


Figure 28.10: Set S

The image of the function is the annular region between the red and blue semicircles.

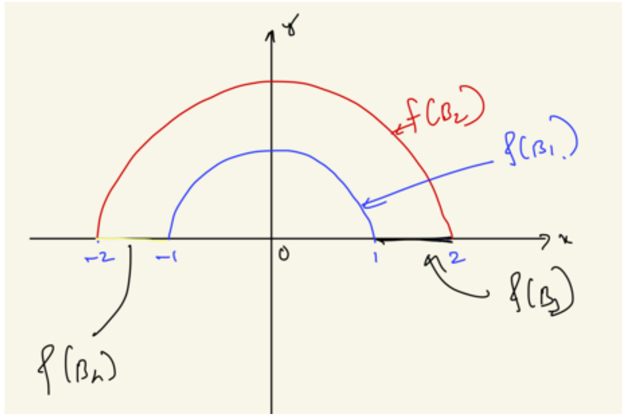


Figure 28.11: Image of F for S

Theorem 28.1.4: Chain rule

Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ and $g : B \to \mathbb{R}^p$ with $f(A) \subset B$. Suppose $f(a) = b$. If f is differentiable at a and g is differentiable at b, then the composite function $g \circ f$ is differentiable at a. Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a)$$

where $\cdot$ denotes matrix multiplication.

Corollary 28.1.5

If f and g are C^r, so is the composition $g \circ f$.

$$D(g \circ f)(x) = Dg(f(x)) \cdot Df(x), \quad \forall x \in A.$$

Theorem 28.1.6: **Let A be open in $\mathbb{R}^m$ and $f : A \to \mathbb{R}$ be differentiable on A. If A contains the line segment with endpoints a and $a+h$, then there is a point $c = a + t_0 h$ with $0 < t_0 < 1$ on the line segment such that**

$$f(a+h) - f(a) = Df(c) \cdot h.$$

Derivative of inverse of a function

Let A be an open subset of $\mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$. Let $f(a) = b$. Suppose g maps a neighborhood of b into $\mathbb{R}^n$, and $g(b) = a$ and

$$g(f(x)) = x \quad \text{for all } x \text{ in a neighborhood of } a.$$

If f is differentiable at a and g is differentiable at b, then

$$Dg(b) = [Df(a)]^{-1}.$$

Example 28.1.4. *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ satisfy*

$$f(a) = (1, 2), \quad \textit{and} \quad Df(0) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$g(x, y) = (x + 2y + 1, 3xy).$$

Find $D(g \circ f)(0)$.

Solution.

$$D(g \circ f)(0) = Dg(f(0)) \cdot Df(0)$$

$$\begin{bmatrix} 1 & 2 \\ 3y & 3x \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 12 & 21 \end{bmatrix}.$$

Example 28.1.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3, g : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as*

$$f(x_1, x_2) = \begin{bmatrix} e^{2x_1 + x_2} \\ 3x_2 - \cos x_1 \\ x_1^2 + x_2 + 2 \end{bmatrix}$$

$$g(y_1, y_2, y_3) = \begin{bmatrix} 3y_1 + 2y_2 + y_3^2 \\ y_1^2 - y_3 + 1 \end{bmatrix}$$

a. If $F(x) = g(f(x))$, find $DF(x)$.

b. If $G(y) = f(g(y))$, find $DG(y)$.

Solution:

$$DF(x) = \begin{bmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{bmatrix}; DG(y) = \begin{bmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{bmatrix}$$

Given $F(x) = g(f(x))$

$$DF(x) = Dg(f(x)) \cdot Df(x)$$

$$f(0) = (1, -1, 2)$$

$$DF(0) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 13 \\ 4 & 1 \end{bmatrix}.$$

Example 28.1.6. *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $g(x, y) = (x, y + x^2)$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Let $h = f \circ g$. Show that the directional derivatives of f and g exist everywhere, but there is a direction $u \neq 0$ for which $h'(0; u)$ does not exist.
Solution:
For the function f :

$$f'(0; u) = \lim_{t \to 0} \frac{f(0 + tu) - f(0)}{t}$$
$$f'(0; u) = \lim_{t \to 0} \frac{t^3 u_1^2 u_2}{t^5 u_1^4 + t^3 u_2^2}$$
$$f'(0; u) = \lim_{t \to 0} \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2}$$
$$= \begin{cases} \frac{u_1^2}{u_2} & \text{if } (u_2) \neq 0 \\ 0 & \text{if } (u_2) = 0 \end{cases}$$

Therefore $f'(0;u)$ exists. For the function g :

$$g'(0;u) = \lim_{t\to 0} \frac{g(0+tu)-g(0)}{t}$$
$$g'(0;u) = \lim_{t\to 0} \frac{\begin{bmatrix} tu_1 \\ tu_2 + t^2u_1^2 \end{bmatrix}}{t}$$
$$g'(0;u) = \lim_{t\to 0} \begin{bmatrix} u_1 \\ u_2 + tu_1^2 \end{bmatrix}$$
$$g'(0;u) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Directional derivative exist for f and g exists for all $u \neq 0$. Consider $h = f \circ g$.

$$\begin{aligned} h'(0;u) =& \lim_{t\to 0} \frac{h(0+tu)-h(0)}{t} \\ =& \lim_{t\to 0} \frac{f(g(tu))-f(g(0))}{t} \end{aligned}$$

and $g(0) = (0,0), f(g(0)) = 0$. Now,

$$f(g(tu)) = \frac{t^2u_1^2(tu_2+t^2u_1^2)}{tu_1^4 + (tu_2+t^2u_1^2)^2}$$

and

$$\begin{aligned} h'(0;u) &= \lim_{t\to 0} \frac{t^3u_1^2u_2 + t^4u_1^4}{t^5u_1^4 + t^3u_2^2 + t^5u_1^4 + 2t^4u_2u_1^2} \\ &= \lim_{t\to 0} \frac{u_1^2u_2 + tu_1^4}{t^2u_1^4 + u_2^2 + t^2u_1^4 + 2tu_2u_1^2} \end{aligned}$$

If $u_2 = 0$

$$\begin{aligned} h'(0;u) &= \lim_{t\to 0} \frac{tu_1^4}{t^2u_1^4 + t^2u_1^4} \\ &= \lim_{t\to 0} \frac{1}{t} \end{aligned}$$

which does not exist.

Differentiability and the inverse function

Theorem 28.1.7

Let A be an open subset in $\mathbb{R}^n$, let $f : A \to \mathbb{R}^n$ and $f(a) = b$. Suppose that g maps a neighborhood of b into $\mathbb{R}^n$, and that $g(b) = a$ and $g(f(x)) = x$ for all x in a neighborhood of a. If f is differentiable at a and if g is differentiable at b, then

$$Dg(b) = [Df(a)]^{-1}.$$

and $g(f(x)) = x \quad \forall\, x$ in an open set around x.

Theorem 28.1.8: Inverse Function Theorem

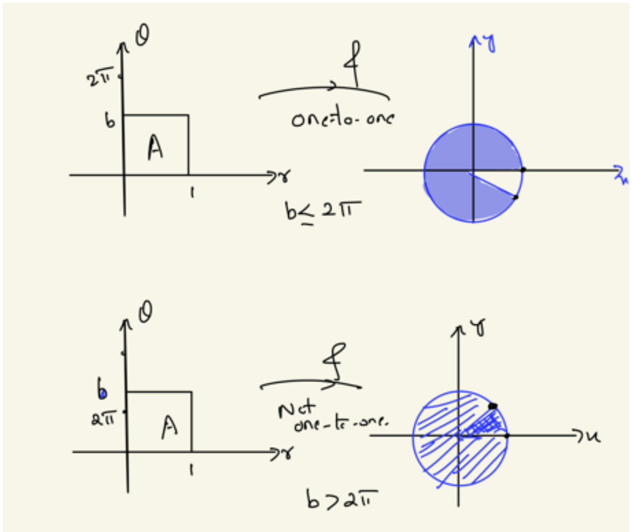
Let A be open in $\mathbb{R}^n$ and $f : A \to \mathbb{R}^n$ be of class C^r. If $Df(x)$ is non-singular at the point x of A, there is a neighborhood U of this point such that f carries U in a one-to-one fashion onto an open set V of $\mathbb{R}^n$ and the inverse function is of class C^r.

To show f need not be one-to-one on all of A.

Example 28.1.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then*

$$Df(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

and $\det(Df(r,\theta)) = r > 0$. *Let $A = \{(r,\theta) : r \in (0,1), \theta \in (0,b)\}$. Note that $Df(r,\theta)$ is non-singular at every point in A. However, f is one-to-one on A only if $b \le 2\pi$.*



Example 28.1.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^2 - y^2, 2xy)$.*

(a) *Show that* f *is one-to-one on* $A \triangleq \{(x,y) : x > 0\}$.

(b) *What is the set* $B = f(A)$?

(c) *If* g *is the inverse function, find* $Dg(0,1)$?

Solution:

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

and $\det(Df(x,y)) = 4x^2 + 4y^2 = 4\|(x,y)\|^2 > 0$ *if* $(x,y) \neq (0,0)$. *To show one-to-one of* f *consider*

$$\begin{aligned} f(x,y) &= f(a,b) \\ \Rightarrow ||f(x,y)||^2 &= ||f(a,b)||^2 \\ \Rightarrow (x^2-y^2)^2 + 4x^2y^2 &= (a^2-b^2)^2 + 4a^2b^2 \\ x^4 + y^4 - 2x^2y^2 + 4x^2y^2 &= a^4 + b^4 - 2a^2b^2 + 4a^2b^2 \\ (x^2+y^2)^2 &= (a^2+b^2)^2 \\ x^2+y^2 &= a^2+b^2. \end{aligned}$$

We also have

$$\begin{aligned} x^2 - y^2 &= a^2 - b^2 \\ 2xy &= 2ab. \end{aligned}$$

From

$$\begin{aligned} x^2 - y^2 &= a^2 - b^2 \\ x^2 + y^2 &= a^2 + b^2 \\ x^2 &= a^2. \end{aligned}$$

But it is given $x > 0$, *thus* $x = a, y = b$. *On the set* A *defined by* $A = \{(x,y) : x > 0\}$, f *is one-to-one and* $Df(x,y)$ *is invertible. It is clear* $0 \notin f(A)$ *and let* $(a,b) \neq 0, \quad (a,b) \in f(A)$. *Given* $(x,y) \in A$, *we have* $x^2 - y^2 = a, \quad 2xy = b$. *Thus,*

$$\begin{aligned} x^2 - \frac{b^2}{4x^2} &= a \\ 4x^4 - b^2 &= 4x^2a \\ 4x^4 - 4ax^2 &= b^2. \end{aligned}$$

Let $\bar{x} = x^2$, *then* $\bar{x} = \frac{a \pm \sqrt{a^2+b^2}}{2}$. *Since* $a - \sqrt{a^2+b^2} < 0$, *we discard it and are left with* $\bar{x} = \frac{a+\sqrt{a^2+b^2}}{2}$ *which implies* $x = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$. *This is well-defined since* $|a| < \sqrt{a^2+b^2} \quad \forall\, (a,b) \in \mathbb{R}^2$.

$$y^2 = \frac{b^2}{4x^2} \quad or \quad y = \frac{b}{2x}$$

Thus

$$f(x,y) = (a,b)$$
$$\therefore \quad B = \mathbb{R}^2 \backslash \{0\}.$$

c)

$$f(x,y) = (0,1)$$
$$x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow x = y$$
$$2xy = 1 \Rightarrow 2x^2 = 1$$
$$x = \frac{1}{\sqrt{2}}$$
$$y = \frac{1}{\sqrt{2}}$$
$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (0,1)$$
$$Dg(0,1) = (Df(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},))^{-1} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix}.$$

Example 28.1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be be given by $f(x) = \|x\|^2 x$. Show that $f \in C^\infty$ and f carries the unit ball $B(0,1)$ onto itself in a one-to-one fashion. Show, however, that the inverse function is not differentiable at 0.*
Solution:
For simplicity take $n = 2$.

$$f(x) = \|x\|^2 x$$
$$f(x) = \begin{bmatrix} (x_1^2 + x_2^2)x_1 \\ (x_1^2 + x_2^2)x_2 \end{bmatrix}$$

Since each component $f_i \in C^\infty, f \in C^\infty$. Next,

$$Df(x) = \begin{bmatrix} 3x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + 3x_2^2 \end{bmatrix} = \begin{bmatrix} x_1^2 & 2x_1x_2 \\ 2x_1x_2 & 2x_2^2 \end{bmatrix} + \begin{bmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{bmatrix} = 2\mathbf{x}\mathbf{x}^T + \|\mathbf{x}\|^2 I$$

At $(x_1, x_2) = (0,0)$,

$$Df(x) = 0$$

We will next show f is one-to-one on $B(0,1)$. Let $x, y \in B(0,1)$ such that $f(x) = f(y)$. If $x = 0$, then $f(x) = f(y) \Rightarrow 0 = 2yy^T + \|y\|^2 I \Rightarrow x = y$. Similarly, if $y = 0, x = y$. Therefore, we can assume x and y are both non-zero. Next, $f(x) = f(y) \Rightarrow \|f(x)\| = \|f(y)\| \Rightarrow \|\|x\|^2 x\| = \|\|y\|^2 y\| \Rightarrow \|x\|^3 = \|y\|^3 \Rightarrow x = y$. The last assertion follows by using the identity $(a^3 - b^3) = (a-b)(a^2 + b^2 - 2ab)$. Therefore, f is one-to-one on $B(0,1)$.
(c) If $x = 0$, $f(x) = 0$. So assume $x \neq 0$. Let

$$y = \|x\|^{-2/3} x.$$

Then

$$\|y\| = \frac{\|x\|}{\|x\|^{2/3}} = \|x\|^{1/3} < 1 \quad \text{for } \|x\| < 1.$$

Therefore, $y \in B(0,1)$.

$$\begin{aligned} f(y) &= \|y\|^2 y = \|y\|^2 \|x\|^{-2/3} x \\ &= \|\|x\|^{-2/3} x\|^2 \|x\|^{-2/3} x \\ &= \|x\|^{-4/3} \|x\|^2 \|x\|^{-2/3} x \\ &= x \in B(0,1). \end{aligned}$$

Therefore, f *is one-to-one on* $B(0,1)$.
c) The inverse of the function f is given by

$$g(y) = ||x||^{-2/3} x$$

We have,

$$\begin{aligned} f(0) &= 0 \\ g(0) &= 0 \\ g(f(x)) &= x \quad \text{for all } x \text{ in a neighborhood of } 0 \end{aligned}$$

If g was differentiable at 0, then

$$Dg(0) = (Df(0))^{-1}$$

But Df(0) is singular. $\therefore$ *g is not diff at 0.*

LECTURE-29

29.1 Implicit function Theorem

Example 29.1.1. *Consider: $f(x,y) = x^3y + 2xe^{xy} = 0$. Determine y as a function of x. Find $\frac{dy}{dx}$.*

$$\begin{aligned} 3x^2y + x^3\frac{dy}{dx} + 2e^{xy}\left(y + x\frac{dy}{dx}\right) &= 0 \\ \frac{dy}{dx}\left(x^3 + 2xe^{ny}\right) &= -2e^{xy}y - 3x^2y \\ \frac{dy}{dx} &= -\frac{2e^{xy}y + 3x^2y}{x^3 + 2xe^{xy}} \end{aligned}$$

provided, $x^3 + 2ne^{xy} \neq 0$. Suppose, we have $f(x,y) = 0$ and y is a function of x, say $y = g(x)$. Then, the differential of f w.r.t. x :

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}g'(x) = 0.$$

We can solve for $g'(x)$:

$$g'(x) = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{(provided } \frac{\partial f}{\partial y} \neq 0\text{)}.$$

Thus, if we are given $f(x,y) = 0$ and $\frac{\partial f}{\partial y}$ is invertible at say (a,b), then locally around a, there exists a differentiable function g such that $f(x, g(x)) = 0$.

Theorem 29.1.1:

Let A be an open set in $\mathbb{R}^{k+n}$. Let $f : A \to \mathbb{R}^n$ be differentiable, with f as $f(x, y)$, where $x \in \mathbb{R}^k, y \in \mathbb{R}^n$. Then,

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

Suppose there is a differentiable function $g : B \to \mathbb{R}^n$ defined on an open set $B \subset \mathbb{R}^k$, such that $f(x, g(x)) = 0 \quad \forall\, x \in B$. Then for $x \in B$,

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot Dg(x) = 0.$$

If $\frac{\partial f}{\partial y}(x, g(x))$ is invertible, then,

$$Dg(x) = -\left(\frac{\partial f}{\partial y}\right)^{-1} \left(\frac{\partial f}{\partial x}\right)\Bigg|_{(x,g(x))}.$$

Theorem 29.1.2: Implicit Function Theorem

Let A be an open set in $\mathbb{R}^{k+n}$; let $f : A \to \mathbb{R}^n$ be of class C^r, with $f = f(x, y)$, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^n$. Suppose that (a, b) is a point in A such that $f(a, b) = 0$ and

$$\det\left(\frac{\partial f}{\partial y}(a, b)\right) \neq 0.$$

Then there is a neighborhood B of a in $\mathbb{R}^k$ and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0 \quad \forall\, x \in B$.The function g is, in fact, of class C^r.

Example 29.1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x, y) = x^2 + y^2 - 5.$$

The point $(1, 2)$ satisfies f, since

$$f(1, 2) = 0.$$

Our objective is to solve for y in terms of x. That is, does there exist $g : B \to \mathbb{R}$ such that $f(x, g(x)) = 0 \quad \forall\, x \in B$. Note that,

$$\frac{\partial f}{\partial y} = 2y$$

and

$$\frac{\partial f}{\partial y}(1, 2) = 4 \neq 0.$$

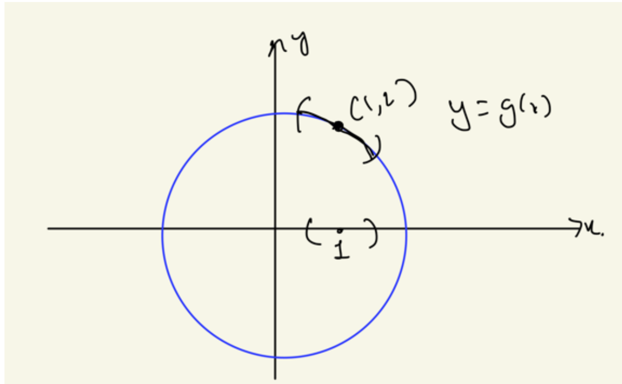


Figure 29.12: Unique function g

Thus, locally around $(1,2)$*, we can solve for* y *in terms of* x *using*

$$y = g(x) \approx \sqrt{5 - x^2}.$$

Note that the solution is not unique unless we specify that the function g *is unique in the neighborhood of* $x = 1$*. For example, the function*

$$h(x) = \begin{cases} \sqrt{5 - x^2}, & x > 1 \\ -\sqrt{5 - x^2}, & x < 1 \end{cases}$$

satisfies $f(x, h(x)) = 0$*, but is not continuous at* $x = 1$*.*

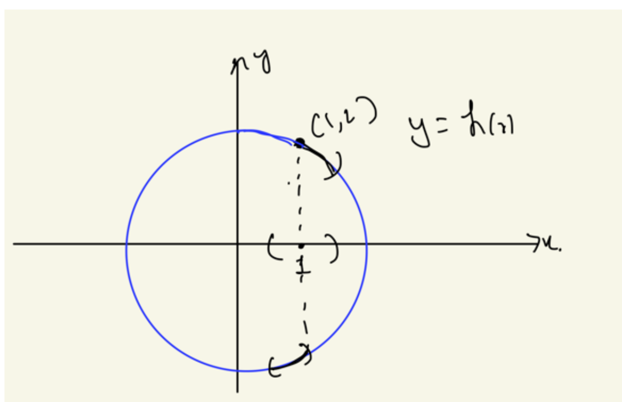


Figure 29.13: Discontinuous function h

Example 29.1.3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f(x, y) = x^2 + y^2 - 5.$$

The point $(\sqrt{5}, 0)$ *satisfies the equation, i.e* $f(\sqrt{5}, 0) = 0$*. But,*

$$\frac{\partial f}{\partial y} = 2y$$

is not invertible at $(\sqrt{5}, 0)$*. Therefore we cannot solve* y *in terms of* x *in a neighborhood of* $(\sqrt{5}, 0)$*.*

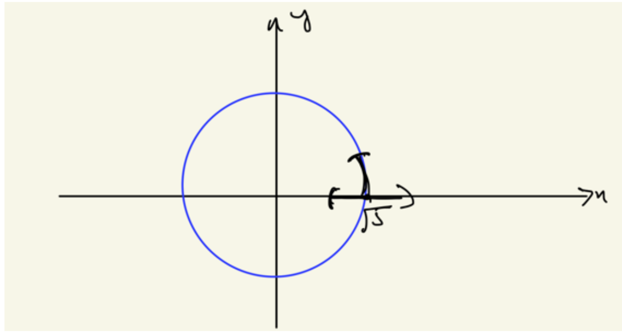


Figure 29.14: Nonexistence of inverse

Example 29.1.4. *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *be of class* C^1*. Suppose* $f(x, y_1, y_2) = 0$ *and* $(a, b, c) = (3, -1, 2)$ *such that* $f(a, b, c) = 0$.

$$Df(3, -1, 2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Show there is a function $g : B \to \mathbb{R}^2$ *of class* C^1 *defined on an open set* $B \subset \mathbb{R}$ *such that* $f(x, g_1(x), g_2(x)) = 0 \quad \forall x \in B$ *and* $g(3) = (-1, 2)$.

(b) Find $Dg(3)$.

Solution:

$$Df(x, y_1, y_2) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{bmatrix} \Bigg|_{(x, y_1, y_2)}$$

For the existence of g, where $g : \mathbb{R} \to \mathbb{R}^2$ i.e,

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

such that $f(x, g(x)) = 0$ and we need $\begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{bmatrix}$ to be invertible around $(3, -1, 2)$.

$$\begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix},$$

This matrix is invertible. Therefore, locally around x, there exists a unique C^1 function: $g = (g_1, g_2) : \mathbb{R} \to \mathbb{R}^2$ such that $f(x, g_1(x), g_2(x)) = 0$.

(b) We have, $f(x, g_1(x), g_2(x)) = 0$. So,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y_1} Dg_1(x) + \frac{\partial f}{\partial y_2} Dg_2(x) = 0.$$

$$\frac{\partial f}{\partial x} + \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{bmatrix} \begin{bmatrix} Dg_1(x) \\ Dg_2(x) \end{bmatrix} = 0.$$

Therefore,

$$\begin{aligned}\begin{bmatrix} Dg_1(x) \\ Dg_2(x) \end{bmatrix} &= -\begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial x} \end{bmatrix} \\ &= -\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}.\end{aligned}$$

Example 29.1.5. *Given $f : \mathbb{R}^5 \to \mathbb{R}^2$ of class C^1. Let $a = (1, 2, -1, 3, 0)$. Suppose that $f(a) = 0$ and*

$$Df(a) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}.$$

(a) Show that there is a function $g : B \to \mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}^3$ such that

$$f(x_1, g_1(x), g_2(x), x_2, x_3) = 0$$

for $x = (x_1, x_2, x_3) \in B$ and $g(1, 3, 0) = (2, -1)$.

(b) Find $Dg(1, 3, 0)$.

Solution:
Given,

$$Df(a) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}$$

Since,

$$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

is invertible, there exists $g : B \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(x_1, y_1, y_2, x_2, x_3) = 0$ and

$$\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{at } (1, 3, 0).$$

Now,

$$f(x_1, g_1(x), g_2(x), x_2, x_3) = 0$$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} + \frac{\partial f}{\partial y_1} Dg_1(x) + \frac{\partial f}{\partial y_2} Dg_2(x) = 0$$

$$\begin{aligned}
\begin{bmatrix} Dg_1(x) \\ Dg_2(x) \end{bmatrix} &= -\left[\begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \end{bmatrix}\right]^{-1} \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} \\
&= -\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix} \\
Dg(1,3,0) &= -\frac{1}{3}\begin{bmatrix} 1 & -3 & 6 \\ 0 & 6 & -12 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}
\end{aligned}$$

LECTURE-30

30.1 Riemann Integration

Definition 30.1.1: Bounded function

A function $f : [a,b] \to \mathbb{R}$ is called **bounded** if there exists a real number M such that

$$|f(x)| \le M \quad \text{for all } x \in [a,b].$$

We denote the set of all bounded functions on $[a,b]$ by $\mathcal{B}[a,b]$.

Example 30.1.1. *Let $f(x) = x$ on the interval $[0,1]$. This function is bounded since $|f(x)| \le 1$ for all $x \in [0,1]$.*

Example 30.1.2. *Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x}$. This function is **not bounded** on $[0,1]$. However, $f \in \mathcal{B}[1,2]$ since $|f(x)| \le 1 \quad$ for all $x \in [1,2]$.*

Example 30.1.3. *Let $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Suppose for contradiction that f is bounded. Let $M > 1$, then $\frac{1}{M+1} \in [0,1]$ and $f(\frac{1}{M+1}) = M + 1 > M$, a contradiction. Hence, $f \notin \mathcal{B}[0,1]$.

Example 30.1.4. *The Dirichlet Function*

Consider

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

This function is bounded, since $|\chi_{\mathbb{Q}}(x)| \le 1$ for all $x \in [0,1]$. Therefore, $\chi_{\mathbb{Q}} \in \mathcal{B}[0,1]$.

30.1.1 Continuity and Boundedness

In the previous example (the Dirichlet function), the function was not continuous, yet it was bounded. We will now show that if a function f is continuous on a closed interval $[a,b]$, then it must be bounded on that interval.

Lemma 30.1.2

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f \in \mathcal{B}[a, b]$.

Proof. The function f is continuous on the interval $[a, b]$. The interval $[a, b]$ is a compact set in $\mathbb{R}$. A well-known theorem in analysis states that the continuous image of a compact set is compact. Therefore, the set

$$f([a, b]) = \{f(x) : x \in [a, b]\}$$

is a compact subset of $\mathbb{R}$. Since $f([a, b])$ is bounded, there exists some $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. $\square$

Definition 30.1.3: Partion

A **partition** P of an interval $[a, b]$ is a finite set of points

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

such that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

We can think of a partition as dividing the interval $[a, b]$ into a union of n closed subintervals:

$$[a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i].$$

The i-th subinterval is denoted by $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

Recall the definitions of the **infimum** (greatest lower bound) and **supremum** (least upper bound) of a function f over a non-empty bounded subset A:

$$\inf_A f = \inf\{f(x) : x \in A\}, \qquad \sup_A f = \sup\{f(x) : x \in A\}.$$

30.2 Lower and Upper Riemann Sums

Definition 30.2.1: Lower and Upper Riemann Sums

Let $f \in \mathcal{B}[a, b])$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. For each subinterval $I_j = [x_{j-1}, x_j]$, define:

- $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$ (infimum on the j-th subinterval)
- $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$ (supremum on the j-th subinterval)

Then the **lower Riemann sum** of f with respect to P is defined as

$$L(f, P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),$$

and the **upper Riemann sum** of f with respect to P is defined as

$$U(f, P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}).$$

Example 30.2.1. *Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the partition P_n which divides the interval $[0, 1]$ into n equal subintervals:*

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}.$$

The j-th subinterval is

$$I_j = [x_{j-1}, x_j] = \left[\frac{j-1}{n}, \frac{j}{n}\right],$$

and the length of each subinterval is

$$\Delta x_j = x_j - x_{j-1} = \frac{1}{n}.$$

Since $f(x) = x^2$ is an increasing function on $[0, 1]$, the infimum on I_j occurs at the left endpoint and the supremum at the right endpoint:

- $m_j = \inf\{f(x) : x \in I_j\} = f\left(\frac{j-1}{n}\right) = \frac{(j-1)^2}{n^2}$
- $M_j = \sup\{f(x) : x \in I_j\} = f\left(\frac{j}{n}\right) = \frac{j^2}{n^2}$

for $j = 1, 2, \ldots, n.$

$$\begin{aligned} L(f, P_n) &= \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \frac{(j-1)^2}{n^2} \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{j=1}^{n} (j-1)^2 \\ &= \frac{1}{n^3} \sum_{k=1}^{n-1} k^2. \end{aligned}$$

$$\begin{aligned} U(f, P_n) &= \sum_{j=1}^{n} M_j (x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \frac{j^2}{n^2} \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{j=1}^{n} j^2. \end{aligned}$$

Recall the standard formulas for sums of powers:

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

Using this, we obtain

$$\begin{aligned} L(f, P_n) = \frac{1}{n^3} \sum_{j=1}^{n} (j-1)^2 &= \frac{1}{n^3} \left[\frac{n(2n^2 - 3n + 1)}{6} \right] \\ &= \frac{2n^2 - 3n + 1}{6n^2}. \end{aligned}$$

$$\begin{aligned} U(f, P_n) = \frac{1}{n^3} \sum_{j=1}^{n} j^2 &= \frac{1}{n^3} \left[\frac{n(2n^2 + 3n + 1)}{6} \right] \\ &= \frac{2n^2 + 3n + 1}{6n^2}. \end{aligned}$$

Proposition 30.2.2

Let $f \in \mathcal{B}[a,b]$ and

$$m = \inf_{x\in[a,b]} f(x) \quad \text{and} \quad M = \sup_{x\in[a,b]} f(x).$$

Then, for any partition P of $[a,b]$, the following inequality holds:

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

Proof. (The detailed proof is omitted here.)

The proposition implies that the set of all lower sums

$$\{\, L(f,P) : P \text{ is a partition of } [a,b] \,\}$$

and the set of all upper sums

$$\{\, U(f,P) : P \text{ is a partition of } [a,b] \,\}$$

are bounded sets. This allows us to take the supremum of all lower sums and the infimum of all upper sums. □

Definition 30.2.3: Lower and Upper Integrals

Let $f \in \mathcal{B}[a,b]$. The **Lower Riemann Integral** of f on $[a,b]$ is defined as the supremum of all lower sums:

$$\underline{\int_a^b} f(x)\,dx = \sup\{\, L(f,P) : P \text{ is a partition of } [a,b] \,\}.$$

The **Upper Riemann Integral** of f on $[a,b]$ is defined as the infimum of all upper sums:

$$\overline{\int_a^b} f(x)\,dx = \inf\{\, U(f,P) : P \text{ is a partition of } [a,b] \,\}.$$

Note that the lower and upper integrals of a bounded function need not be equal.

Example 30.2.2. *The Dirichlet Function*

Consider the Dirichlet function $\chi_{\mathbb{Q}}$ on $[0,1]$:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

For any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$, and for each subinterval $I_j = [x_{j-1}, x_j]$:

- $m_j = \inf\{\chi_{\mathbb{Q}}(x) : x \in I_j\} = 0$, since every interval contains an irrational number.
- $M_j = \sup\{\chi_{\mathbb{Q}}(x) : x \in I_j\} = 1$, since every interval contains a rational number.

Hence,

$$L(\chi_{\mathbb{Q}}, P) = \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0,$$

and

$$U(\chi_{\mathbb{Q}}, P) = \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = (x_n - x_0) = 1.$$

Since this holds for *any* partition P:

$$\underline{\int_0^1} \chi_{\mathbb{Q}}(x)\,dx = \sup\{L(\chi_{\mathbb{Q}}, P\} = 0,$$

$$\overline{\int_0^1} \chi_{\mathbb{Q}}(x)\,dx = \inf\{U(\chi_{\mathbb{Q}}, P)\} = 1.$$

Thus, the lower and upper integrals are not equal.

Definition 30.2.4: Riemann Integrability

Let $f \in \mathcal{B}[a,b]$. We say that f is **Riemann integrable** on $[a,b]$ if its lower and upper integrals are equal:

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

In this case, we call this common value the **Riemann integral** of f from a to b, denoted by

$$\int_a^b f(x)\,dx.$$

That is,

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Refinement of Partitions

Definition 30.2.5: Refinement

A partition P^* of $[a,b]$ is called a **refinement** of a partition P if every point of P is also in P^*; that is,

$$P \subseteq P^*.$$

A partition P^* is a **common refinement** of two partitions P_1 and P_2 if it is a refinement of both. A simple way to construct a common refinement is by taking the union of their points and arranging them in increasing order:

$$P^* = P_1 \cup P_2.$$

Example 30.2.3. *Let the interval be* $[0, 4]$.

- $P = \{0, 2, 4\}$
- $P^* = \{0, 1, 2, 4\}$

Since every point of P is also in P^*, we say that P^* is a refinement of P.

Example 30.2.4. *Let the interval be* $[0, 5]$.

- $P_1 = \{0, 1, 2, 3.5, 5\}$
- $P_2 = \{0, 2, 4, 5\}$

A common refinement is obtained by taking the union (and ordering it):

$$P^* = P_1 \cup P_2 = \{0, 1, 2, 3.5, 4, 5\}.$$

Example 30.2.5. *Let the interval be* $[0, 1]$. *Consider the partitions:*

$$P_1 = \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1\}, \quad P_2 = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\}.$$

We can form a common refinement P^* by taking their union and ordering the points:

$$P^* = P_1 \cup P_2 = \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, 1\}.$$

By definition, P^* is a refinement of both P_1 and P_2.

Lemma 30.2.6

Let $f \in \mathcal{B}[a, b]$.

(i) If P^* is a refinement of the partition P of $[a, b]$, then

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P).$$

(ii) If P_1 and P_2 are any two partitions of $[a, b]$, then

$$L(f, P_1) \leq U(f, P_2).$$

Proof. (i.) Let $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \ldots < x_n = b$. Let's first consider the case where P^* is a refinement of P obtained by adding just one additional point. Let $P = \{x_0, x_1, \ldots, x_n\}$. Suppose $P^* = \{x_0, \ldots, x_{j-1}, x^*, x_j, \ldots, x_n\}$, where $x_{j-1} < x^* < x_j$. Most of the terms in the sum for $L(f, P^*)$ will also appear in $L(f, P)$. The j-th term in $L(f, P)$, which is $m_j(x_j - x_{j-1})$, is replaced in $L(f, P^*)$ by two terms. Let

- $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$
- $m_1^* = \inf\{f(x) : x \in [x_{j-1}, x^*]\}$
- $m_2^* = \inf\{f(x) : x \in [x^*, x_j]\}$

Since $[x_{j-1}, x^*] \subset [x_{j-1}, x_j]$ and $[x^*, x_j] \subset [x_{j-1}, x_j]$, the infimum over the larger interval must be less than or equal to the infimum over the smaller subintervals. Therefore, we have:

$$m_j \leq m_1^* \quad \text{and} \quad m_j \leq m_2^*.$$

Now, let us analyze the Lower Sum $L(f, P)$. We split the j-th term, which is the one containing the new point x^*:

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i(x_i - x_{i-1}) + m_j(x_j - x_{j-1}). \end{aligned}$$

We can rewrite the length of the j-th interval as

$$(x_j - x_{j-1}) = (x_j - x^*) + (x^* - x_{j-1}).$$

Hence,

$$\begin{aligned} L(f, P) &= \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i(x_i - x_{i-1}) + m_j\big((x_j - x^*) + (x^* - x_{j-1})\big) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i(x_i - x_{i-1}) + m_j(x_j - x^*) + m_j(x^* - x_{j-1}). \end{aligned}$$

Since the interval lengths $(x_j - x^*)$ and $(x^* - x_{j-1})$ are positive, we can write:

$$\begin{aligned} L(f, P) &\leq \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i(x_i - x_{i-1}) + m_2^*(x_j - x^*) + m_1^*(x^* - x_{j-1}) \\ &= L(f, P^*). \end{aligned}$$

Thus, we have shown that

$$L(f, P) \leq L(f, P^*).$$

By a similar argument (using the supremums $M_j \geq M_1^*$ and $M_j \geq M_2^*$), we obtain

$$U(f, P) \geq U(f, P^*).$$

The general result follows by induction on the number of points added to P to obtain P^*.

ii. Let $P^\star$ be the common refinement of P_1 and P_2. Then, by part (i.),

$$L(f, P_1) \le L(f, P^\star) \le U(f, P^\star)$$

and

$$L(f, P^\star) \le U(f, P^\star) \le U(f, P_2).$$

The result then follows. □

Corollary 30.2.7

For any bounded function f on $[a, b]$, the lower Riemann integral is less than or equal to the upper Riemann integral:

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Proof. Let P_1 and P_2 be any two partitions of $[a, b]$. For any partition P, the lower sum is less than or equal to the upper sum, so $L(f, P_1) \le U(f, P_1)$.

If P is a common refinement of P_1 and P_2, then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

Thus, for any two partitions P_1, P_2, we have $L(f, P_1) \le U(f, P_2)$.

This means $U(f, P_2)$ is an upper bound for the set of all lower sums $\{L(f, P) \mid P \text{ is a partition}\}$. Since the lower integral is the supremum of this set,

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(f, P) \le U(f, P_2).$$

This holds for any partition P_2. Hence the lower integral is a lower bound for the set of all upper sums. Since the upper integral is the infimum of this set,

$$\underline{\int_a^b} f(x)\,dx \le \inf_P U(f, P) = \overline{\int_a^b} f(x)\,dx.$$

□

Theorem 30.2.8: Darboux's criterion for integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Proof. ($\Longrightarrow$) Assume f is Riemann integrable on $[a, b]$. By definition,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Let $\epsilon > 0$. By definition of supremum, there exists a partition P_1 such that

$$L(f, P_1) > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} = \int_a^b f(x)\,dx - \frac{\epsilon}{2}.$$

Similarly, by definition of infimum, there exists a partition P_2 such that

$$U(f, P_2) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} = \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

Let P be a common refinement of P_1 and P_2. Then

$$L(f, P) \geq L(f, P_1), \qquad U(f, P) \leq U(f, P_2).$$

Therefore,

$$U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1) < \left(\int_a^b f(x)\,dx + \frac{\epsilon}{2}\right) - \left(\int_a^b f(x)\,dx - \frac{\epsilon}{2}\right) = \epsilon.$$

Thus $U(f, P) - L(f, P) < \epsilon$.

($\Longleftarrow$) Conversely, assume that for every $\epsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

For any partition P,

$$L(f, P) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f, P).$$

So for this P we have

$$\overline{\int_a^b} f(x)\,dx \leq U(f, P) < L(f, P) + \epsilon \leq \underline{\int_a^b} f(x)\,dx + \epsilon.$$

Thus

$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon.$$

Since this holds for every $\epsilon > 0$, the difference must be 0:

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Hence f is Riemann integrable. $\square$

Example 30.2.6. *Let $f : [0,1] \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 5, & \text{if } x = \frac{1}{2}, \\ 2, & \text{if } x \in [0,1] \setminus \left\{\frac{1}{2}\right\}. \end{cases}$$

We will show that f is Riemann integrable on $[0,1]$ and

$$\int_0^1 f(x)\,dx = 2.$$

Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{7}$ (any $\delta < \epsilon/3$ would work). Assume ϵ small enough so that $0 < \frac{1}{2} - \delta$ and $\frac{1}{2} + \delta < 1$. Define the partition

$$P_\epsilon = \left\{0,\, \frac{1}{2} - \delta,\, \frac{1}{2} + \delta,\, 1\right\} = \left\{0,\, \frac{1}{2} - \frac{\epsilon}{7},\, \frac{1}{2} + \frac{\epsilon}{7},\, 1\right\}.$$

This produces three subintervals

$$I_1 = \left[0, \frac{1}{2} - \frac{\epsilon}{7}\right], \quad I_2 = \left[\frac{1}{2} - \frac{\epsilon}{7}, \frac{1}{2} + \frac{\epsilon}{7}\right], \quad I_3 = \left[\frac{1}{2} + \frac{\epsilon}{7}, 1\right].$$

Their lengths are

$$\Delta x_1 = \frac{1}{2} - \frac{\epsilon}{7}, \quad \Delta x_2 = \frac{2\epsilon}{7}, \quad \Delta x_3 = \frac{1}{2} - \frac{\epsilon}{7}.$$

On I_1 and I_3, $f(x) = 2$, so $m_1 = M_1 = m_3 = M_3 = 2$. On I_2, f takes values 2 and 5, so $m_2 = 2$ and $M_2 = 5$.

$$L(f, P_\epsilon) = 2\left(\frac{1}{2} - \frac{\epsilon}{7}\right) + 2\left(\frac{2\epsilon}{7}\right) + 2\left(\frac{1}{2} - \frac{\epsilon}{7}\right) = 2.$$

$$U(f, P_\epsilon) = 2\left(\frac{1}{2} - \frac{\epsilon}{7}\right) + 5\left(\frac{2\epsilon}{7}\right) + 2\left(\frac{1}{2} - \frac{\epsilon}{7}\right) = 2 + \frac{6\epsilon}{7}.$$

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \frac{6\epsilon}{7} < \epsilon.$$

Thus, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$, so f is Riemann integrable. Moreover,

$$L(f, P_\epsilon) \le \int_0^1 f(x)\,dx \le U(f, P_\epsilon) \Rightarrow 2 \le \int_0^1 f(x)\,dx \le 2 + \frac{6\epsilon}{7}.$$

Since this holds for every $\epsilon > 0$, we conclude

$$\int_0^1 f(x)\,dx = 2.$$

LECTURE-31

31.1 Sequential characterization of Riemann integral

Theorem 31.1.1

Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if there exists a partition P_n of $[a,b]$ such that
$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

Example 31.1.1. *Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$. We will show that $f \in \mathcal{R}[0,1]$. Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$. Then*

$$L(f,P_n) = \sum_{i}^{n} \frac{(i-1)}{n^2} = \frac{(n-1)}{2n}$$
$$U(f,P_n) = \sum_{i}^{n} \frac{i}{n^2} = \frac{(n+1)}{2n}$$

Thus,
$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = \lim_{n\to\infty} \frac{1}{2n} = 0.$$

Theorem 31.1.2

Let $f, g \in \mathcal{R}[a,b]$. If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Proof. Let f, g be Riemann integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. For any partition P and any subinterval $I \in P$ define
$$m_f(I) = \inf_I f,\ M_f(I) = \sup_I f, \qquad m_g(I) = \inf_I g,\ M_g(I) = \sup_I g.$$

Since $f \leq g$ on I we have $m_f(I) \leq m_g(I)$ and $M_f(I) \leq M_g(I)$. Summing over $I \in P$ and multiplying by the lengths $|I|$ yields
$$L(f,P) \leq L(g,P), \qquad U(f,P) \leq U(g,P).$$

Taking suprema over partitions for the lower sums and infima over partitions for the upper sums gives

$$\underline{\int_a^b} f dx \leq \underline{\int_a^b} g dx, \qquad \overline{\int_a^b} f dx \leq \overline{\int_a^b} g dx.$$

Since f and g are integrable, the lower and upper integrals coincide with the Riemann integrals, so

$$\int_a^b f dx \leq \int_a^b g dx.$$

□

LECTURE-32

2.1 Properties of the Integral

Theorem 2.1.1

If f and g are integrable on $[a,b]$, then $f+g$ is also integrable on $[a,b]$, and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. Given $\epsilon > 0$, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$U(P_1,f) - L(P_1,f) < \frac{\epsilon}{2}, \quad U(P_2,g) - L(P_2,g) < \frac{\epsilon}{2}.$$

Let $P^* = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then,

$$\begin{aligned} U(P^*, f+g) &\le U(P^*, f) + U(P^*, g) \\ &\le U(P_1, f) + U(P_2, g), \\ L(P^*, f+g) &\ge L(P_1, f) + L(P_2, g). \end{aligned}$$

Therefore,

$$\begin{aligned} U(P^*, f+g) - L(P^*, f+g) &\le \big[U(P_1,f) + U(P_2,g)\big] - \big[L(P_1,f) + L(P_2,g)\big] \\ &= \big[U(P_1,f) - L(P_1,f)\big] + \big[U(P_2,g) - L(P_2,g)\big] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, $f+g$ is integrable on $[a,b]$, and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

□

Theorem 2.1.2

If f is integrable on $[a, b]$ and c is a constant, then cf is integrable on $[a, b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem 2.1.3

If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. Note that $g(x) - f(x) \ge 0$ for all $x \in [a, b]$. This implies

$$\underline{\int_a^b}(g(x) - f(x))\,dx \ge 0.$$

Note that it need not be true that the upper integral is non-negative. Therefore,

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g - f)(x)\,dx = \underline{\int_a^b}(g(x) - f(x))\,dx \ge 0$$

□

Theorem 2.1.4

If f is integrable on $[a, b]$, then so is $|f|$, and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$. Define

$$M_j = \sup_{x\in[x_{j-1},x_j]} f(x), \quad m_j = \inf_{x\in[x_{j-1},x_j]} f(x),$$

and

$$\overline{M}_j = \sup_{x\in[x_{j-1},x_j]} |f(x)|, \quad \overline{m}_j = \inf_{x\in[x_{j-1},x_j]} |f(x)|.$$

Now, for each subinterval $[x_{j-1}, x_j]$ we have

$$\overline{M}_j - \overline{m}_j = \sup_{x\in[x_{j-1},x_j]} |f(x)| - \inf_{x\in[x_{j-1},x_j]} |f(x)|.$$

Recall that for any function h on an interval I,

$$\sup_{x\in I} h(x) - \inf_{x\in I} h(x) = \sup_{x,x'\in I} \big(h(x) - h(x')\big).$$

Applying this to $h(x) = |f(x)|$, we get

$$\overline{M}_j - \overline{m}_j = \sup_{x,x'\in[x_{j-1},x_j]} \big(|f(x)| - |f(x')|\big).$$

Since for all real numbers a, b,

$$\big|\,|a| - |b|\,\big| \le |a - b|,$$

it follows that

$$\sup_{x,x'\in[x_{j-1},x_j]} \big(|f(x)| - |f(x')|\big) \le \sup_{x,x'\in[x_{j-1},x_j]} \big(f(x) - f(x')\big) = M_j - m_j.$$

Hence,

$$\overline{M}_j - \overline{m}_j \le M_j - m_j.$$

Summing over all subintervals, we obtain

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P).$$

Since $f \in R[a, b]$, given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Then it follows that

$$U(|f|, P) - L(|f|, P) < \epsilon.$$

Therefore, $|f| \in R[a, b]$. Next, since $f(x) \le |f(x)|$ and $-f(x) \le |f(x)|$ for all $x \in [a, b]$, by we have

$$\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx, \qquad -\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

Combining both inequalities gives

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

□

Theorem 2.1.5

If $f, g \in R[a, b]$, then $fg \in R[a, b]$.

Proof. Assume WLG $f \neq 0$ and $g \neq 0$. Also, consider the case f and g are non-negative. Let $\epsilon > 0$. Since $f, g \in R[a, b]$, there exists partions P_1, P_2 of $[a, b]$ s.t Let

$$M_f = \sup_{x \in [a,b]} |f(x)|, \qquad M_g = \sup_{x \in [a,b]} |g(x)|.$$

Both are finite since f, g are bounded on $[a, b]$. We assumed $f, g \geq 0$ on $[a, b]$. For any subinterval $[x_{j-1}, x_j]$, we have

$$\sup(fg) \leq \sup(f)\sup(g), \qquad \inf(fg) \geq \inf(f)\inf(g).$$

Thus,

$$M_{fg,j} \leq M_{f,j}M_{g,j}, \qquad m_{fg,j} \geq m_{f,j}m_{g,j}.$$

Consider the difference:

$$M_{fg,j} - m_{fg,j} \leq M_{f,j}M_{g,j} - m_{f,j}m_{g,j}.$$

Rewriting the right-hand side,

$$M_{f,j}M_{g,j} - m_{f,j}m_{g,j} = (M_{f,j} - m_{f,j})M_{g,j} + m_{f,j}(M_{g,j} - m_{g,j}).$$

Summing over all subintervals with $\Delta x_j = x_j - x_{j-1}$,

$$\sum_{j=1}^{n}(M_{fg,j} - m_{fg,j})\Delta x_j \leq \sum_{j=1}^{n} M_{g,j}(M_{f,j} - m_{f,j})\Delta x_j + \sum_{j=1}^{n} m_{f,j}(M_{g,j} - m_{g,j})\Delta x_j.$$

Then

$$U(P, fg) - L(P, fg) \leq M_g\big(U(P, f) - L(P, f)\big) + M_f\big(U(P, g) - L(P, g)\big).$$

Take $P = P_1 \cup P_2$. Then

$$U(P, fg) - L(P, fg) < M_g \cdot \frac{\epsilon}{2M_g} + M_f \cdot \frac{\epsilon}{2M_f} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $fg \in R[a, b]$. □

Theorem 2.1.6: First Mean Value Theorem for Integrals

Suppose u is continuous on $[a, b]$ and v is integrable and non-negative on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$\int_a^b u(x)v(x)\,dx = u(c)\int_a^b v(x)\,dx.$$

Proof. Since u is continuous on the closed interval $[a, b]$, it attains its minimum and maximum values. Let

$$m = \min\{u(x) : x \in [a, b]\}, \quad M = \max\{u(x) : x \in [a, b]\}.$$

This means that for all $x \in [a, b]$, we have $m \leq u(x) \leq M$.

Since $v(x)$ is non-negative, we can multiply the inequality by $v(x)$ without changing the direction of the inequality:

$$mv(x) \leq u(x)v(x) \leq Mv(x).$$

Now, integrate from a to b:

$$\int_a^b mv(x)\,dx \leq \int_a^b u(x)v(x)\,dx \leq \int_a^b Mv(x)\,dx.$$

The constants m and M can be factored out:

$$m\int_a^b v(x)\,dx \leq \int_a^b u(x)v(x)\,dx \leq M\int_a^b v(x)\,dx.$$

1. **Case 1: If** $\int_a^b v(x)\,dx = 0$
   Then the inequality becomes
   $$0 \leq \int_a^b u(x)v(x)\,dx \leq 0,$$
   which implies
   $$\int_a^b u(x)v(x)\,dx = 0.$$
   The theorem holds for any $c \in [a, b]$.

2. **Case 2: If** $\int_a^b v(x)\,dx > 0$
   Divide the inequality by $\int_a^b v(x)\,dx$:
   $$m \leq \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx} \leq M.$$
   Let
   $$k = \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx}.$$
   Since $m \leq k \leq M$, by the Intermediate Value Theorem, there exists a number $c \in [a, b]$ such that $u(c) = k$. Therefore,
   $$u(c) = \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx}.$$
   Rearranging gives the desired result:
   $$\int_a^b u(x)v(x)\,dx = u(c)\int_a^b v(x)\,dx.$$

□

1. If we let $v(x) \equiv 1$, then $\int_a^b v(x)\,dx = (b-a)$. The theorem becomes:
$$\int_a^b u(x)\,dx = u(c)(b-a).$$
This is the standard Mean Value Theorem for Integrals.

2. The expression
$$\frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx}$$
is called the weighted average of u over $[a,b]$ with respect to v.

Theorem 2.1.7

If $f \in R[a,b]$ and $a \leq a_1 < b_1 \leq b$, then $f \in R[a_1, b_1$.

Proof. Given $\epsilon > 0$, since $f \in R[a,b]$, there exists a partition P of $[a,b]$ s.t
$$U(P,f) - L(P,f) = \sum_{j=1}^{n}(M_j - m_j)(x_j - x_{j-1}) < \epsilon$$
Assume that a_1 and b_1 are partition points of P, if not they can be inserted to obtain a refinement of P, denoted by $P^\star$ and we have
$$U(P^\star, f) - L(P^\star, f) \leq U(P,f) - L(P,f) < \epsilon.$$
$P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ and let $a_1 = x_r, b_1 = x_s, r < s$. In
$$U(P,f) - L(P,f) = \sum_{j=1}^{n}(M_j - m_j)(x_j - x_{j-1}) < \epsilon$$
every term is nonnegative, and hence
$$\sum_{j=r+1}^{s}(M_j - m_j)(x_j - x_{j-1}) < \epsilon$$
Thus $\bar{P} = \{a_1 = x_r, x_{r+1}, \ldots, x_s = b_1\}$ is a partion of $[a_1, b_1]$ s.t
$$U(\bar{P}, f) - L(\bar{P}, f) < \epsilon$$
This implies $f \in R[a_1, b_1]$. □

Theorem 2.1.8

If f is integrable on $[a,b]$ and $a < c < b$, then f is integrable on $[a,c]$ and $[c,b]$.

Proof. Since $f \in R[a,b]$, given $\epsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon.$$

We can assume, without loss of generality, that the partition P includes the point c. If not, we can form a new partition $P^* = P \cup \{c\}$, which is a refinement of P. We know that for a refinement P^*, $U(P^*,f) \leq U(P,f)$ and $L(P^*,f) \geq L(P,f)$, so

$$U(P^*,f) - L(P^*,f) \leq U(P,f) - L(P,f) < \epsilon.$$

Therefore, we can work with the partition P that contains c.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Let c be a point in (a,b). We can write the difference between the upper and lower sums for the integral over $[a,b]$ as:

$$U(P,f) - L(P,f) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}).$$

Since f is integrable, for every $\epsilon > 0$ there exists a partition P such that

$$U(P,f) - L(P,f) < \epsilon.$$

Because each term in the sum is non-negative, the whole sum being less than ϵ implies each partial sum is also bounded by ϵ.

Now, consider the partitions of $[a,c]$ and $[c,b]$ that are induced by P. Let

$$P_1 = P \cap [a,c], \qquad P_2 = P \cap [c,b].$$

The difference between the upper and lower sums for the integral over $[a,c]$ is

$$U(P_1,f) - L(P_1,f) = \sum_{x_j \leq c} (M_j - m_j)(x_j - x_{j-1}),$$

and for the integral over $[c,b]$:

$$U(P_2,f) - L(P_2,f) = \sum_{x_j \geq c} (M_j - m_j)(x_j - x_{j-1}).$$

Since all terms in the sums are non-negative, and their total sum is less than ϵ, it follows that each partial sum must also be less than ϵ:

$$U(P_1,f) - L(P_1,f) < \epsilon, \qquad U(P_2,f) - L(P_2,f) < \epsilon.$$

Therefore, f is integrable on $[a,c]$ and on $[c,b]$. □

Theorem 2.1.9

Suppose $f \in R[a,b]$ and $a < c < b$. Then $f \in R[a,b]$ iff $f \in R[a,c]$ and $f \in R[c,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Let $f \in R[a, b]$. Given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon.$$

Let $P\prime$ be a refinement of P that includes the point c, i.e.,

$$P\prime = P \cup \{c\}.$$

If $c \in P$, then $P' = P$, otherwise, $P' = Q \cup R$, where $Q = [a, c] \cap P'$ and $R = [c, b] \cap P'$. Then we still have

$$U(P', f) - L(P', f) < \epsilon.$$

Now, let

$$Q = P' \cap [a, c], \qquad R = P' \cap [c, b].$$

Then Q and R are partitions of $[a, c]$ and $[c, b]$, respectively. By properties of upper and lower sums, we have

$$U(f, P') = U(f, Q) + U(f, R),$$

and

$$L(f, P') = L(f, Q) + L(f, R).$$

Subtracting these, we obtain

$$U(f, P') - L(f, P') = \big(U(f, Q) - L(f, Q)\big) + \big(U(f, R) - L(f, R)\big).$$

Since all terms are non-negative, and the left-hand side is less than ϵ, it follows that

$$U(f, Q) - L(f, Q) < \epsilon, \qquad U(f, R) - L(f, R) < \epsilon.$$

Thus $f \in R[a, c]$ and $f \in R[c, b]$.
Conversely, let f be integrable on $[a, c]$ and $[c, b]$. Given $\epsilon > 0$, there exist partitions Q of $[a, c]$ and R of $[c, b]$ such that

$$U(Q, f) - L(Q, f) < \tfrac{\epsilon}{2}, \qquad U(R, f) - L(R, f) < \tfrac{\epsilon}{2}.$$

Now, let $P = Q \cup R$. Then P is a partition of $[a, b]$. The upper and lower sums for f on $[a, b]$ with respect to P are

$$U(P, f) = U(Q, f) + U(R, f), \qquad L(P, f) = L(Q, f) + L(R, f).$$

Subtracting, we get

$$U(P, f) - L(P, f) = \big(U(Q, f) - L(Q, f)\big) + \big(U(R, f) - L(R, f)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

This proves that f is integrable on $[a, b]$.
Finally,

$$\begin{aligned}\int_a^b f(x)\,dx &\le U(P, f) = U(Q, f) + U(R, f) < L(Q, f) + L(R, f) + \epsilon \\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \epsilon\end{aligned}$$

Similarly,

$$\int_a^b f(x)\,dx \geq L(P,f) = L(Q,f) + L(R,f) > U(Q,f) + U(R,f) + \epsilon$$
$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \epsilon$$

Thus,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \epsilon < \int_a^b f(x)\,dx < \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

□

Theorem 2.1.10

If f is integrable on $[a,b]$ and $a \leq c \leq b$, then the function defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition and is therefore continuous on $[a,b]$.

Proof. Let $x, y \in [a,b]$. We have

$$F(x) - F(y) = \int_c^x f(t)\,dt - \int_c^y f(t)\,dt$$
$$= \int_c^x f(t)\,dt + \int_y^c f(t)\,dt$$
$$= \int_y^x f(t)\,dt.$$

Now, consider the absolute value:

$$|F(x) - F(y)| = \left|\int_y^x f(t)\,dt\right| \leq \int_y^x |f(t)|\,dt.$$

Since f is integrable on $[a,b]$, it is bounded. Thus, there exists a constant M such that $|f(t)| \leq M$ for all $t \in [a,b]$. Hence,

$$|F(x) - F(y)| \leq \int_y^x M\,dt = M|x-y|.$$

Therefore,

$$|F(x) - F(y)| \le M|x - y|.$$

□

Theorem 2.1.11: The Fundamental Theorem of Calculus, Part I

If f is integrable on $[a,b]$, and $a \le c \le b$, then

$$F(x) = \int_c^x f(t)\,dt$$

is differentiable at any point $x_0 \in (a,b)$, where f is continuous with $F'(x_0) = f(x_0)$. If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Proof. Consider the case $x_0 \in (a,b)$. Since $f \in R[a,b]$, f is continous on (a,b). Since

$$= \frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,dt = f(x_0)$$

We can write

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0}\left[\int_c^x f(t)\,dt - \int_c^{x_0} f(t)\,dt\right] - f(x_0) \\
&= \frac{1}{x - x_0}\left[\int_c^x f(t)\,dt - \int_c^{x_0} f(t)\,dt\right] - \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt \\
&= \frac{1}{x - x_0}\left[\int_{x_0}^x (f(t) - f(x_0))\,dt\right]
\end{aligned}$$

Using the definition of $F(x)$, we have

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} &= \frac{1}{x - x_0}\left(\int_c^x f(t)\,dt - \int_c^{x_0} f(t)\,dt\right) \\
&= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt.
\end{aligned}$$

Subtracting $f(x_0)$, we get

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt - f(x_0) \\
&= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt - \frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt \\
&= \frac{1}{x - x_0}\int_{x_0}^x \big(f(t) - f(x_0)\big)\,dt.
\end{aligned}$$

Taking the absolute value:

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right| = \left|\frac{1}{x-x_0}\int_{x_0}^{x}\big(f(t)-f(x_0)\big)\,dt\right| \le \frac{1}{|x-x_0|}\int_{x_0}^{x}\big|f(t)-f(x_0)\big|\,dt.$$

Since f is continuous at x_0, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|t-x_0| < \delta$, then $|f(t)-f(x_0)| < \epsilon$. If $|x-x_0| < \delta$, then for all t between x_0 and x we have $|f(t)-f(x_0)| < \epsilon$. Hence,

$$\le \frac{1}{|x-x_0|}\int_{x_0}^{x}\epsilon\,dt = \frac{1}{|x-x_0|}\cdot\epsilon|x-x_0| = \epsilon.$$

Since we can make the difference arbitrarily small, it follows that

$$\lim_{x\to x_0}\left(\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right) = 0.$$

Therefore,

$$F'(x_0) = f(x_0).$$

□

Example 2.1.1. *Let*

$$f(x) = \begin{cases} x, & x\in[0,1],\\ x+1, & x\in(1,2].\end{cases}$$

Then the function

$$F(x) = \begin{cases} \dfrac{x^2}{2}, & x\in[0,1],\\ \dfrac{x^2}{2}+x-1, & x\in(1,2].\end{cases}$$

is continuous on $[0,2]$.

The function $f(x)$ is defined on $[0,2]$ and has a discontinuity at $x=1$. We define

$$F(x) = \int_0^x f(t)\,dt.$$

For $x\in[0,1]$, we have

$$F(x) = \int_0^x t\,dt = \frac{x^2}{2}.$$

For $x\in(1,2]$, we compute

$$F(x) = \int_0^1 t\,dt + \int_1^x (t+1)\,dt = \left[\tfrac{t^2}{2}\right]_0^1 + \left[\tfrac{t^2}{2}+t\right]_1^x = \tfrac{1}{2} + \left(\tfrac{x^2}{2}+x\right) - \left(\tfrac{1}{2}+1\right).$$

Thus,

$$F(x) = \frac{x^2}{2}+x-1.$$

So,

$$F(x) = \begin{cases} \dfrac{x^2}{2}, & x\in[0,1],\\ \dfrac{x^2}{2}+x-1, & x\in(1,2].\end{cases}$$

Continuity at $x = 1$

$$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} \frac{x^2}{2} = \frac{1}{2}, \quad \lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} \frac{x^2}{2} + x - 1 = \frac{1}{2}.$$

Hence, $F(x)$ is continuous at $x = 1$.

Derivatives at the endpoints

At $x = 0$ (right derivative):

$$F'_+(0) = \lim_{x\to 0^+} \frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0^+} \frac{\frac{x^2}{2}}{x} = \lim_{x\to 0^+} \frac{x}{2} = 0 = f(0).$$

At $x = 2$ (left derivative):

$$F'_-(2) = \lim_{x\to 2^-} \frac{F(x) - F(2)}{x - 2}.$$

Since $F(2) = \frac{2^2}{2} + 2 - 1 = 3$, we have

$$F'_-(2) = \lim_{x\to 2^-} \frac{\frac{x^2}{2} + x - 1 - 3}{x - 2} = \lim_{x\to 2^-} \frac{x^2 + 2x - 8}{2(x - 2)}.$$

Factorizing:

$$= \lim_{x\to 2^-} \frac{(x - 2)(x + 4)}{2(x - 2)} = \lim_{x\to 2^-} \frac{x + 4}{2} = \frac{6}{2} = 3 = f(2).$$

Now, let's consider the derivative of $F(x)$ at $x = 1$. We need to calculate the left-hand and right-hand derivatives.

Right-hand derivative at $x = 1$

$$F'_+(1) = \lim_{x\to 1^+} \frac{F(x) - F(1)}{x - 1}$$

Using the formula for $F(x)$ for $x > 1$:

$$F'_+(1) = \lim_{x\to 1^+} \frac{\left(\frac{x^2}{2} + x - 1\right) - \frac{1^2}{2}}{x - 1} = \lim_{x\to 1^+} \frac{\frac{x^2}{2} + x - \frac{3}{2}}{x - 1}$$

$$= \lim_{x\to 1^+} \frac{x^2 + 2x - 3}{2(x - 1)} = \lim_{x\to 1^+} \frac{(x - 1)(x + 3)}{2(x - 1)} = \lim_{x\to 1^+} \frac{x + 3}{2} = \frac{1 + 3}{2} = 2.$$

This value is equal to $f(1^+)$, which is the limit of $f(x)$ as x approaches 1 from the right.

Left-hand derivative at $x = 1$

$$F'_-(1) = \lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1}$$

Using the formula for $F(x)$ for $x < 1$:

$$F'_-(1) = \lim_{x \to 1^-} \frac{\frac{x^2}{2} - \frac{1^2}{2}}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{2(x - 1)}$$

$$= \lim_{x \to 1^-} \frac{(x-1)(x+1)}{2(x-1)} = \lim_{x \to 1^-} \frac{x+1}{2} = \frac{1+1}{2} = 1.$$

This value is equal to $f(1^-)$, which is the limit of $f(x)$ as x approaches 1 from the left.

Since $F'_-(1) \neq F'_+(1)$, the function $F(x)$ does not have a derivative at $x = 1$. This is because $f(x)$ is discontinuous at $x = 1$.

Theorem 2.1.12: The Fundamental Theorem of Calculus, Part II

Suppose that F is continuous on $[a, b]$ and differentiable on (a, b), and f is intgrable on $[a, b]$. Suppose that $F'(x) = f(x)$ for $x \in (a, b)$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We can write the difference $F(b) - F(a)$ as a telescoping sum:

$$F(b) - F(a) = \sum_{j=1}^{n} \big[F(x_j) - F(x_{j-1})\big].$$

By the Mean Value Theorem, for each subinterval $[x_{j-1}, x_j]$, there exists a point $c_j \in (x_{j-1}, x_j)$ such that

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1}).$$

Since $F'(x) = f(x)$, this becomes

$$F(x_j) - F(x_{j-1}) = f(c_j)(x_j - x_{j-1}).$$

Substituting into the sum, we obtain

$$F(b) - F(a) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}).$$

Since $f \in R[a, b]$, it is bounded and hence

$$m_j \leq f(c_j) \leq M_j$$
$$m_j(x_j - x_{j-1}) \leq F(x_j) - F(x_{j-1}) \leq M_j(x_j - x_{j-1})$$

Summing over $i = 1, \dots, n$

$$L(P, f) \le F(b) - F(a) \le U(P, f)$$

for every partition P of $[a, b]$. But $f \in R[a, b]$, hence $U(P, f) = L(P, f) = \int_a^b f(t)\,dt$. Therefore, $F(b) - F(a) = \int_a^b f(t)\,dt$. □

Definition 2.1.13: Anti-derivative

A function F is an antiderivative of f on $[a, b]$ if F is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem 2.1.14: Integration by Parts

Suppose u and v are differentiable on $[a, b]$ and their derivatives u' and v' are integrable on $[a, b]$. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

Proof. We are given that u and v are differentiable on $[a, b]$, which implies they are continuous on $[a, b]$. Since u and v' are integrable, and u is continuous, the product uv' is integrable. Similarly, since u' and v are integrable, and v is continuous, the product $u'v$ is integrable.

Consider the function

$$F(x) = u(x)v(x).$$

By the product rule, we have

$$F'(x) = u'(x)v(x) + u(x)v'(x).$$

Since both $u'v$ and uv' are integrable, their sum $F'(x)$ is also integrable.

Applying the Fundamental Theorem of Calculus, Part II, to F, we obtain

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

That is,

$$\int_a^b \big(u'(x)v(x) + u(x)v'(x)\big)\,dx = u(b)v(b) - u(a)v(a) = u(x)v(x)\Big|_a^b.$$

By linearity of the integral,

$$\int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b.$$

Rearranging terms, we obtain the formula for integration by parts:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

□

Theorem 2.1.15: Second mean value theorem for Integrals

Suppose f' is nonnegative and integrable and g is continuous on $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx \tag{2.6}$$

for some $c \in [a, b]$.

Proof. Since f is differentiable on (a, b), it is continuous on $[a, b]$. Also, g is continuous on $[a, b]$, hence the product fg is continuous on $[a, b]$. So all the integrals in (2.6) exist. If $G(x) = \int_x^a g(t)dt$, then $G'(x) = g(x)$ for $x \in (a, b)$. Therefore,

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &= \int_a^b f(x)G'(x)\,dx \\ &= f(x)G(x)|_a^b - \int_a^b f'(x)G(x)\,dx.\end{aligned}$$

Since f' is nonnegative and G is continuous on $[a, b]$, by first mean value theorem,

$$\int_a^b f'(x)G(x)\,dx = G(c)\int_a^b f'(x)\,dx$$

for some $c \in [a, b]$. By the fundamental theorem of Calculus-II,

$$\begin{aligned}\int_a^b f'(x)G(x)\,dx &= f(b)G(b) - f(a)G(a) - G(c)(f(b) - f(c)) \\ &= f(b)\int_a^b g(x)\,dx - f(a)\int_a^a g(x)\,dx - f(b)\int_a^c g(x)\,dx + f(a)\int_a^c g(x)\,dx \\ &= f(b)\int_c^b g(x)\,dx + f(a)\int_a^c g(x)\,dx.\end{aligned}$$

□

Theorem 2.1.16: Change of Variables

Suppose $g : I \to \mathbb{R}$ is differentiable on an open interval I. Let g' be integrable on I. Further, $f : J \to \mathbb{R}$ be continuous, where $J = g(I)$. Then for all $a, b \in I$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Let

$$F(x) = \int_a^x f(t)\,dt \quad \text{for some } c \in I.$$

Since f is continuous, F is an antiderivative of f on $[a,b]$, so $F'(x) = f(x)$.

Consider the function $H(x) = F(g(x))$. By the chain rule,

$$H'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Since $H(x)$ is an antiderivative of $f(g(x))g'(x)$, by the Fundamental Theorem of Calculus we have:

$$\int_a^b f(g(x))g'(x)\,dx = H(b) - H(a) = F(g(b)) - F(g(a)).$$

Using the definition of $F(u)$, we get

$$F(g(b)) - F(g(a)) = \int_c^{g(b)} f(t)\,dt - \int_c^{g(a)} f(t)\,dt = \int_{g(a)}^{g(b)} f(t)\,dt.$$

□

Example 2.1.2. *Evaluate the integral*

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1-2x^2}{\sqrt{1-x^2}}\,dx.$$

Proof. Let $x = \sin t$. Then $dx = \cos t\,dt$. The limits of integration change as follows: - When $x = -1/\sqrt{2}$, we have $\sin t = -1/\sqrt{2}$, so $t = -\pi/4$. - When $x = 1/\sqrt{2}$, we have $\sin t = 1/\sqrt{2}$, so $t = \pi/4$.

Substituting these into the integral, we get

$$I = \int_{-\pi/4}^{\pi/4} \frac{1-2\sin^2 t}{\sqrt{1-\sin^2 t}}(\cos t)\,dt.$$

Using the identities $\cos^2 t = 1 - \sin^2 t$ and $\cos(2t) = 1 - 2\sin^2 t$, we have

$$I = \int_{-\pi/4}^{\pi/4} \frac{\cos(2t)}{\sqrt{\cos^2 t}}(\cos t)\,dt = \int_{-\pi/4}^{\pi/4} \frac{\cos(2t)}{|\cos t|}(\cos t)\,dt.$$

Since $\cos t > 0$ on $(-\pi/2, \pi/2)$, we have $|\cos t| = \cos t$ on $(-\pi/4, \pi/4)$. Hence

$$I = \int_{-\pi/4}^{\pi/4} \cos(2t)\,dt.$$

Evaluating the integral:

$$\int \cos(2t)\,dt = \tfrac{1}{2}\sin(2t),$$

so

$$I = \left[\tfrac{1}{2}\sin(2t)\right]_{-\pi/4}^{\pi/4} = \tfrac{1}{2}\sin\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sin\left(-\tfrac{\pi}{2}\right).$$

$$I = \tfrac{1}{2}(1) - \tfrac{1}{2}(-1) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

□

Example 2.1.3. *Evaluate the integral*

$$I = \int_0^{\sqrt{\pi}} \frac{\sin t}{2t + \cos t}\, dt.$$

Proof. Let $u = \cos t$. Then $du = -\sin t\, dt$. The limits of integration change as follows: - When $t = 0$, $u = \cos(0) = 1$, - When $t = \pi$, $u = \cos(\pi) = -1$.

Thus,

$$\int_0^{\pi} \frac{\sin t}{2 + \cos t}\, dt = \int_1^{-1} \frac{-du}{2 + u} = \int_{-1}^{1} \frac{du}{2 + u}.$$

Evaluating,

$$\int_{-1}^{1} \frac{du}{2 + u} = [\ln|2 + u|]_{-1}^{1} = \ln(3) - \ln(1) = \ln(3).$$

□

documentclass[../Main.tex]subfiles

LECTURE-33

33.1 Metric spaces

We are familiar with real valued functions. Ex: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. In $\mathbb{R}^2$, $f(x, y) = x^2 + y^2$, $f : \mathbb{R}^2 \to \mathbb{R}$. On the real line, we have the notion of "distance" between two points, denoted by $d : \mathbb{R} \to \mathbb{R}$ and defined by $d(x, y) = |x - y| \;\; x, y \in \mathbb{R}$. $d(\cdot, \cdot)$ is a finite, non-negative real number.

Let X be an abstract space (set). By abstract space it is meant that the nature of the elements of the set X are unspecified. The functions that we will study is the notion of distance on set X.

Definition 33.1.1: A metric space is a pair (X, d) s.t $d : X \times X \to \mathbb{R}$ satisfies the following axioms:

M1. d **is real-valued, finite and non-negative.**

M2. $d(x, y) = 0$ **if and only if** $x = y$

M3. $d(x, y) = d(y, x)$ **(symmetry)**

M4. $d(x, y) \leq d(x, z) + d(z, y)$ **(Triangle inequality)** $x, y, z \in X$

N

ext we will be give examples of the pair (X, d) and verify that the pair (X, d) is a metric space.

Example 33.1.1. $X = \mathbb{R}$, $d(x, y) = |x - y|$.

M1 It is a real, finite and non-negative.

M2 $d(x, y) = 0 \implies |x - y| = 0 \Leftrightarrow x = y$.

M3 $d(x, y) = |x - y| = |y - x| = d(y, x)$.

M4 $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$.

$\therefore (X, d)$ *is a metric space.*

Axioms $M1$ and $M2$ are easily verified.
We need to show
$d(x,y) \leq d(x,z) + d(z,y)$
Consider $d(x,z) + d(z,y)$
$\sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} + \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2}$
$\sqrt{r} + \sqrt{p} \geq \sqrt{r + p}$ (we will use this fact)
$\geq \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (z_1 - y_1)^2 + (z_2 - y_2)^2}$
$= \sqrt{(x_1 - z_1)^2 + (z_1 - y_1)^2 + (x_2 - z_2)^2 + (z_2 - y_2)^2}$
$\geq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
$= d(x,y)$
$\therefore (X,d)$ is a Metric space.

Example 33.1.2. $X = \mathbb{R}^2, d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
$M1$ to $M3$ are easy to verify.
$d(x,y) + d(y,z)$
$=|x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2|$
$=|x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$
$\geq |x_1 - z_1| + |x_2 - z_2|$ *we have used the triangle for real numbers.*
$= d(x,z)$
$\therefore d(x,z) \leq d(x,y) + d(y,z)$
$\therefore (X,d)$ *is a Metric space.*

Example 33.1.3. $X = \mathbb{R}$,
$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$

Example 33.1.4. $X = \mathbb{C}^n$,*(n-dimensional unitary space) be an ordered n-tuples of complex numbers*
$x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$
$y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$
$d(x,y) = \sqrt{|(x_1 - y_1)|^2 + \cdots + |(x_n - y_n)|^2}$
$(\mathbb{C}^n, d(x,y))$ *is a metric space.*

Example 33.1.5. *Function space $X = C[a,b]$, the set of real-valued functions of an independent variable 't' that are continuous on a given closed interval $[a,b]$, that is, $C[a,b] = \{f : [a,b] \to \mathbb{R} : f \in C^0\}$. Let $d(x,y) = \max |x(t) - y(t)|\ t \in [a,b]$. We will verify $M4$ as other axioms hold.*
$|x(t) - y(t)| = |x(t) - z(t) + z(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)| \leq \max\limits_{t\in[a,b]} |x(t) - z(t)| + \max\limits_{t\in[a,b]} |z(t) - y(t)|$. *Take max over both sides.*

$$\max_{t\in[a,b]} |x(t) - y(t)| \leq \max_{t\in[a,b]} |x(t) - z(t)| + \max_{t\in[a,b]} |z(t) - y(t)|$$
$$d(x,y) \leq d(x,z) + d(z,y)$$

$(X, d(x,y))$ *is a metric space.*

Example 33.1.6. *Let X be a non-empty set and define*

$$d(x,y) = \begin{cases} 1, & \textit{if } x \neq y \\ 0, & \textit{if } x = y. \end{cases}$$

It is easy to see that M1-M3 hold. To check M4, we note that

$$d(x,z) + d(z,y) = \begin{cases} 2, & \textit{if } x \neq y \neq z \\ 0, & \textit{if } x = y = z \\ 1, & \textit{if } x = y \neq z \end{cases}$$

Thus $d(x,y) + d(y,z) \geq d(x,z)$. (X,d) is called discrete metric space.

documentclass[../Main.tex]subfiles

LECTURE-34

34.1 Infinite dimensional metric spaces

34.1.1 Sequence space S

This space consists of the set of all (bounded and unbounded) sequences $x = \{\zeta_i\}_{i=1}^{\infty}$, where $\zeta_i \in \mathbb{R}$ or $\zeta_i \in \mathcal{C}$ and the metric is defined as

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|}.$$

It is straightforward to see that M1-M3 hold. Let us show $M4$ also holds. In doing so we need an auxillary function: $f : \mathbb{R} \to \mathbb{R}$ defined as $f(t) = \frac{t}{1+t}$. We will show $f(\cdot)$ is monotonically increasing, that is, if $x, y \in \mathbb{R}$ s.t $x \leq y \implies f(x) \leq f(y)$. Note that $f'(t) = \frac{1}{(1+t)^2} > 0 \; \forall \, t \in \mathbb{R}$. Therefore, $f(.)$ is monotonically increasing. For $a, b \in \mathbb{R}$, we have $|a + b| \leq |a| + |b|$. By the monotonicity of f, it follows that

$$\begin{aligned} f(|a+b|) &\leq f(|a| + |b|) \\ \frac{|a+b|}{1 + |a+b|} &\leq \frac{|a| + |b|}{1 + |a| + |b|} \\ &= \frac{|a|}{1 + |a| + |b|} + \frac{|b|}{1 + |a| + |b|} \\ &\leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|} \end{aligned}$$

Now, define $a = \zeta_i - \sigma_i, b = \sigma_i - \eta_i$. Then,

$$\frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|} \leq \frac{|\zeta_i - \sigma_i|}{1 + |\zeta_i - \sigma_i|} + \frac{|\sigma_i - \eta_i|}{1 + |\sigma_i - \eta_i|}$$

Multiply both the sides by $\frac{1}{2^i}$ and sum over by i from 1 to ∞.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\zeta_i - \sigma_i|}{1 + |\zeta_i - \sigma_i|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\sigma_i - \eta_i|}{1 + |\sigma_i - \eta_i|}$$

With $z = \sigma_i, d(x, y) \leq (x, z) + d(z, y)$ and thus (S, d) is a metric space.

34.1.2 l^p space

Space l^p where $p \geq 1$ is a real number. The l^p space consists of sequences of numbers (real or complex) $x = \{\zeta_i\}$ such that $|\zeta_1|^p + |\zeta_2|^p + \dots$ converges, that is, $\sum_{i=1}^{\infty} |\zeta|^p < \infty$ and the metric is defined by $d(x, y) = (\sum_{i=1}^{\infty} |\zeta_i - \eta_i|^p)^{\frac{1}{p}}$, where $x = \zeta_i$ and $y = \eta_i$. Note that in $d(x, y)$, an infinite sum is involved, so we need to show that the series converges.

If $\zeta_i \in \mathbb{R}$, we get real l^p space. If $\zeta_i \in \mathbb{C}$, we get Complex l^p space. If $p = 2, l^2$ is called Hilbert space, introduced by David Hilbert in 1912.

Let us prove l^p is a metric space with the metric

$$d(x, y) = (\sum_{i=1}^{\infty} |\zeta_i - \eta_i|^p)^{\frac{1}{p}}. \tag{34.7}$$

We need to show

i The series in (34.7) is convergent.

ii $M1$ to $M4$ also holds.

The proof involves showing the following:

a An auxillary equation

b Holder inequality

c Minkowski inequality

d Triangle inequality

Let $p > 1$ and define q as

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{34.8}$$

The numbers p, q are called conjugate exponents. Equation (34.8) can be rewritten as $(p - 1)(q - 1) = 1$. Define a curve $u = t^{p-1}$ and using the conjugate exponent, it can be written as $t = u^{q-1}$. Let α, β be any positive constants. Then $\alpha\beta$ is the area of the rectangle and it is less than the area under the curve $u = t^{p-1}$ integrating from 0 to α plus the area under the curve $t = u^{q-1}$ integrating from 0 to β as shown in Figure 34.15.

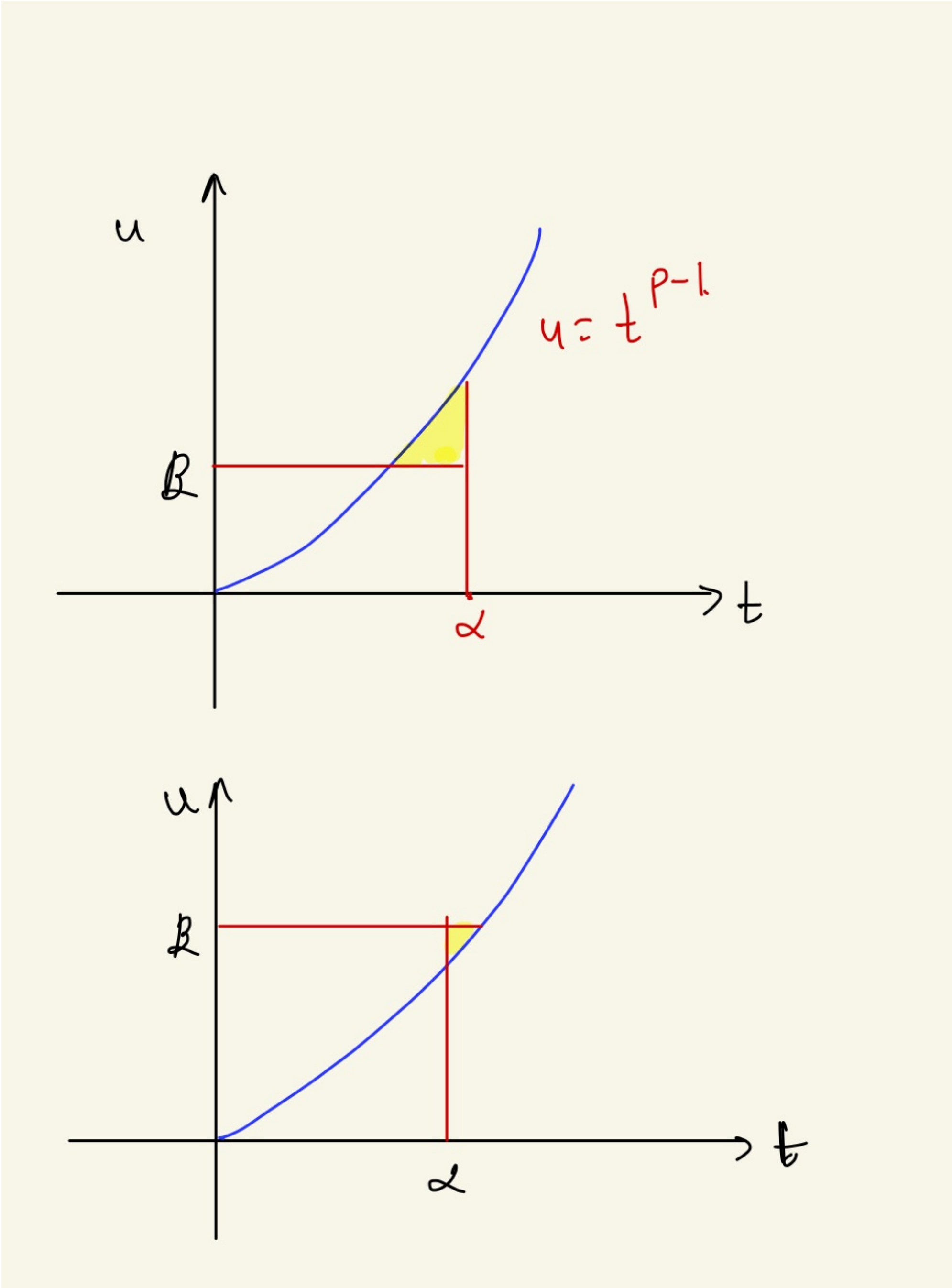


Figure 34.15: Overestimate of the area

$$\alpha\beta \leq \underbrace{\int_0^\alpha t^{p-1}dt + \int_0^\beta u^{q-1}du}_{\text{overestimate of the area of the rectangle formed by } \alpha \text{ and } \beta}$$
$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

If $\alpha = 0$ or $\beta = 0$, the equality holds trivially. Next, consider the sequences $\{\tilde{\zeta}_i\}$ and $\{\tilde{\eta}_i\}$ s.t $\sum_{i=1}^{\infty} |\tilde{\zeta}_i|^p = 1$ and $\sum_{i=1}^{\infty} |\tilde{\eta}_i|^p = 1$. Setting $\alpha = |\tilde{\zeta}_i|$ and $\beta = |\tilde{\eta}_i|$, we have

$$|\tilde{\zeta}_i \tilde{\eta}_i| \leq \frac{|\tilde{\zeta}_i|^p}{p} + \frac{|\tilde{\eta}_i|^q}{q}$$

Summing over from 1 to ∞ we get

$$\sum_{i=1}^{\infty} |\tilde{\zeta}_i \tilde{\eta}_i| \leq \sum_{i=1}^{\infty} \frac{|\tilde{\zeta}_i|^p}{p} + \sum_{i=1}^{\infty} \frac{|\tilde{\eta}_i|^q}{q} = 1$$

Next, take any non-zero $x = \{\zeta_i\}$ and $y = \{\eta_i\}$ such that

$$\begin{aligned} \tilde{\zeta}_i &= \frac{\zeta_i}{(\sum_{k=1}^{\infty} |\zeta_k|^p)^{\frac{1}{p}}} \\ \tilde{\eta}_i &= \frac{\eta_i}{(\sum_{m=1}^{\infty} |\eta_m|^q)^{\frac{1}{q}}}. \end{aligned}$$

It is easy to verify that $\sum_{i=1}^{\infty} |\tilde{\zeta}_i|^p = 1$ and $\sum_{i=1}^{\infty} |\tilde{\eta}_i|^q = 1$. Consider

$$\begin{aligned} \sum_{i=1}^{\infty} |\tilde{\zeta}_i \tilde{\eta}_i| &\leq 1 \\ \sum_{i=1}^{\infty} \left| \frac{\zeta_i}{(\sum_{k=1}^{\infty} |\zeta_k|^p)^{\frac{1}{p}}} \frac{\eta_i}{(\sum_{m=1}^{\infty} |\eta_m|^q)^{\frac{1}{q}}} \right| &\leq 1 \\ \sum_{i=1}^{\infty} |\zeta_i \eta_i| &\leq \left(\sum_{k=1}^{\infty} |\zeta_k|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} |\eta_m|^q \right)^{\frac{1}{q}} \end{aligned}$$

This inequality is called Hölder's inequality. For $p = q = 2$, we get the Cauchy-Schwartz inequlaity.

We next derive the Minkowski inequality for the sum. Let $x = \{\zeta_i\}, y = \{\eta_i\}$ and $x, y \in l^p$. For $p = 1$, the Minskoswki inequality follows from the trianagle inequality of the reals. Consider $p > 1$. Define $w_i = \zeta_i + \eta_i$. Then

$$\begin{aligned} |w_i| &= |\zeta_i + \eta_i| \\ |w_i|^p |w_i|^{1-p} &= |\zeta_i + \eta_i| \\ |w_i|^p &= (|\zeta_i + \eta_i|)|w_i|^{1-p} \\ &\leq (|\zeta_i|)|w_i|^{1-p} + (|\eta_i|)|w_i|^{1-p} \\ \sum_{i=1}^{n} |w_i|^p &\leq \underbrace{\sum_{i=1}^{n} (|\zeta_i|)|w_i|^{1-p}}_{A} + \underbrace{\sum_{i=1}^{n} (|\eta_i|)|w_i|^{1-p}}_{B}. \end{aligned}$$

Applying Hölder's inequality to the terms A and B we get

$$\begin{aligned}\sum_{i=1}^{n}(|\zeta_i|)|w_i|^{1-p} &\leq (\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}}(\sum_{m=1}^{n}|w_m|^{(1-p)q})^{\frac{1}{q}}\\ &= (\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}}(\sum_{m=1}^{n}|w_m|^p)^{\frac{1}{q}} \text{ since } (1-p)q = p.\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n}(|\eta_i|)|w_i|^{1-p} \leq (\sum_{k=1}^{n}|\eta_k|^p)^{\frac{1}{p}}(\sum_{m=1}^{n}|w_m|^p)^{\frac{1}{q}}.$$

Thus

$$\begin{aligned}\sum_{i=1}^{n}|w_i|^p &\leq (\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}}(\sum_{m=1}^{n}|w_m|^p)^{\frac{1}{q}} + (\sum_{r=1}^{n}|\eta_r|^p)^{\frac{1}{p}}(\sum_{m=1}^{n}|w_m|^p)^{\frac{1}{q}}\\ &= \left\{(\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}} + (\sum_{r=1}^{n}|\eta_r|^p)^{\frac{1}{p}}\right\}(\sum_{m=1}^{n}|w_m|^p)^{\frac{1}{q}}\\ \left(\sum_{i=1}^{n}|w_i|^p\right)^{1-\frac{1}{q}} &\leq (\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}} + (\sum_{r=1}^{n}|\eta_r|^p)^{\frac{1}{p}}\\ \left(\sum_{i=1}^{n}|w_i|^p\right)^{\frac{1}{p}} &\leq (\sum_{k=1}^{n}|\zeta_k|^p)^{\frac{1}{p}} + (\sum_{r=1}^{n}|\eta_r|^p)^{\frac{1}{p}}\end{aligned}$$

Letting $n \to \infty$, we get

$$\left(\sum_{i=1}^{\infty}|\zeta_i + \eta_i|^p\right)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}|\zeta_k|^p)^{\frac{1}{p}} + (\sum_{r=1}^{\infty}|\eta_r|^p)^{\frac{1}{p}} \tag{34.9}$$

Since $x, y \in l^p, \sum_{k=1}^{\infty}|\zeta_k| < \infty, \sum_{r=1}^{\infty}|\eta_r| < \infty$. The RHS in (34.9) is bounded. Therefore, the series on the left side of (34.9) converges. Finally, set $z = \{\sigma_i\}$. Then

$$\begin{aligned}d(x,y) = \left(\sum_{i=1}^{\infty}|\zeta_i - \eta_i|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{\infty}|\zeta_i - \sigma_i + \sigma_i + \eta_i|^p\right)^{\frac{1}{p}}\\ &\leq \left(\sum_{i=1}^{\infty}|\zeta_i - \sigma_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}|\sigma_i - \eta_i|^p\right)^{\frac{1}{p}}\\ &= d(x,z) + d(z,y)\end{aligned}$$

Thus, (l^p, d) is a metric space.

ASSIGNMENT-1

1.1 Point set topology

1. Consider the following subsets of $\mathbb{R}$ and decide whether the sets are open or closed. Find out all the interior, closure and limit points of the following sets.

   (a) All integers $\mathbb{Z}$.

   (b) The set of all rational numbers $\mathbb{Q}$.

   (c) The set of all numbers of the form $\frac{1}{n}$, $n \in \mathbb{N}$ or, $A = \{x = \frac{1}{n} \mid n \in \mathbb{N}\}$.

   (d) $B = \{x = \frac{1}{n} + \frac{1}{m} \mid n, m \in \mathbb{N}\}$

   (e) $C = (0, 1]$, $\quad D = (-\infty, 1]$

   (f) Cantor Set

2. Consider the following sets in $\mathbb{R}^2$ and plot the set in your note book. Find out the interior, closure and boundary of the following sets. Check if any of the set is either open or closed or both.

   (a) $A = \{(x, y) \mid y = 0\}$.

   (b) $A = \{(x, y) \mid x > 0 \text{ and } y \neq 0\}$.

   (c) $A = \{(x, y) \mid 0 < x^2 - y^2 \leq 1\}$.

   (d) $A = \{(x, y) \mid x \neq 0 \text{ and } y \leq \frac{1}{x}\}$.

3. Consider the space $\mathbb{R}^2$. Plot the unit open ball centred around the origin with respect to 1, 2 and ∞ norm in $\mathbb{R}^2$. Check if the open ball with respect to one norm is also open with respect to all other norms also.

4. We study that the finite intersection of open sets is open. Give an counter example to show that any arbitrary intersection of open sets is not always open. Similarly, give an counter example to show that any arbitrary union of closed set is not always closed.

5. Consider the following statements and decide whether the following statements are True or False. If the statement is true, prove it. If the statement is false, give a counter example to justify your answer.

(a) $A \subseteq \mathbb{R}^n$, $\text{int}(A)$ is always an open set. Also, $\text{int}(A)$ is the largest open set contained in A.

(b) $A \subseteq \mathbb{R}$, $\text{int}(A) = \text{int}(\bar{A})$, where, $\bar{A}$ is the closure of the set A.

(c) A and $\text{int}(A)$ always have the same closures.

(d) $A \subseteq \mathbb{R}^n$, $(\text{int}(A))^c = (\bar{A^c})$ and $\bar{A} = \{x |\ d(x, A) = 0\}$.

(e) Every point in an open set is also the limit point of that set.

(f) Every point in a closed set is also the limit point of that set.

(g) A' is the set of all limit points of the set A (derived set), A and A' always have the same limit points.

(h) Let U is a open set in $\mathbb{R}^n$. Consider $A \subseteq \mathbb{R}^n$, if $A \cap U = \phi$ then $\bar{A} \cap U = \phi$ also.

(i) If U is open set in $\mathbb{R}^n$ then $U = \text{int}(\bar{U})$

6. Prove the following statements.

(a) $\{A_i\ |i \in \mathcal{I}\}$ is any arbitrary collections of subsets in $\mathbb{R}^n$. Show that $\text{int}(\cap_{i \in \mathcal{I}}\ A_i) \subseteq \cap_{i \in \mathcal{I}}\ \text{int}(A_i)$. Give an example to show that the equality sign in this relation does not hold.

(b) $A_1, A_2, \ldots A_n$ are any finite collections of subsets of $\mathbb{R}^n$. Show that $\cup_{i=1}^n\ \text{int}(A_i) \subseteq \text{int}(\cup_{i=1}^n\ A_i)$. Give an example to show that the equality sign in this relation does not hold.

(c) If ∂A is the set of all boundary points in $A \subseteq \mathbb{R}^n$, Show that $\partial A = \bar{A} \cap \bar{A^c}$, $\partial A = \partial(A^c)$ and $\bar{A} = \text{int}(A) \cup \partial A$.

(d) Prove the following relations: $\partial(A \cup B) \subseteq \partial A \cup \partial B$. When the equality sign holds in this relation. Give an example when the equality sign does not hold.

7. Give an example of the following statement.

(a) $A, B \subseteq \mathbb{R}$, $\text{int}(A) = \text{int}(B) = \phi$ and $\text{int}(A \cup B) = \mathbb{R}$.

(b) A set $A \subseteq \mathbb{R}^n$ which is both open and closed.

(c) A set $A \subseteq \mathbb{R}^n$ which is neither open nor closed.

(d) A set $A \subseteq \mathbb{R}^n$ which contains a point not a limit point of A.

(e) A set $A \subseteq \mathbb{R}^n$ which contains no point which is not a limit point of A.

(f) A countable set in $\mathbb{R}$ which is closed and a countable set which is not closed.

(g) A non-empty set A in $\mathbb{R}$ where $\partial A = \phi$.

8. Prove the following statements.

(a) Consider $n > m$, show that $A \subseteq \mathbb{R}^m$ is an open subset in $\mathbb{R}^m$ if and only if A can be written as an intersection of an open subset in $\mathbb{R}^n$ and $\mathbb{R}^m$. Consider any unit ball in $\mathbb{R}^2$, What is the corresponding open set in $\mathbb{R}^3$?

(b) Consider $x, y \in \mathbb{R}^n$, where $x \neq y$, show that there always exist open sets U and V in $\mathbb{R}^n$ such that $x \in U$ and $y \in V$ and $U \cap V = \phi$. The metric spaces which possess this property are called Hausdorff space.

(c) A subset $A \subseteq \mathbb{R}$ is said to be be bounded if and only if it is contained in some closed interval in $\mathbb{R}$. In a more general sense, a set is bounded if and only if it is contained in some closed ball.

(d) A point x is the limit point of the set A, show that every open balls centered on x contain an infinite number of distinct points of A.

ASSIGNMENT-2

2.1 Suprema and Infima

1. Warm up by recalling the definition of supremum and infimum, maximum and minimum and recollecting the proof of the following facts.

   - Supremum and infimum, if exists, is unique
   - Maximum, if exists, is supremum and minimum, if exists, is infimum
   - *Least upper bound property of* $\mathbb{R}$: Every set in $\mathbb{R}$ that has an upper bound has a supremum and so and so for lower bounds and infimum

2. Find the suprema and infima of the following subsets of $\mathbb{R}$ (if they exist). Also, determine if they are maxima and minima respectively.

   - $(a, b]$
   - $[a, b]$
   - (a, b)
   - $\{x|x > 0\}$
   - $\{x|x^2 < 2\}$

3. Find out the supremum and infimum of the following subsets of $\mathbb{R}$ (if they exists) and check that the supremum and infimum are maximum or minimum respectively with proper justification.

   (a) $A = \{x : x^2 - 5x + 6 < 0\}$ and $A' = \{x : x^2 - 5x + 6 > 0\}$.

   (b) $B = \{\frac{m}{|m|+n}|n \in \mathbb{N}, m \in \mathbb{Z}\ \}$.

   (c) $C = \{2^{-x} + 3^{-y} + 5^{-z}|x, y, z > 0\ \}$.

   (d) $D = \{1 - \frac{(-1)^n}{n}|n \in \mathbb{N}\}$.

   (e) $E = \{\frac{1}{n} - \frac{1}{m}|n, m \in \mathbb{N}\}$.

   (f) $F = \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$

4. Let $A \subseteq \mathbb{R}$ be a non-empty bounded above subset, define $-A = \{-x | x \in A\}$.
   (a) Show that $-A$ is bounded below subset of $\mathbb{R}$.
   (b) Show that $\inf(-A) = -\sup(A)$.
   (c) From (b) show that least upper bound properties of $\mathbb{R}$ actually implies greatest lower bound property of $\mathbb{R}$ and vice versa.
5. Prove the following statements.
   (a) Suppose $x, y \in \mathbb{R}$ and $x < y$. Show that $\exists\ n \in \mathbb{N}$ such that $x < x + \frac{1}{n} < y$. Similarly, show that $\exists\ n \in \mathbb{N}$ such that $x < y - \frac{1}{n} < y$. From this statement show that if $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ is an upper bound of A if $\forall\ n \in \mathbb{N}\ \exists\ a_n \in A$ such that $a_n > a - \frac{1}{n}$ then $a = \sup A$.
   (b) Consider $A \subseteq \mathbb{R}$ is bounded. Assume that $a = \sup A$ and $b = \sup A$ then prove that $a = b$. Similarly, show that infimum of a set (if it exists) is always unique.
   (c) Let A be the non-empty bounded subset of $\mathbb{R}$ and a be an upper bound of A. Show that if $a \in A$ that implies $a = \sup A$.
   (d) Consider A be a non-empty subset of $\mathbb{R}$, Show that if $\sup A \notin A$, then $\sup A$ must be a limit point of A. What if $\sup A \in A$, then does this always be a limit point of A? From this statement, show that if $a = \sup A$ and $b = \inf A$ then $d(a, A) = 0$ and $d(b, A) = 0$ where, $d(a, A)$ is the distance between the point a and the set A.
   (e) From part (d) show that if $A \subseteq \mathbb{R}$ is a closed set then $\sup A$ and $\inf A$ (if they exists) must belong to the set A. That is $\sup A \in A$ and $\inf A \in A$.
6. Prove the following statements to show that the set of rational numbers $\mathbb{Q}$ do not have the least upper bound property.
   (a) Consider $x \in \mathbb{Q}$ and $x > 0$. If $x^2 < 2$, show that there exists $n \in \mathbb{N}$ such that $(x + \frac{1}{n})^2 < 2$. Similarly, if $x^2 > 2$ show that there exists $n \in \mathbb{N}$ such that $(x - \frac{1}{n})^2 > 2$.
   (b) From part (a) we can show that the set $A = \{r \in \mathbb{Q} | r > 0, r^2 < 2\}$ is bounded above in $\mathbb{Q}$. Show that the set does not have the least upper bound in $\mathbb{Q}$ that is there does not exist any $x \in \mathbb{Q}$ such that $x = \sup A$.
   (c) From part (a) and (b) we can conclude that the $\mathbb{Q}$ does not possess the least upper bound property. On the other hand show that if $x \in \mathbb{R}$ such that $x = \sup A$ then $x^2 = 2$.

      In a similar way, you can show that the set of rational numbers $\mathbb{Q}$ do not have greatest lower bound property.
7. A set $A \subset \mathbb{R}$ satisfies $\sup(A) = \inf(A)$. What can you say about such an A?
8. Prove that a subset A of $\mathbb{R}$ is bounded (considering $\mathbb{R}$ as a normed space with the usual norm) if and only if it has supremum and infimum.

9. Give examples of functions $f : I = (0, 1) \to \mathbb{R}$ such that
   - $f(I)$ is unbounded
   - $f(I)$ is bounded
10. Give examples of functions $f : I = [0, 1] \to \mathbb{R}$ such that
    - $f(I)$ is unbounded
    - $f(I)$ is bounded
11. Give examples of functions $f : I = \mathbb{R} \to \mathbb{R}$ such that
    - $f(I)$ is unbounded
    - $f(I)$ is bounded
12. - Give an example of a set $I \subset \mathbb{R}$ and a function $f : I \to \mathbb{R}$ such that I has a supremum but not a maximum, but $f(I)$ has maximum.
    - Give an example of a set $I \subset \mathbb{R}$ and a function $f : I \to \mathbb{R}$ such that I has a maximum but $f(I)$ has only a supremum and not a maximum
13. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. Is it true for any f that

    $$\sup_{x \in I} f(x) = f(\sup(I))$$

    If yes, prove. If not, give a counterexample and a restricted class of functions for which this is true.
14. Let I be a closed subset of $\mathbb{R}$. Then prove that its supremum (if exists) is its maximum and infimum (if exists) is its minimum.
15. Prove that $\mathbb{N} \subset \mathbb{R}$ does not have a supremum in $\mathbb{R}$.
16. Let $f : I \to \mathbb{R}$ be a linear function defined as $f(x) = ax + b$ where $a, b \in \mathbb{R}$. Prove that $f(I)$ is bounded (has a supremum and infimum) if and only if $a = 0$ or I is bounded.
17. Give an example of a set $A \subseteq \mathbb{R}$ such that $\partial(A) \neq \emptyset$ but $\sup(A), \inf(A)$ do not exist.

    Extra Questions:
18. Prove that for a set $A \subset \mathbb{R}$, $\sup(A)$ (assuming it exists) is a boundary point of A (considering $\mathbb{R}$ with the usual norm).
19. Prove that the empty set cannot have a supremum or an infimum in $\mathbb{R}$ (Use the fact that empty set is contained in every subset)
20. Diameter of a set A in $\mathbb{R}^n$ is defined as $dia(A) = \sup_{x,y \in A} ||x - y||$. Prove that if $A \subset \mathbb{R}$, then its diameter is given by $dia(A) = \sup(A) - \inf(A)$.

21. Let A, B be subsets of $\mathbb{R}$ and $f : A \times B \to \mathbb{R}$ be real valued functions. Then prove that

$$\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y)$$

(*Food for thought:* This result is important and arises in decision theory which has applications in zero-sum game theory, economics and optimization. Google minimax rule for more information and applications)

22. A subset $A \subseteq \mathbb{R}$ is called an **initial segment** if it satisfies that

$$x \in A, y < x \Longrightarrow y \in A$$

Show that any initial segment of $\mathbb{R}$ can be only among the four possibilities mentioned below

(a) Empty

(b) $A = \mathbb{R}$

(c) $A = (-\infty, a)$ for some a

(d) $A = (-\infty, a]$ for some a

ASSIGNMENT-3

3.1 Sequences

1. Recollect the following from class
   (a) Definitions of bounded, Cauchy and Convergent sequences
   (b) Convergent $\implies$ Cauchy $\implies$ bounded
   (c) Limit of a a sequence, if exists, is unique
   (d) Spaces where Cauchy $\implies$ Convergent are called complete spaces. $\mathbb{R}$ is complete.
2. If x_n, y_n are convergent sequences in $\mathbb{R}$, prove that
   (a) $\lim\ (x_n \pm y_n) = \lim\ x_n \pm \lim\ y_n$
   (b) $\lim\ (x_n \times y_n) = \lim\ x_n \times \lim\ y_n$
   (c) $\lim\ \frac{x_n}{y_n} = \frac{\lim\ x_n}{\lim\ y_n}$ if $\lim\ y_n \neq 0$
   (d) if $z_n = x_{n+n_0}$ for any given n_0, then $\lim\ z_n = \lim\ x_n$
3. Consider the following sequences $\{a_n\}$ in $\mathbb{R}$ and determine whether the following sequences are convergent or not.
   (a) $a_n = \frac{a^2-2n+3}{5n^3}$.
   (b) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{2n}$
   (c) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}$.
   (d) $a_n = (2^n + 3^n)^{\frac{1}{n}}$.
   (e) $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = \frac{a_n + a_{n+1}}{2}$ $\forall\ n \geq 1$.
   (f) $a_n = n^{\frac{1}{n}}$.
   (g) $a_n = \frac{n!}{n^n}$
   (h) $a_n = (n+1)^{\frac{2}{7}} - n^{\frac{2}{7}}$.
   (i) $a_n = \frac{2^n}{n!}$.

4. Answer the following questions.

   (a) Consider the sequence $\{a_n\} = \{(-1)^n\}\ \forall\ n$. Show that the sequence $\{a_n\}$ does not converge.

   (b) Suppose that $\{a_n\}$ and $\{b_n\}$ two sequences in $\mathbb{R}$ such that $a_n \to 0$ and $\{b_n\}$ is bounded. Does $\{a_n b_n\}$ converge? if so, what is the limit?

   (c) If $\{a_n\}$ converges to a, does $\{a_n^2\}$ also converge to a^2?
   What about the converse? that is if $\{a_n^2\}$ converge to a^2 does $\{a_n\}$ also converge to a?

   (d) Consider $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers. Assume that $\lim\limits_{n\to\infty} b_n = 0$ and $a_{n+1} \leq \frac{1}{2}a_n + b_n$. Show that $\lim\limits_{n\to\infty} a_n = 0$.

   (e) Consider $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers such that $0 < b_1 < a_1$. Define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}\ \forall\ n > 1$. Show that both $\{a_n\}$ and $\{b_n\}$ converge.

   (f) Define the Fibonacci sequence as follows: $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-1} + a_{n-2}$ $\forall\ n > 2$. Show that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and evaluate the limit. This limit is called by the famous name "*Golden Ratio*".

   (g) Consider the sequence $\{a_n\}$ as $a_n = (1+\frac{1}{n})^n$. Show that the sequence $\{a_n\}$ is an increasing sequence and bounded above.

5. Consider the following sequence $\{a_n\}$ in $\mathbb{R}$ and let M_n be their arithmetic mean as follows: $M_n = \frac{a_1+a_2+a_3+\ldots a_n}{n}$. Answer the following questions.

   (a) If $\{a_n\}$ converges to a, show that $\{M_n\}$ also converges to a.

   (b) What about the converse? If the sequence $\{M_n\}$ converges, does the sequence $\{a_n\}$ also converge? If yes, prove the statement or give an example to disprove the statement.

   (c) If the sequence $\{M_n\}$ converges, the sequence $\{a_n\}$ is bounded. Prove the statement or give a counterexample to disprove this statement.

6. Consider the following statements and decide whether the following statements are True or False. If the statement is true, prove it. If the statement is false, give a counter example to justify your answer.

   (a) Consider a convergent sequence $\{a_n\}$ in $\mathbb{R}$ that has infinitely many distinct points. Define the set $A = \{a_n | n \geq 1\}$ and $a = \lim\limits_{n\to\infty} a_n$. Then a must be a limit point of A.
   What if $\{a_n\}$ has only finitely many distinct points?
   Is the converse true? If a is a limit point of A, is it necessarily true that $a_n \longrightarrow a$?

   (b) Consider any set $A \subseteq \mathbb{R}$ and L is a limit point of A. Then there exists a sequence $\{a_n\} \subseteq A$ such that $\{a_n\}$ converges to L.

(c) Consider the set $A = (0, 1)$ and suppose any sequence $\{a_n\} \subseteq A$ is a Cauchy sequence. That is $\forall\ \epsilon > 0\ \exists\ \ n_0 \in \mathbb{N}$ such that $|a_n - a_m| \leq \epsilon\ \forall\ \ n, m \geq n_0$. Then $\lim_{n\to\infty} a_n = a \in A$.
What if we take A is any closed set suppose $A = [0, 1]$.

(d) Consider a bounded sequence $\{a_n\}$ in $\mathbb{R}$. Define another sequence $\{b_n\}$ as follows $b_n = \sup\{a_k | k \geq n\}$. Then the sequence $\{b_n\}$ converges.
Similarly, define another sequence $\{c_n\}$ as follows: $c_n = \inf\{a_k | k \geq n\}$. Then the sequence $\{c_n\}$ converges.

(e) Consider a bounded sequence $\{a_n\}$ in $\mathbb{R}$. Then there exists atleast one subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which converges to one limit point of the sequence.

(f) Consider the set of rational numbers $\mathbb{Q}$ and a sequence $\{q_n\} \subseteq \mathbb{Q}$. Suppose $q_n \to q$ then $q \in \mathbb{Q}$. (*Hints: See qns 4.(h)*)

7. *Subsequences:* Given a sequence x_n in a metric space, a subsequence of x_n , denoted by x_i^S is a selection of natural numbers $n_1 < n_2 < \cdots n_i < \cdots$ such that $x_i^S = x_{n_i}$. Prove the following statements.

(a) Any subsequence of a convergent subsequence is convergent and converges to the same limit as the original sequence

(b) If a sequence is Cauchy in a metric space and it has a convergent subsequence, then the original sequence itself is convergent and converges to the same limit of the subsequence

(c) The Cauchy assumption in the above subdivision is crucial. Give a simple counterexample to disprove the above subdivision without the Cauchy assumption of the original sequence

8. Look at the following definitions for sequences in $\mathbb{R}$ and answer the questions that follow

- *Divergent sequence:* A sequence a_n is said to diverge if for every $R \in \mathbb{R}$, there exists $M' \in \mathbb{N}$ such that $|x_{M'}| \geq R$.
- *Discrete sequence:* A sequence a_n is called discrete if there exists a number $r > 0$ such that for all $i, j \in \mathbb{N}$, we have $x_i = x_j$ or $|x_i - x_j| > r$.

(a) Prove that a divergent sequence cannot be convergent

(b) Are all non convergent sequences divergent? If yes, prove. If no, give a counterexample

(c) Prove that a discrete sequence is convergent if and only if there exists a number $N \in \mathbb{N}$ and another constant C such that $x_n = C$ for all $n \geq N$ (eventually constant). Give as many examples of discrete sequnces as possible.

9. All the sequences in this question are in $\mathbb{R}$. Prove whatever is asked in each subdivision.

(a) If a sequence x_n satisfies $x_n \leq a$ for all n, then prove that $\lim_n x_n \leq a$ if the limit exists for the sequence x_n. Does the claim hold if we replace $\leq$ by $<$? (prove, if yes and give a counterexample, if not). Similarly, for $x_n \geq a$ for all n, what can you conclude?

(b) Let three sequences x_n, y_n, z_n satisfy $x_n \leq y_n \leq z_n$ for all n and let x_n, z_n are convergent. If y_n is also convergent, then prove that it satisfies $\lim\ x_n \leq \lim\ y_n \leq \lim\ z_n$

(c) *Sandwich theorem*: If three sequences x_n, y_n, z_n satisfy $x_n \leq y_n \leq z_n$ for all n and x_n, z_n are convergent and also satisfy $\lim\ x_n = \lim\ z_n$, then prove that y_n is also convergent with the same common limit.

(d) If for two sequences a_n, b_n, we have $a_n \leq b_n$ for all n, and if both the sequences are convergent. Then prove that $\lim\ a_n \leq \lim\ b_n$.

(e) *Monotone Convergence Theorem*: A sequence is called **non decreasing** if it satisfies $x_1 \leq x_2 \leq x_3 \leq \cdots$. i.e. $i \leq j \Longrightarrow x_i \leq x_j$. Let $X = \{x_n\}$ (collecting all the terms of the sequence in the set X). Then, prove that a non decreasing sequence is convergent if and only if $\sup(X)$ exists and $x_n \to \sup(X)$ if the supremum exists. Similarly, define an appropriately meaningful notion of non increasing sequence and replace suprema by infima and conclude appropriately. (a restatement of the result would be that non decreasing sequence in $\mathbb{R}$ is convergent if and only if it is bounded above)

10. The problems below illustrate that the order of limits cannot be interchanged generically. Evaluate the following

(a) $\lim_m\ \lim_n\ 2^{m-n}$ and $\lim_n\ \lim_m\ 2^{m-n}$

(b) $\lim_m\ \lim_n\ \left(\frac{1}{n}\right)^{\frac{1}{m}}$ and $\lim_n\ \lim_m\ \left(\frac{1}{n}\right)^{\frac{1}{m}}$

(c) *Dominated convergence theorem*: Let $a_{nm} \geq 0$ and $b_{nm} \geq 0$ be bisequences such that $a_{nm} \leq b_{nm}$ and where $\lim_m\ \lim_n\ b_{nm} = \lim_n\ \lim_m\ b_{nm}$. Then, show that $\lim_m\ \lim_n\ a_{nm} = \lim_n\ \lim_m\ a_{nm}$

Given a sequence a_n, its partial sum S_n is defined as $S_n = \sum_{i=1}^{n} a_i$ and is called the series corresponding to the sequence a_n. The limit of S_n, if exists, is called the infinite sum of the sequence. If the infinite sum exists, the series is called summable. Now, prove the following.

1. The geometric series $S_n = 1 + a + a^2 + a^3 + \cdots + a^n$ is summable if and only if $|a| < 1$ and its infinite sum is given by $\frac{1}{1-a}$ for $|a| < 1$

2. *The sequence of a summable series converges to zero:* Prove that if a series S_n is summable, then the sequence a_n satisfies *lim* $a_n = 0$. (*Hint: Start with* $a_{n+1} = S_{n+1} - S_n$ *, take limit both sides and use appropriate previously proved results*

3. We will now show that the converse to the above statement is false. i.e. if $a_n \to 0$, it is not necessary that S_n converges. Consider the sequence $a_n = \frac{1}{n}$ and the corresponding

series $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, called the *harmonic series*. Prove that $S_n \geq \int_1^n \frac{dx}{x} = log(n)$ and using this, show that S_n cannot converge.

4. Prove that if $0 \leq a_n \leq b_n$ for all n and if b_n is summable, then a_n is summable as well.

Given the sequence a_n and its corresponding series S_n, the sequence is called *absolutely summable* if the sequence $|a_n|$ is summable.

1. Prove that an absolutely summable sequence is summable
2. The converse of the above statement is false. There are summable sequences that are not absolutely summable. For example, consider the following alternating harmonic series given by $a_n = (-1)^{n+1} \times \frac{1}{n}$. It is not absolutely summable as its magnitude sequence $|a_n| = \frac{1}{n}$, was shown to be not summable. But we will show now that the sequence itself is summable and it turns out that $S_n \to ln\, 2$.
3. Prove that the odd sequence $S_{2n+1} = S_{2n-1} - \frac{1}{2n} + \frac{1}{2n+1}$ is a non increasing sequence.
4. Since $S_{2n+1} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots + (\frac{1}{2n-1} - \frac{1}{2n}) + \frac{1}{2n+1}$, we have that $S_{2n+1} \geq 0$. Now use monontone convergence theorem to conclude the convergence of odd terms of the series.
5. Do similar manipulations for concluding the convergence of even terms of the series S_{2n}.
6. Now that we have shown convergence of odd and even terms of the series separately, use the fact that $S_{2n+1} - S_{2n} = \frac{1}{2n+1} \to 0$ to conclude that the odd and even sums converge to the same limit.
7. Using the above result that the odd and even terms of the series converge to the same limit, prove that the series itself converges to that limit

ASSIGNMENT-4

4.1 Series

1. Use all the tests that you know to test whether the following non-negative series are convergent or divergent

   (a) $\sum_n \frac{n^n}{n!}$

   (b) $\sum_n \frac{2^n}{n!}$

   (c) $\sum_n \log(n)$

2. **Limit comparison test:** Let a_n, b_n be two non-negative sequences. Prove the following.

   - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_n b_n$ converges, $\implies \sum_n a_n$ converges
   - If $\lim_{n\to\infty} \frac{a_n}{b_n} = K > 0$, then $\sum_n a_n$ converges $\iff \sum_n b_n$ converges

3. Prove that the series defined by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is actually power of a number e^x where $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ using the binomial theorem and rigorously, verifying that all limits are well defined. (You never bothered to prove this in school, but the series seriously does not look like a power of some number at first glance!)

4. Prove that if a series made of a sequence of non-negative numbers has a divergent subseries (series corresponding to a subsequence), then the entire series is divergent.

5. Check the convergence, divergence, absolute convergence, and conditional convergence of the series

   - $\sum_n \frac{1}{n^2+4n+3}$
   - $\sum_n \frac{5n^2-6n+3}{n^3-7n+8}$
   - $\sum_n \left(1 + \frac{1}{n}\right)^n$
   - $\sum_n \frac{\cos(n\pi)}{\sqrt{n}}$

- $\sum_n \frac{3^{n-1}}{n2^n}$
- $\sum_n \left(\frac{2n}{5n-1}\right)^n$
- $\sum_n \frac{(-1)^n}{ln(n)}$

6. *Root test:* Prove that if for a non negative series $\sum_n a_n$, if $\lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ exists and equal to K , then
   - The series converges if $0 \leq K < 1$
   - The series diverges if $K > 1$

   What can you say if $K = 1$?

7. Using integral test, given an $R \in \mathbb{R}_+$, provide a N so that the n^{th} partial sum of the harmonic series $\sum_n \frac{1}{n}$ exceeds R.

8. A series is called telescoping if its n^{th} term can be written as $a_n = f(n) - f(n-1)$ for some $f : \mathbb{N} \to \mathbb{R}$. Give conditions on f that can guarantee the convergence and divergence of the series $\sum_n a_n$

9. Answer the following:
   - If the sum of even terms and the sum of odd terms of a series are individually convergent, does that imply the series itself is convergent?
   - If the sum of even terms and the sum of odd terms of a series are individually absolutely convergent, does that imply the series itself is absolutely convergent?
   - If the sum of even terms and the sum of odd terms of a series are individually absolutely convergent, does that imply the series itself is convergent?
   - If the sum of even terms and the sum of odd terms of a series are individually divergent, does that imply the series itself is divergent?

10. Answer the following:
    - If the even and the odd partial sums of a series are individually convergent, does that imply the series itself is convergent?
    - If the even and the odd partial sums of a series are individually absolutely convergent, does that imply the series itself is absolutely convergent?
    - If the even and the odd partial sums of a series are individually absolutely convergent, does that imply the series itself is convergent?
    - If the even and the odd partial sums of a series are individually divergent, does that imply the series itself is divergent?

    NOTE: An even partial sum is given by $\sum_{i=1}^{2n} a_i$ whereas the sum of even terms is given by $\sum_{i=1}^{n} a_{2i}$. Both are not the same thing!

ASSIGNMENT-5

5.1 Limits and continuity

1. Is the intersection of a finite number of neighborhoods of a point c a neighborhood of c?

2. Is the intersection of an infinite number of neighborhoods of a point c a neighborhood of c?

3. Can a finite set have a limit point? Does $\mathbb{N}$ have a limit point?

4. If $A \subseteq \mathbb{R}$, Is every interior point of A, a limit point of A?

5. Suppose that f is a continuous function such that

$$\begin{array}{ll} S_1 = \{x \in \mathbb{R} : f(x) > 0\}, & S_2 = \{x \in \mathbb{R} : f(x) < 0\}, \\ S_3 = \{x \in \mathbb{R} : f(x) = 0\}, & S_4 = \{x \in \mathbb{R} : f(x) \neq 0\}. \end{array} \tag{5.10}$$

   Is either of these sets open in $\mathbb{R}$?

6. Is it true that if $\lim\limits_{x\to x_0} f(x)$ does not exist and $\lim\limits_{x\to x_0} g(x)$ does not exist, then $\lim\limits_{x\to x_0}(f+g)(x)$ does not exist? How about the existence of $\lim\limits_{x\to x_0} f(x)g(x)$?

7. When we deal with a quotient f/g of functions, what is the domain of f/g ?

8. Suppose that $\lim\limits_{x\to x_0} f(x)$ exists but $\lim\limits_{x\to x_0} g(x)$ does not exist. Can $\lim\limits_{x\to x_0}(f+g)(x)$ exist?

9. Suppose that $f(x)$ is a function defined on $\mathbb{R}$ such that $\lim\limits_{x\to x_0} |f(x)|$ exists for each $x_0 \in \mathbb{R}$. Must $\lim\limits_{x\to x_0} f(x)$ exist?

10. Does $\lim\limits_{x\to 0} \sqrt{x}$ exist? Does $\lim\limits_{x\to 0+} \sqrt{x}$ exist?

11. If $\lim\limits_{x\to a} f(x) = \ell > 0$, do we have $\lim\limits_{x\to a} \sqrt{f(x)} = \sqrt{\ell}$?

12. Let f be defined on (a, ∞) for some $a \in \mathbb{R}$ and $\ell \in \mathbb{R}$. Must

$$\lim_{x \to \infty} f(x) = \ell \Longleftrightarrow \lim_{x \to 0+} f\left(\frac{1}{x}\right) = \ell?$$

13. Let f be defined on $(-\infty, a)$ for some $a \in \mathbb{R}$ and $\ell \in \mathbb{R}$. Must

$$\lim_{x \to -\infty} f(x) = \ell \Longleftrightarrow \lim_{x \to 0-} f\left(\frac{1}{x}\right) = \ell?$$

14. Suppose that f and g are defined on (c, ∞) for some $c \in \mathbb{R}$ such that $\lim\limits_{x \to \infty} f(x) = \ell$ for some real ℓ and $\lim\limits_{x \to \infty} g(x) = \infty$. Does the limit $\lim\limits_{x \to \infty} (f \circ g)(x)$ exist? If so, what is the limit?

15. Suppose that f and g are defined in a deleted neighborhood of a such that $\lim\limits_{x \to a} f(x) = \ell$ for some non negative real number $\ell \geq 0$ and $\lim\limits_{x \to a} g(x)$. Must $\lim\limits_{x \to a} f(x)g(x)\Big) = \infty$ if $\ell > 0$? What can you say about the limit $\lim\limits_{x \to a} f(x)g(x)$ if $\ell = 0$?

16. Suppose that $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = \infty$. Does either $\lim_{x \to 0} f(x)g(x) = 0$ or $\lim_{x \to 0} f(x)g(x) = \infty$ hold?

17. Suppose that $\lim_{x \to 0} f(x) = \infty = \lim\limits_{x \to 0} g(x)$. Must $\lim\limits_{x \to 0} (f(x) - g(x)) = \infty$?

18. For what values of α does $\lim\limits_{x \to \infty} (\sin x / |x|^{\alpha})$ exist? When does it not?

19. Does the sequence $\{\sin n\}_{n \geq 1}$ converge? Does it have a convergent sub sequence?

20. For what values of $t \in \mathbb{R}$ does $\{\sin(nt)\}$ converge?

21. For what values of $t \in \mathbb{R}$ does $\{\cos(nt)\}$ converge?

22. Does $\{\cos(n\pi)\}$ converge?

23. Does $\{\sin(n\pi/2)\}$ converge? Does $\{(1/n)\sin(n\pi/2)\}$ converge? How about the sequences $\{(1/n)\sin(n\pi/4)\}$ and $\{(1/n)\sin(n\pi/5)\}$?

24. What can be said about the convergence of the sequences $\{(1/n)\sin n\}$ and $\{(1/n)\cos n\}$?

25. What is meant by a limit point of a sequence? How does a limit point differ from a limit of sequence?

26. For each of the following sets determine the set of all limit points: $\mathbb{N}, \mathbb{Z}$, $\mathbb{Q}, \mathbb{R}, \emptyset, \mathbb{R} \setminus \mathbb{Q}$.

27. Show that $\lim\limits_{x \to 0} 3^{1/x}$ does not exist.

28. Define $f(x) = (1/x)\sin(1/x)$ for $x \neq 0$. Determine $\lim_{x \to 0} f(x)$ if it exists. If not, explain why the limit does not exist.

29. Suppose that f is bounded and monotone on (a, b) and $c \in (a, b)$. Show that $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ both exist.

30. Draw the graph of

$$f(x) = \begin{cases} |x| + \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

for $x \in \mathbb{R}$. Determine $\lim_{x \to 0} f(x)$ if it exists. If not, explain why it does not exist.

31. State and prove the squeeze rule for functions f, g, and h defined on (a, ∞) (respectively $(-\infty, a)$).

32. Compute the following limits if they exist: (a) $\lim_{x \to \infty} \frac{x + \sin x}{x}$. (b) $\lim_{x \to 2} \frac{1}{(1-x)^2}$. (c) $\lim_{x \to 0} \frac{\sin 3x - 3x}{x^3}$. (d) $\lim_{x \to 0} \frac{\cos(|x|) - 1}{x}$. (e) $\lim_{x \to 0} \frac{1}{|x|}$. (f) $\lim_{x \to 0} \frac{1}{x}$. (g) $\lim_{x \to 0} \frac{x e^{1/x}}{1 + e^{1/x}}$. (h) $\lim_{x \to 0} \frac{\sin x}{|x|}$. (i) $\lim_{x \to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$. In the cases in which the limit does not exist, determine the left- and right-hand limits if they exist. Note: Those who are not familiar with the exponential function can wait until we introduce $\exp x$.

33. Draw the graph of

$$f(x) = \frac{1}{1 + e^{1/x}}$$

and determine the following limits: (a) $\lim_{x \to 0^-} f(x)$. (b) $\lim_{x \to 0^+} f(x)$.

34. Let f be defined in a deleted neighborhood B' of x_0. Prove or disprove the following: $\lim_{x \to x_0} f(x)$ exists if and only if given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every pair of points x, y in B' such that $|x - y| < \delta$.

35. 1. Let $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x} \sin\left(\frac{1}{x^2}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that f is not continuous at the origin.

36. Using the $\epsilon - \delta$ definition, show that $f(x) = \sqrt{x + 2}$ is continuous at $x = 2$.

37. Give an example of a function that is continuous on $[0, 2)$ and $(2, 3]$, but not on any open interval containing 2.

38. Prove that the following functions are continuous at the indicated points by finding δ for a given $\epsilon > 0$: (a) $f(x) = \sqrt{x}$ at $x = 4$.
(b) $f(x) = \sqrt{x^2 - 9}$ at $x = 3$.

(c) $f(x) = \frac{1}{x}$ at $x = a \neq 0$.
(d) $f(x) = \frac{1}{\sqrt{x}}$ at $x = a > 0$.

39. For $x > 0$ or $-1 < x < 0$, prove the Bernoulli inequality

$$(1+x)^n > 1 + nx \text{ for } n \geq 2.$$

Using this, prove that $\{r^n\}$ converges if and only if $-1 < r \leq 1$.

40. For what values of a does $\lim_{x \to a}[x]$ exist? Determine the domain where $[x]$ is continuous.

41. If $f(x) = \sqrt{x - [x]}$ on $(0, 2)$, determine $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$. Determine whether f is continuous at $x = 1$.

42. Define

$$f(x) = \frac{(1 + \sin \pi x)^n - 1}{(1 + \sin \pi x)^n + 1}, \quad x \in \mathbb{R}.$$

Determine the points where f is discontinous.

43. Determine the constants a and b such that f defined by

$$f(x) = \begin{cases} ax + 3 & \text{for } x > 4 \\ 7 & \text{for } x = 4 \\ x^2 + bx + 3 & \text{for } x < 4 \end{cases}$$

is continuous on $\mathbb{R}$.

44. Suppose that f is uniformly continuous on a set E and $\{x_n\}$ is a Cauchy sequence in E. Show that $\{f(x_n)\}$ is a Cauchy sequence. Using this, show that $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$.

45. Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3|x|$. Let $\epsilon > 0$ be given. Find $\delta(\epsilon) > 0$ such that $|x - y| < \delta(\epsilon)$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$.

46. Suppose that f is continuous on $[0, \infty)$ such that $\lim_{x \to \infty} f(x) = \ell$ for some $\ell \in \mathbb{R}$. Prove that f is uniformly continuous.

47. Suppose that f is continuous on $\mathbb{R}$ and

$$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x).$$

Prove that f is uniformly continuous on $\mathbb{R}$.

48. Define

$$f(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases} \quad \text{and } g(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Prove or disprove the following: f is uniformly continuous on $\mathbb{R}$, but g is not.

ASSIGNMENT-6

1 Riemnann Integration

Example 1.1. *Let $f, g \in B[a,b]$. Show that for any partition P of $[a,b]$,*

$$\begin{aligned} L(f,P) + L(g,P) &\leq L(f+g,P) \\ U(f+g,P) &\leq U(f,P) + U(g,P). \end{aligned}$$

Example 1.2. *Let $f, g \in R[a,b]$. Prove that $f+g \in R[a,b]$ and*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Example 1.3. *Let $f \in R[a,b]$ and $k \in \mathbb{R}$. Then $kf \in R[a,b]$ and*

$$\int_a^b (kf)(x)\,dx = k\int_a^b f(x)\,dx.$$

Example 1.4. *Give an example of a function f with $f \notin R[0,1]$ but $f^2 \in R[0,1]$.*

Example 1.5. *Let*

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 2, \\ 2, & 2 < x \leq 3. \end{cases}$$

(a) Show that $f \in R[0,3]$.

(b) Compute $\int_0^3 f(x)\,dx$ using the definition of the Riemann integral.

Example 1.6. *Prove or give a counterexample: Suppose $f \in R[a,b]$ and there exists $K > 0$ such that $f(x) > K$ for all $x \in [a,b]$. Then $\frac{1}{f} \in R[a,b]$.*

Example 1.7. *Prove or give a counterexample: Let $f, g \in R[a,b]$ and $h \in R[a,b]$. If $f(x) = h(x) \leq g(x)$ for all $x \in [a,b]$, then $h \in R[a,b]$.*

Example 1.8. *Let $\{f_n\}$ be a sequence of functions with $f_n \in R[a,b]$ for each n. Suppose the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$. Show that $f \in R[a,b]$.*

Example 1.9. *Let $\{r_1, r_2, r_3, \ldots, r_k\}$ be a counting of the rational numbers in the interval $[0,1]$. For each natural number k, define the function f_k by*

$$f_k(x) = \begin{cases} 1, & x \in \{r_1, r_2, \ldots, r_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find f, the point-wise limit of the sequence $\{f_k\}$.

(b) Show that $f_k \in R[0,1]$ for each k.

(c) In general, if $\{f_n\}$ is a sequence of Riemann integrable functions which converge point-wise to f, is f Riemann integrable?

Example 1.10. *Assume $f \in R[a,b]$.*

(a) Let $c \in [a,b]$. Suppose g is defined on $[a,b]$ and

$$g(x) = \begin{cases} f(x), & x \neq c, \\ \neq f(c), & x = c. \end{cases}$$

Show that $g \in R[a,b]$

(b) Suppose g differs from f at a finite number of points. Show that $g \in R[a,b]$.

(c) Does this extend to the case where g and f differ at a countable number of points? Prove or give a counterexample.

Example 1.11. *Prove or modify and then prove: Let $f \in B[a,b]$.*

Define

$$f^+(x) = \begin{cases} f(x), & f(x) > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad f^-(x) = \begin{cases} 0, & f(x) > 0, \\ -f(x), & \text{otherwise.} \end{cases}$$

Then

$$f \in R[a,b] \quad \text{if and only if both } f^+ \in R[a,b] \text{ and } f^- \in R[a,b].$$

ASSIGNMENT-7

7.1 Metric Spaces

1. Show that the real line is a metric space.

2. Does $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers?

3. Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of all real numbers.

4. Find all metrics on a set X consisting of two points. Consisting of one point.

5. Let d be a metric on X. Determine all constants k such that

   (a) kd

   (b) $d + k$

   is a metric on X.

6. Let X be the set of all bounded sequences of complex numbers; that is, every element of X is a complex sequence

   $$x = (\xi_1, \xi_2, \cdots) \text{ briefly } x = \{\xi_i\}$$

   such that $j = 1, 2, \cdots$ we have
   $$|\xi_j| \le C_x$$
   where C_x is a real number which may depend on X, but does not depend on j. The metric d is defined as
   $$d(x, y) = \sup_{j \in \mathbf{N}} |\xi_j - \eta_j|$$
   where $y = (\eta_j) \in X$ and $\mathbf{N}$ is the set of all natural numbers. Show that d satisfies the triangle inequality.

7. If A is the subspace of l^∞ consisting of all sequences of zeros and ones, what is the induced metric on A?

8. Show that $\tilde{d}$ on the set X defined by

   $$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)|\, dt$$

is a metric, where X is the set of all real-valued functions $x, y, \cdots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$.

9. Show that the d defined by

$$d(x, x) = 0, \quad d(x, y) = 1 \text{ for } x \neq y.$$

defined for any set X is a metric.

10. **(Hamming distance)** Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by $d(x, y) =$ number of places where x and y have different entries. (This space and similar spaces of n-tuples play a role in switching and automata theory and coding. $d(x, y)$ is called the Hamming distance between x and y.)

11. Prove the generalized triangle inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

12. **(Triangle inequality)** The triangle inequality has several useful consequences. For instance, using the inequality from *Qn.11*, show that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$

13. Using the triangle inequality, show that

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

14. **(Axioms of a metric)** (M1) to (M4) could be replaced by other axioms (without changing the definition). For instance, show that (M3) and (M4) could be obtained from (M2) and

$$d(x, y) \leq d(z, x) + d(z, y).$$

15. Show that nonnegativity of a metric follows from (M2) to (M4).

BIBLIOGRAPHY

[1] T. Tao, *Analysis I.* Hindustan Book Agency, 3 ed., 2017.

[2] E. Çinlar and R. J. Vanderbei, *Real and Convex Analysis.* Springer, 2 ed., 2013.

[3] T. Apostol, *Mathematical Analysis.* Addison Wesley and Narosa Publishing House, 2 ed., 1993.

[4] M. Spivak, *Calculus on Manifolds.* CRC Press, 2 ed., 2018.

[5] M. D. Crossley, *Essential Toplology.* Springer International Edition, 2 ed., 2015.

[6] G. S. Nelson, *A User-friendly Introduction to Lebesgue Measure and Integration.* AMS, 1 ed., 2010.

[7] J. R. Munkres, *Analysis on Manifolds.* CRC Press, 2 ed., 1991.