Lecture 0

24 July 2019 15:06

EE6140 Multi-antenna Digital Communications

Website: http://www.ee.iitm.ac.in/~skrishna/ee6140/

- Has a brief description of the course contents
- Course notes from the 2012 version
- List of papers that will be discussed in this course.

Course content

Selected papers from the multiple-input-multiple-output (MIMO) literature will be discussed in class.

Main model $\underline{y} = H\underline{x} + \underline{n}$

$\underline{y}$: $n_r \times 1$ received vector
$\underline{x}$: $n_t \times 1$ transmit vector
H : $n_r \times n_t$ channel matrix
$\underline{n}$: $n_r \times 1$ channel noise vector

L0

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Usually $\underline{n}$: Gaussian random vector

H : Deterministic matrix

(or)

Random matrix with i.i.d. Gaussian entries

H may be known at both transmitter and receiver

(or) known only at rx

(or) not known at both tx and rx

(or) known only at tx

First, we will discuss some background material

- Gaussian random vectors
- Detection in vector Gaussian noise channel: Prob. of error and union bound
- Information theory definitions (capacity)

Then, we will discuss papers on (1) capacity of MIMO channels

(2) Detection in spatial multiplexing systems

(3) Space-time coding, diversity multiplexing tradeoff

Lecture 1

29 July 2019 13:51

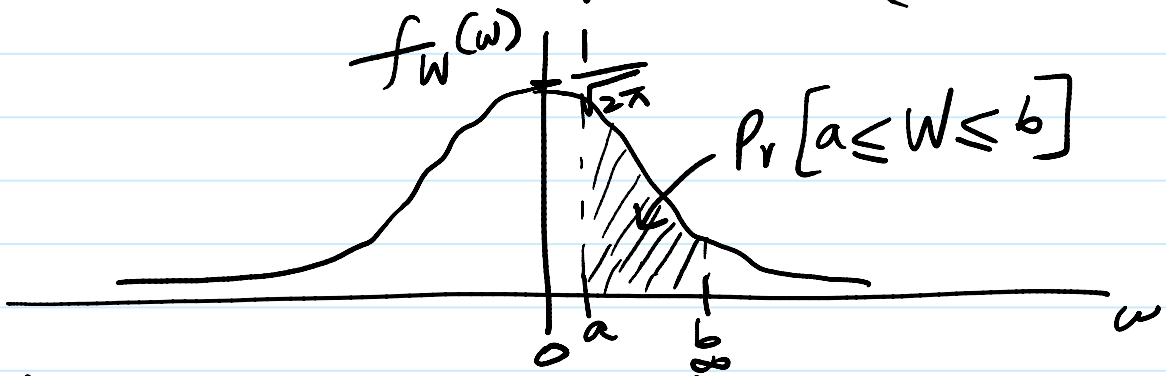
Review of Gaussian random variables and vectors

- Scalar real Gaussian random variables

A <u>Standard Gaussian random variable</u> W has probability density function (pdf)

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) \quad \text{for all } w \in \mathbb{R}$$

↑ Takes values from $\mathbb{R}$



Mean of W: $E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = 0$

Variance of W: $E[(W - E[W])^2] = E[W^2] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = 1.$

Lecture 1

30 July 2019 12:07

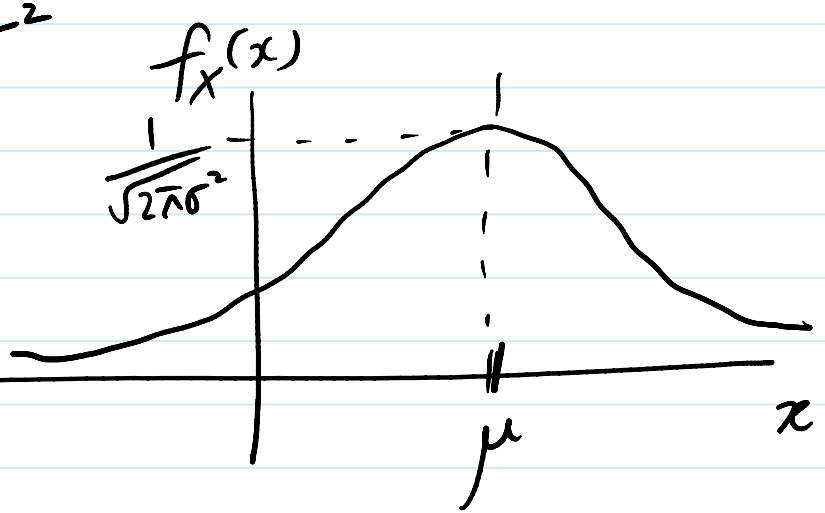
General Gaussian random variable $X = \sigma W + \mu$

$$E[X] = \mu$$

$$\text{Var}(X) = E[(X-\mu)^2] = \sigma^2$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

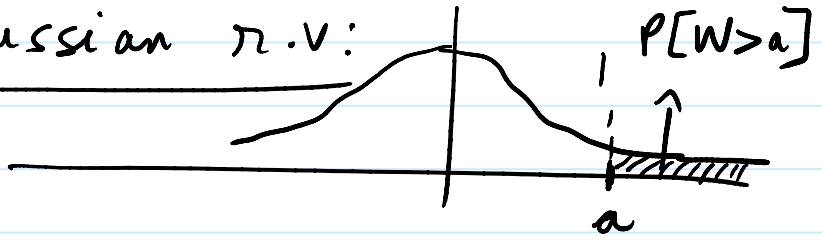
for all $x \in \mathbb{R}$



Notation: $N(\mu, \sigma^2)$

Tail probability of the standard Gaussian r.v:

$$P(W > a) = \int_a^\infty f_W(w)\,dw$$



L1

30 July 2019 12:13

$$= \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$ (cannot be evaluated in closed form)

$$\triangleq Q(a)$$ (Q function)

Tail probability of a general Gaussian r.v.:

$$P(X > a) = P(\sigma W + \mu > a)$$

$X \sim N(\mu, \sigma^2)$

$$= P\left[W > \frac{a-\mu}{\sigma}\right] \quad \text{(assuming } \sigma \neq 0\text{)}$$

$$= Q\left(\frac{a-\mu}{\sigma}\right)$$

L1

30 July 2019 12:17

Useful bounds on $Q(a)$:

(used in analysis later: since $Q(\cdot)$ cannot be written in closed form, we use bounds in closed form)

$$\frac{1}{\sqrt{2\pi}\,a}\left(1-\frac{1}{a^2}\right)e^{-a^2/2} < Q(a) < e^{-a^2/2} \quad \text{for } a>1$$

$$Q(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

Linear transformations of Gaussian r.v.'s.

If $X_1, X_2, \ldots, X_n$ are independent Gaussian r.v's with $X_i \sim N(\mu_i, \sigma_i^2)$, then

$$\sum_{i=1}^n c_i X_i \sim N\left(\sum_{i=1}^n c_i \mu_i,\ \sum_{i=1}^n c_i^2 \sigma_i^2\right)$$

L1

30 July 2019 12:24

– Real Gaussian random vectors

Standard Gaussian random vector $\underline{W}$

* Collection of i.i.d. standard Gaussian r.v's $W_1, W_2, \dots, W_n$ $\quad W_i \sim N(0,1)\ \forall i$

$$\underline{W} = \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{pmatrix}$$

* $$f_{\underline{W}}(\underline{w}) = \frac{1}{(\sqrt{2\pi})^n} \exp\left(-\frac{\sum_{i=1}^{n} w_i^2}{2}\right), \quad \underline{w} \in \mathbb{R}^n$$

$$\|\underline{w}\|^2 \triangleq \sum_{i=1}^{n} w_i^2$$

(Euclidean Distance from $\underline{0}$ to $\underline{w}$)^2

L1

30 July 2019 12:29

Note that $f_{\underline{W}}(\underline{w})$ depends only on $\|\underline{w}\|$

* If $T_{n\times n}$ is an orthonormal transformation (i.e. $TT^T = T^TT = I_n$), then $T\underline{W}$ is also standard Gaussian.

$$\left(\|T\underline{w}\|^2 = \underline{w}^T T^T T \underline{w} = \underline{w}^T\underline{w} = \|\underline{w}\|^2\right)$$

↳ Isotropic transformation

e.g. Rotation

Reflection

Another interpretation:

$$T\underline{W} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$$X_i = \left(i^{th} \text{ row of } T\right)\begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{pmatrix}$$

Projection of $\underline{W}$ along the direction of the ith row.

L1

30 July 2019 12:35

> Projections of the standard Gaussian random vector $\underline{W}$ in orthogonal directions are independent.

$$\|\underline{W}\|^2 = \sum_{i=1}^{n} W_i^2$$ (chi-squared random variable with n degrees of freedom)

for $n = 2$ $\|\underline{W}\|^2 \sim$ exponential distribution with parameter $\frac{1}{2}$.

$$f_{\|\underline{W}\|^2}(a) = \frac{1}{2} e^{-a/2}, \quad a \geq 0.$$

General Gaussian random vector $\underline{X} = A\underline{W} + \underline{\mu}$

($\underline{X}$: $n \times 1$, A: $n \times n$, $\underline{W}$: $n \times 1$, $\underline{\mu}$: $n \times 1$)

$\underline{\mu} \in \mathbb{R}^n$

A : $n \times n$ matrix

Any linear transformation of a Gaussian random vector is also Gaussian.

L1

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If A is invertible, then

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det(AA^T)}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T (AA^T)^{-1}(\underline{x}-\underline{\mu})\right\} \quad \text{for } \underline{x} \in \mathbb{R}^n$$

$$E[\underline{X}] = \underline{\mu}$$

Covariance matrix of $\underline{X}$ $= E\left[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^T\right] = E\left[(A\underline{W})(A\underline{W})^T\right]$

$$= A\,E\left[\underline{W}\,\underline{W}^T\right]A^T$$

$$= AA^T \triangleq K$$

$$\underline{X} \sim N(\underline{\mu}, K)$$

$$\underline{W} \sim N(\underline{0}, I_n)$$

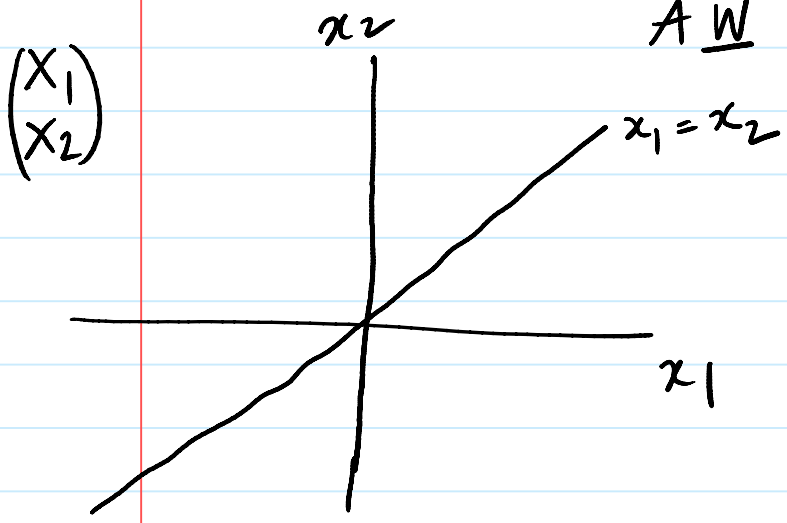
L1

30 July 2019 12:45

If K is diagonal, then the component r.v.'s in $\underline{X}$ are independent.

If $K = I$, then the components of $\underline{X}$ are independent and have the same unit variance.

* If A is not invertible,



$A\underline{W}$ lies in a subspace of dimension $< n$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} W_1 + W_2 \\ W_1 + W_2 \end{pmatrix}$$

Lecture 2

01 August 2019 11:01

If A is not invertible,
some components of $A\underline{W}$ can be expressed as a linear combination of the others.

$\underline{X}$ → $\underline{\tilde{X}}$ which has an invertible covariance matrix

$\underline{X}$ → Linear combination of components in $\underline{\tilde{X}}$.

Complex Gaussian Random vectors :

$$\underline{X} = \underline{X}_R + j\underline{X}_I \qquad \left(\text{fn. of } \begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix}\right)$$

$\underline{X}$: complex random vector

$\underline{X}_R$, $\underline{X}_I$: Real random vectors

L 2

01 August 2019 11:09

Need to describe the joint PDF of $\begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix}$.

$\underline{X}$ is complex Gaussian if $\begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix}$ is a real Gaussian random vector

→ PDF of $\underline{X}$ is specified completely by

Mean and covariance matrix of $\begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix}$

(or)

$$E[\underline{X}] = E[\underline{X}_R] + jE[\underline{X}_I] \triangleq \mu$$

$$K = E[(\underline{X}-\mu)(\underline{X}-\mu)^H] \quad \text{Covariance}$$

$$J = E[(\underline{X}-\mu)(\underline{X}-\mu)^T] \quad \text{Pseudo covariance}$$

$E[\underline{X}_R]$, $E[\underline{X}_I]$, $\text{Cov}\left(\begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix}\right) = \begin{bmatrix} C_R & C_{RI} \\ C_{IR} & C_I \end{bmatrix}$

$C_{RI} = C_{IR}^T$

$$K = (C_R + C_I) + j(C_{IR} - C_{RI})$$

$$J = (C_R - C_I) + j(C_{IR} + C_{RI})$$

L2

01 August 2019 11:15

Circular symmetry

$\underline{X}$ is circularly symmetric if $e^{j\theta}\underline{X}$ has the same distribution as $\underline{X}$ for any θ.

Not only for Gaussian r.v.s

$$E[\underline{X}] = E[e^{j\theta}\underline{X}] \text{ for all } \theta \Rightarrow E[\underline{X}] = \underline{0} \text{ (zero mean)}$$

$$E[\underline{X}\underline{X}^T] = E[e^{j\theta}\underline{X}\,(e^{j\theta}\underline{X})^T] = e^{j2\theta}E[\underline{X}\underline{X}^T] \text{ for all } \theta$$

$$\Rightarrow E[\underline{X}\underline{X}^T] = \underbrace{0}_{\text{matrix}} \quad \text{i.e. } \underline{J = 0}.$$

$C_R = C_I,\ C_{IR} = -C_{RI}$

K fully specifies

If $\underline{X}$ is a circularly symmetric random vector, the second order statistics. Further, if $\underline{X}$ is Gaussian, K specifies the entire distribution.

Circularly symmetric complex Gaussian random vector: $CN(\underline{0}, K)$

L 2

01 August 2019 11:23

$n=1$ circularly symmetric complex Gaussian random variable

$C_R = C_I$, $C_{IR} = -C_{RI} \Rightarrow E[X_R^2] = E[X_I^2] = \sigma^2$

$E[X_I X_R] = -E[X_R X_I] \Rightarrow E[X_I X_R] = 0$.

$\Rightarrow X_R, X_I$ i.i.d. $N(0, \sigma^2)$

$n>1$ $C_R = C_I$ ✓

$C_{IR} = -C_{RI}$ ✓ $\Rightarrow$ diagonal entries of C_{IR} are zero.

✓ Standard circularly symmetric complex Gaussian random vector $\underline{W}$

$\sim CN(\underline{0}, I_n)$ $\rightarrow$ $\begin{bmatrix} \underline{X}_R \\ \underline{X}_I \end{bmatrix} \sim N\left(\underline{0}, \begin{bmatrix} \frac{1}{2} I_n & 0 \\ 0 & \frac{1}{2} I_n \end{bmatrix}\right)$ ✓

$K = C_R + C_I + j(C_{IR} - C_{RI})$

L 2

01 August 2019 11:31

$$f_{\underline{X}_R, \underline{X}_I}(x_R, x_I) = \frac{1}{\left(\sqrt{2\pi\left(\frac{1}{2}\right)}\right)^{2n}} \exp\left(-\frac{\sum_{i=1}^{n}(x_{R_i}^2 + x_{I_i}^2)}{2\left(\frac{1}{2}\right)}\right)$$

$$= \frac{1}{\pi^n} \exp\left(-\sum_{i=1}^{n}(x_{R_i}^2 + x_{I_i}^2)\right)$$

$$= \frac{1}{\pi^n} \exp\left(-\sum_{i=1}^{n}|w_i|^2\right)$$

where $w_i = x_{R_i} + j x_{I_i}$

$\sum_{i=1}^{n}|w_i|^2 = \underline{w}^{H}\underline{w} = \|\underline{w}\|^2$

written as

$$\boxed{f_{\underline{W}}(\underline{w}) = \frac{1}{\pi^n} \exp\left(-\|\underline{w}\|^2\right)}, \quad \underline{w} \in \mathbb{C}^n$$

↑ n-dim complex space.

L2

01 August 2019 11:38

* $U\underline{W}$ has the same distribution as $\underline{W}$ if U is unitary ($U^H U = I$)

  Isotropic property.

* $\|\underline{W}\|^2 \sim$ chi-square r.v. with $2n$ degrees of freedom

* $\underline{X} = A\underline{W}$ is also circularly symmetric complex Gaussian

  → zero-mean

  → $K = AA^H$

Also, any circularly symmetric complex Gaussian r.v. can be written as a linearly transformed version of a standard circularly symmetric complex Gaussian r.v.

If A is invertible, $f_{\underline{X}}(\underline{x}) = \frac{1}{\pi^n \det(K)} \exp\left(-\underline{x}^H K^{-1} \underline{x}\right), \underline{x} \in \mathbb{C}^n$

L2

01 August 2019 11:43

Detection in Gaussian noise

Scalar case: $Y = U + W$ (Real AWGN channel)

- Y: observation
- U: equally likely to be u_A or u_B
- $W \sim N\left(0, \frac{N_0}{2}\right)$

Problem: Observe Y and decide on U

$$\hat{U} = g(Y)$$

- $\hat{U}$: decision
- $g(\cdot)$ specifies the decision/detection rule.

Choose $g(\cdot)$ to minimize $\Pr[\hat{U} \neq U]$

→ Probability of error.

Lecture 3

02 August 2019 10:02

$$P_e = Pr[\hat{U} \neq U] = \int_{-\infty}^{\infty} \underbrace{Pr[\hat{U} \neq U / Y=y]}_{\geq 0} \underbrace{f_Y(y)}_{\geq 0} \, dy$$

To minimize $Pr[\hat{U} \neq U]$, we can minimize $Pr[\hat{U} \neq U / Y=y]$ for each y.

U is either u_A or u_B.

$Y=y$

If $\hat{U} = u_A$, $P_e = Pr[U = u_B / Y=y]$

If $\hat{U} = u_B$, $P_e = Pr[U = u_A / Y=y]$

Suppose we observe $Y=y$,

decide $\hat{U} = u_A$ if $P[U=u_A / Y=y] \geq P[U=u_B / Y=y]$

$\hat{U} = u_B$ otherwise

MAP rule (max a posteriori prob. rule)

L 3

02 August 2019 10:10

$$P[U=u_A|Y=y] = \frac{f_{Y/U=u_A}(y)\,P[U=u_A]}{f_Y(y)}$$

$$P[U=u_B|Y=y] = \frac{f_{Y/U=u_B}(y)\,P[U=u_B]}{f_Y(y)}$$

$Y = U + W$

$f_{Y/U=u_A} \sim N\left(u_A, \frac{N_0}{2}\right)$

$f_{Y/U=u_B} = N\left(u_B, \frac{N_0}{2}\right)$

$P[U=u_A] = P[U=u_B] = \frac{1}{2}$

Decision rule simplifies to:

Decide $\hat{U}=u_A$, if $f_{Y/U=u_A}(y) \geq f_{Y/U=u_B}(y)$

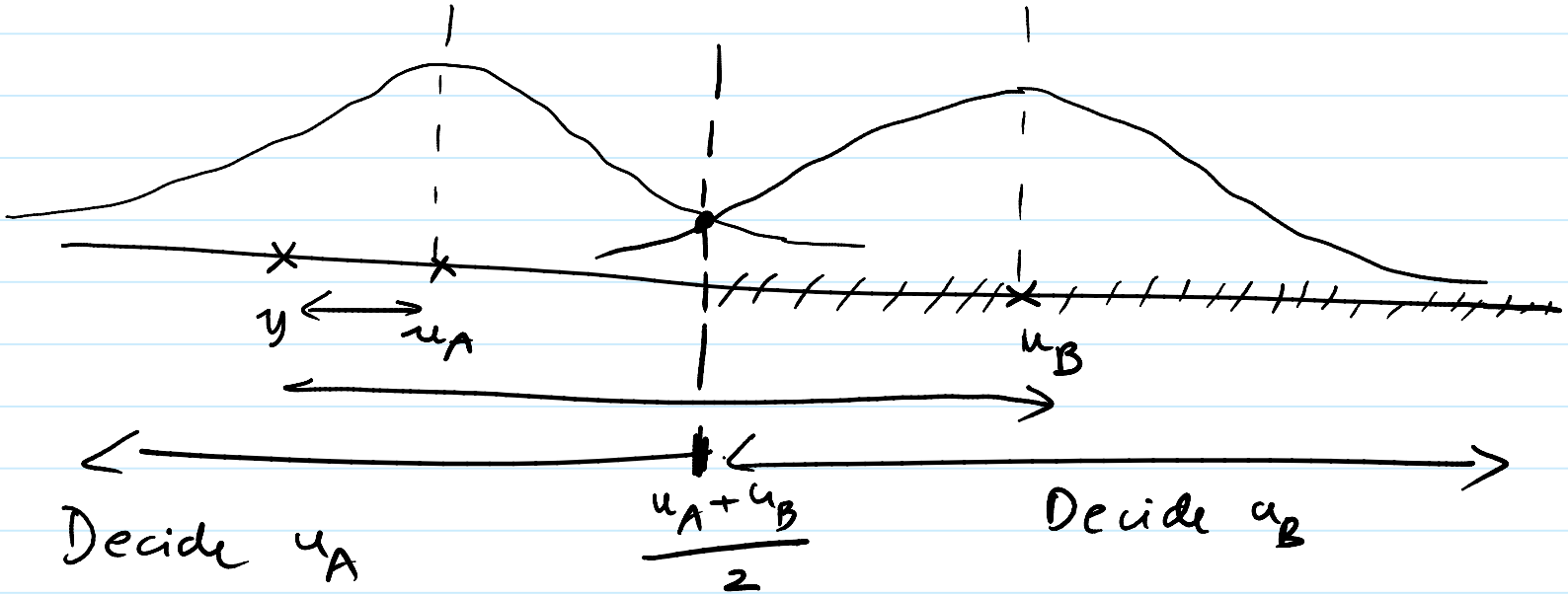
$$\frac{1}{\sqrt{2\pi\left(\frac{N_0}{2}\right)}} \exp\left(-\frac{(y-u_A)^2}{2\left(\frac{N_0}{2}\right)}\right) \geq \frac{1}{\sqrt{2\pi\left(\frac{N_0}{2}\right)}} \exp\left(\frac{-(y-u_B)^2}{2\left(\frac{N_0}{2}\right)}\right)$$

$\hat{U}=u_A$ if $(y-u_A)^2 \leq (y-u_B)^2$ (Nearest neighbour rule)

(or) $|y-u_A| \leq |y-u_B|$

L 3

02 August 2019 10:15



- Easy extension of MAP rule to M-ary signaling

- If $u_1, u_2 \cdots, u_M$ are equally likely, nearest neighbour detection (Gaussian noise)

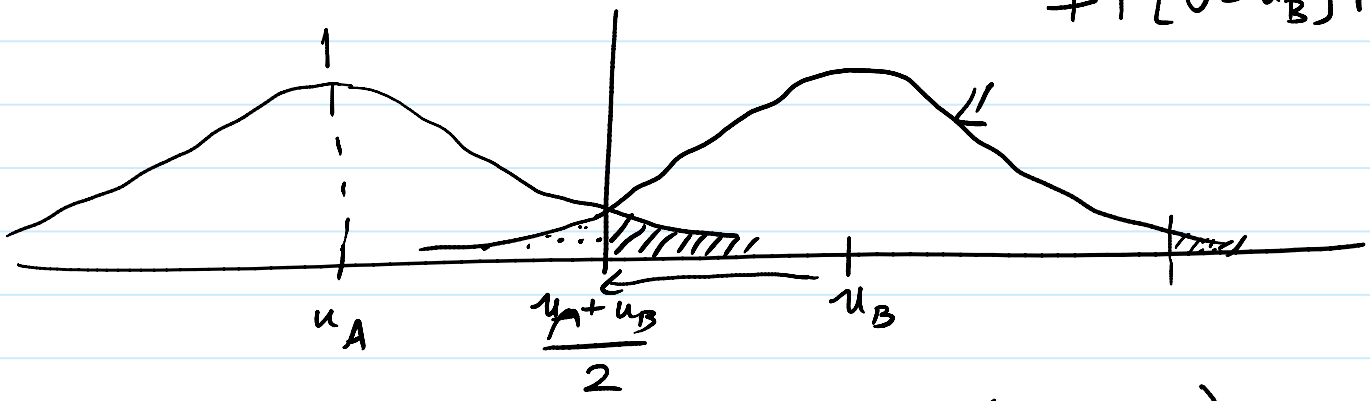
$$\begin{cases} U = u_1 \\ \text{or} \\ u_2 \\ \vdots \\ \text{or} \\ u_M \end{cases} \quad \hat{U} = u_i \text{ if } P[U=u_i/Y=y] \geq P[U=u_j/Y=y] \; \forall j \neq i$$

(MAP rule)

L 3

02 August 2019 10:21

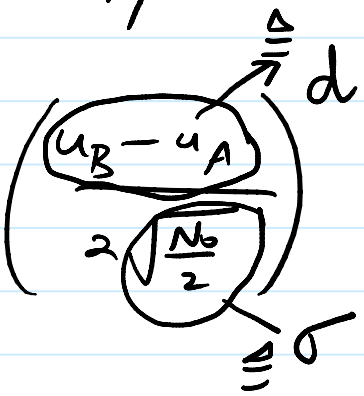
$$P_e = P(\hat{U} \neq U) = P(U=u_A)\,P(\hat{U}=u_B \mid U=u_A) + P[U=u_B]\,P[\hat{U}=u_A \mid U=u_B]$$



$$P_e = \frac{1}{2}\left(Q\left(\frac{\frac{u_A+u_B}{2}-u_A}{\sqrt{\frac{N_0}{2}}}\right)\right) + \frac{1}{2}\left(Q\left(\frac{u_B-\frac{u_A+u_B}{2}}{\sqrt{\frac{N_0}{2}}}\right)\right)$$

$$= \frac{1}{2}Q\left(\frac{u_B-u_A}{2\sqrt{\frac{N_0}{2}}}\right) + \frac{1}{2}Q\left(\frac{u_B-u_A}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\frac{d}{2\sigma}\right)$$

$Y/U=u_A \sim N\left(u_A, \frac{N_0}{2}\right)$



L 3

02 August 2019 10:28

Vector case

$$\underline{y}_{n\times 1} = \underline{U}_{n\times 1} + \underline{W}_{n\times 1}$$

$\underline{U}$: either $\underline{u}_A$ or $\underline{u}_B$ (equally likely)

$\underline{W} \sim N\left(\underline{0}, \frac{N_0}{2} I_n\right)$

MAP rule $\hat{U} = \underline{u}_A$ if $P[\underline{U} = \underline{u}_A / \underline{Y} = \underline{y}] \geq P[\underline{U} = \underline{u}_B / \underline{Y} = \underline{y}]$

$$f_{\underline{Y}/\underline{U}=\underline{u}_A}(\underline{y}) \geq f_{\underline{Y}/\underline{U}=\underline{u}_B}(\underline{y})$$

$$N\left(\underline{u}_A, \frac{N_0}{2} I_n\right) \qquad N\left(\underline{u}_B, \frac{N_0}{2} I_n\right)$$

Nearest neighbour

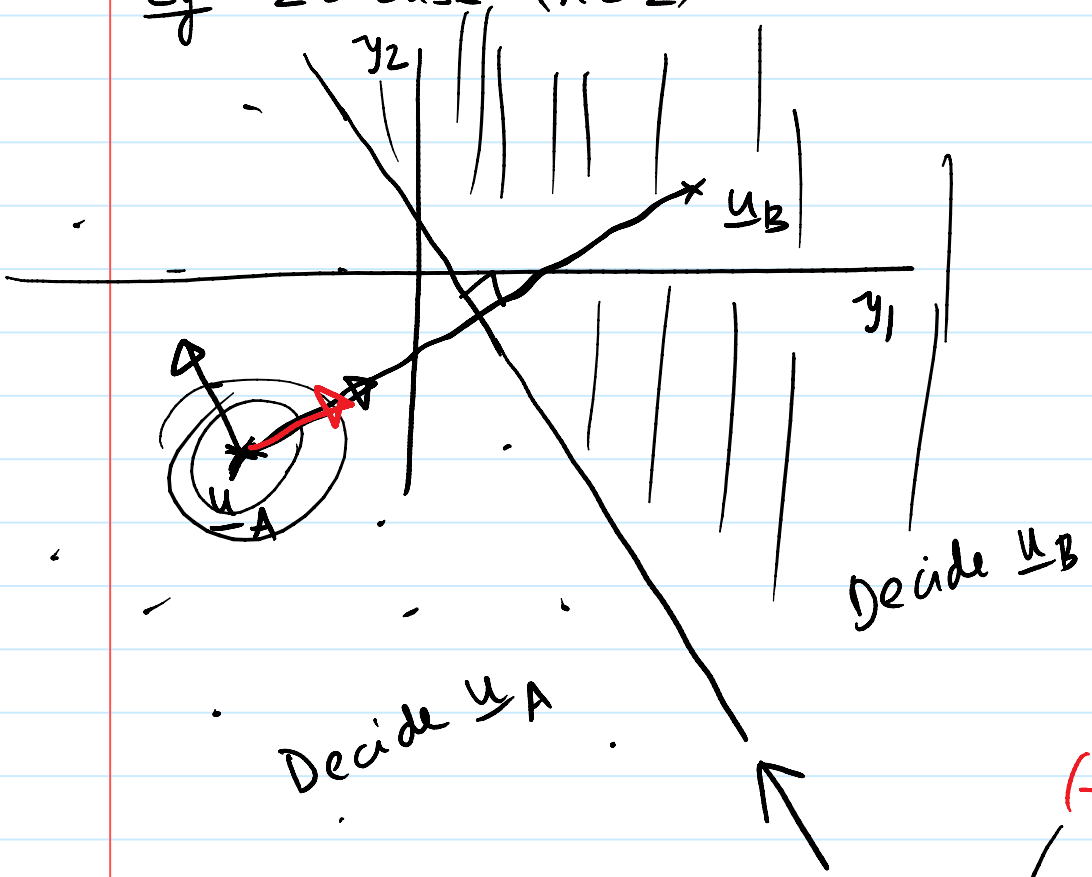
$$\frac{1}{\left(2\pi \frac{N_0}{2}\right)^{n/2}} \exp\left(\frac{-\|\underline{y} - \underline{u}_A\|^2}{2\left(\frac{N_0}{2}\right)}\right)$$

$$\|\underline{y} - \underline{u}_A\|^2 \leq \|\underline{y} - \underline{u}_B\|^2$$

L 3

02 August 2019 10:39

eg 2D case ($n = 2$)



$P_e = ?$

Nearest neighbour rule is optimal

→ Noise orthogonal to $\underline{u}_A - \underline{u}_B$ is irrelevant

(does not affect our decision)

→ Noise along $\underline{u}_A - \underline{u}_B$ matters

$$\frac{(\underline{u}_B - \underline{u}_A)^T}{\|\underline{u}_B - \underline{u}_A\|} \frac{N_0}{2} I_2 \frac{(\underline{u}_B - \underline{u}_A)}{\|\underline{u}_B - \underline{u}_A\|} = \frac{N_0}{2}$$

(→ intuitive)

$$\left(\frac{\underline{u}_B - \underline{u}_A}{\|\underline{u}_A - \underline{u}_B\|}\right)^T \underline{W} \sim N\left(0, \frac{N_0}{2}\right)$$

L 3

02 August 2019 10:48

$$P_e = Q\left(\frac{\|\underline{u}_A - \underline{u}_B\|}{2\left(\sqrt{\frac{N_0}{2}}\right)}\right)$$

Nearest neighbour rule extends to M-ary signalling also.

Lecture 4

05 August 2019 08:00

Complex vector Gaussian case

$$\underline{Y}_{n\times 1} = \underline{U}_{n\times 1} + \underline{W}_{n\times 1}$$

$\underline{W} \sim \mathcal{CN}(\underline{0}, N_0 I_n)$

$\underline{U}$: either $\underline{u}_A$ or $\underline{u}_B$ with equal probability

$$\hat{\underline{U}} = \underline{u}_A \text{ if } \|\underline{Y} - \underline{u}_A\|^2 < \|\underline{Y} - \underline{u}_B\|^2$$ (Nearest neighbour of $\underline{Y}$)

– Projection of $\underline{Y}$ onto $\underline{u}_A - \underline{u}_B$ is sufficient

$$\frac{(\underline{u}_A - \underline{u}_B)^H \underline{Y}}{\|\underline{u}_A - \underline{u}_B\|}$$

L4

05 August 2019 08:07

Binary signaling : $P_e = Q\left(\frac{d}{2\sigma}\right)$

$d = \|\underline{u}_A - \underline{u}_B\|$

σ : sqrt of noise variance in the direction $\underline{u}_A - \underline{u}_B$

M-ary signaling : $P_e = ?$

* $\underline{U}$ can be $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_M$ (equally likely)

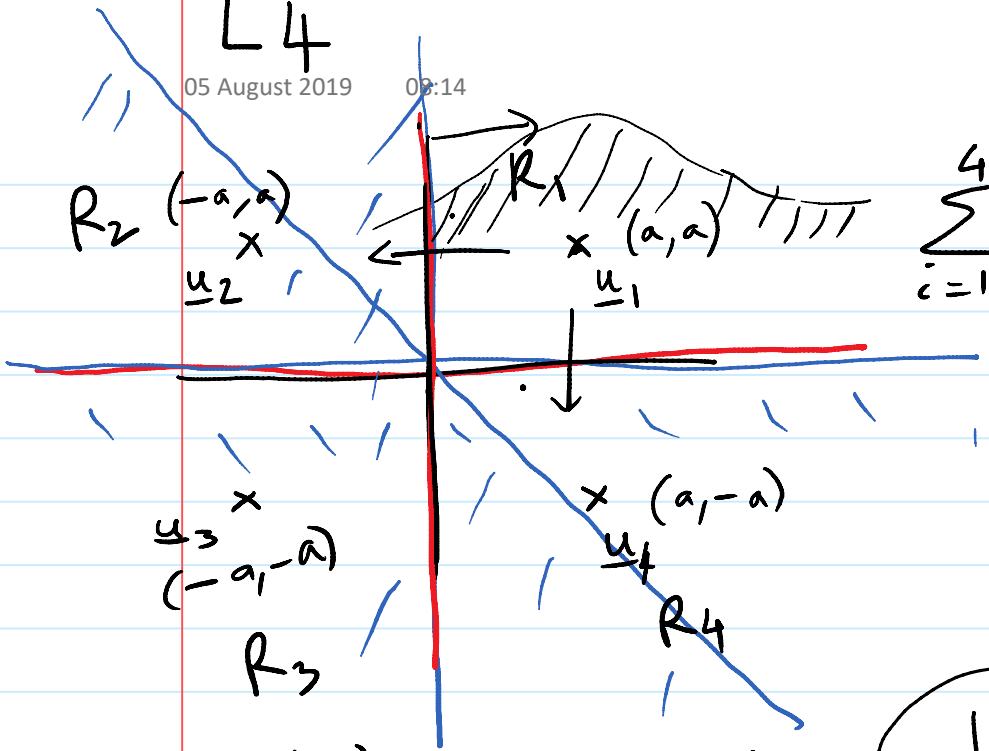
* Decision rule : $\hat{\underline{U}} = \underline{u}_i$ if $\|\underline{y} - \underline{u}_i\|^2 < \|\underline{y} - \underline{u}_j\|^2$ for all $j \neq i$

$$P_e = \sum_{i=1}^{M} P(\text{error}/\underline{u}_i)\, P(\underline{u}_i)$$

$D_i^2 \triangleq \|\underline{y} - \underline{u}_i\|^2$

L4

05 August 2019 08:14



$$R_i = \{\underline{y} : D_i \text{ is min}\}$$

$$\sum_{i=1}^{4} Pr(\text{error}/\underline{u}_i)\frac{1}{4}$$

$Pr(\text{error}/\underline{u}_i)$ same for all i

(by symmetry)

$$Pr(\text{error}/\underline{u}_1) = Pr(\underline{y} \notin R_1/\underline{u}_1)$$

$$\underline{y}/\underline{u}_1 \sim N\left(\underline{u}_1, \frac{N_0}{2}I\right)$$

$$1 - \int_{R_1} f_{\underline{y}/\underline{u}_1}(y)\,dy$$

(or)

$$\int_{R_2 \cup R_3 \cup R_4} f_{\underline{y}/\underline{u}_1}(y)\,dy$$

L4

05 August 2019 08:20

$$\int_{R_1} f_{Y/U_1}(y)\,dy = \left(Q\left(\frac{-a}{\sqrt{\frac{N_0}{2}}}\right)\right)\left(Q\left(\frac{-a}{\sqrt{\frac{N_0}{2}}}\right)\right)$$

R_1 = (Rectangular region) $\longrightarrow$ ( ) ( )

$$\int_{R_2} f_{Y/U_1}(y)\,dy = \left(Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right)\right)\left(Q\left(\frac{-a}{\sqrt{\frac{N_0}{2}}}\right)\right)$$

R_2

Decision regions are rectangular

If decision regions are not rectangular, we cannot break it down into a product of $Q(\cdot)$ fns.

L 4

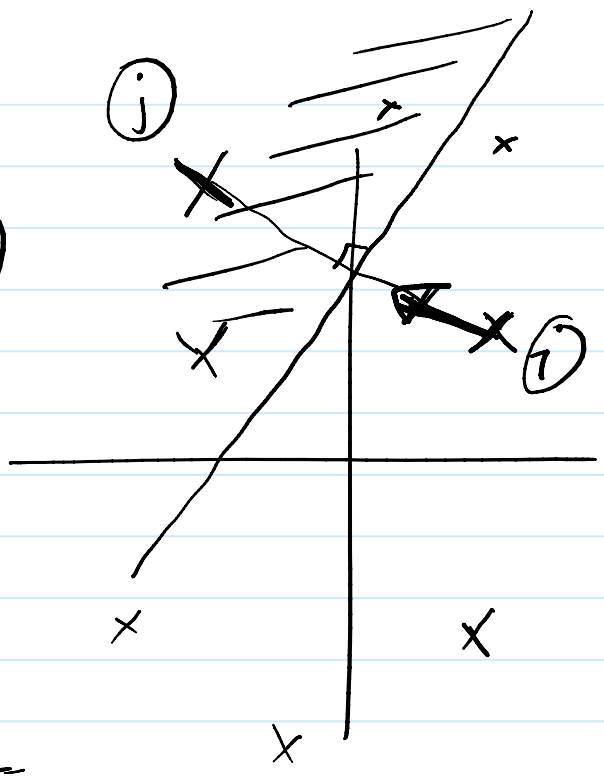
05 August 2019 08:31

Union bound

$$P_r(\text{error}/\underline{u}_i) = P_r\left(\bigcup_{j\neq i}\{D_j < D_i\}/\underline{u}_i\right)$$

$$(\text{Union bound}) \leq \sum_{j\neq i} Pr(\{D_j < D_i\}/\underline{u}_i)$$

$$= \sum_{j\neq i} Q\left(\frac{d_{ij}}{2\sigma}\right)$$



For the previous example

$$2\,Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) + Q\left(\frac{a\sqrt{2}}{\sqrt{\frac{N_0}{2}}}\right)$$

$\sqrt{\frac{2a^2}{N_0}}$

Bound is good at high $\frac{a^2}{N_0}$.

Union bound

Actual Pe $\quad 1 - \left[Q\left(\frac{-a}{\sqrt{\frac{N_0}{2}}}\right)\right]^2 = 1 - \left(1 - Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right)\right)^2$

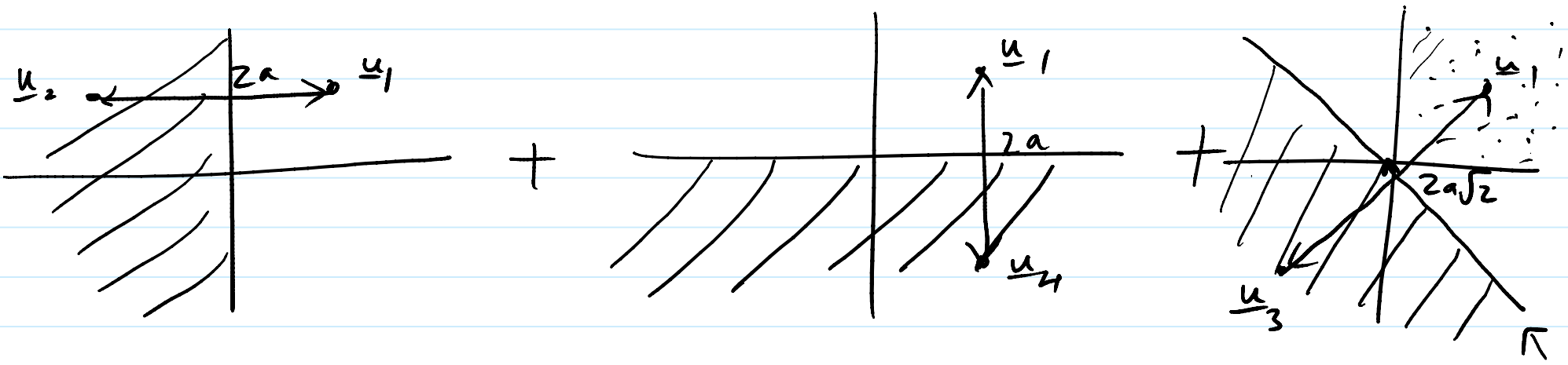
$1-(1-x)^2$

$1-1-x^2+2x$

$= 2x-x^2$

L4

05 August 2019 08:43



Union bound can be loose for large constellations

$P(\{D_j < D_i\}/\underline{u}_i)$: Pairwise error probability for binary detection between signals $\underline{u}_i$ and $\underline{u}_j$

(or)

$P(\underline{u}_i \rightarrow \underline{u}_j)$

L 4

05 August 2019 08:47

Intelligent union bound

$$P(\text{error}/\underline{u}_i) \leq \sum_{j \neq i} P\left(\{D_j < D_i\}/\underline{u}_i\right) \quad \text{union bound}$$

$\Downarrow$

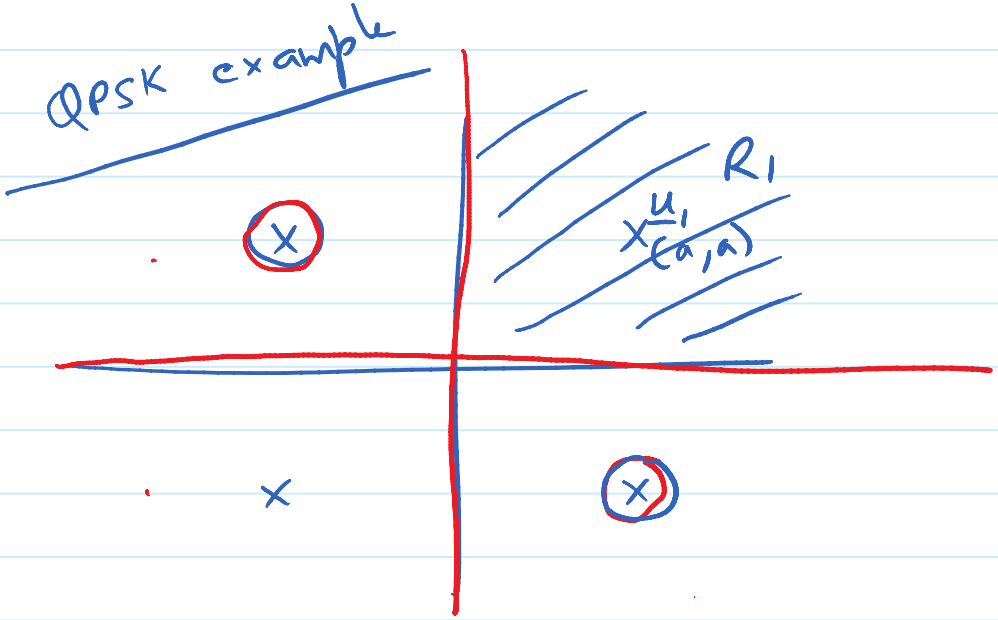
Drop terms (How?)

Lecture 5

06 August 2019 12:00

$$Pr(error/\underline{u}_i) \leq \sum_{j \in N_{ML}(i)} Pr\left[\{D_j < D_i\}/\underline{u}_i\right]$$

$N_{ML}(i)$: Set of neighbours of i that characterize the decision region R_i;



$$P(error/\underline{u}_1) \leq 2Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right)$$

Intelligent union bound.

L5

06 August 2019 12:06

$$Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) \leq 2Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) - Q^2\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) \leq 2Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) \leq 2Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) + Q\left(\frac{a\sqrt{2}}{\sqrt{\frac{N_0}{2}}}\right)$$

Any pairwise error prob

Exact P_e

Intelligent union bound

Union bound

Pairwise error prob corresponding to the nearest neighbour

Approximate P_e : $\approx 2Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right)$

(Need not be a lower bound, Need not be an upper bound)

(Nearest neighbour approximation)

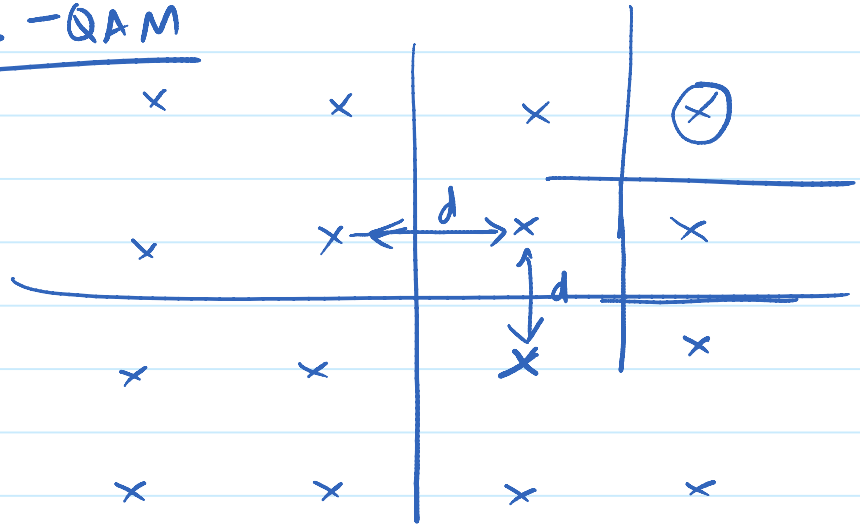
$$\#\text{neighbours at } d_{min} \times Q\left(\frac{d_{min}}{2\sigma}\right)$$

d_{min} → distance from nearest neighbour.

L5

06 August 2019 12:16

16-QAM



$$P_e = \sum_{i=1}^{16} \underbrace{P(\text{error}/\underline{u}_i)}\, \underbrace{P(\underline{u}_i)}_{\frac{1}{16}}$$

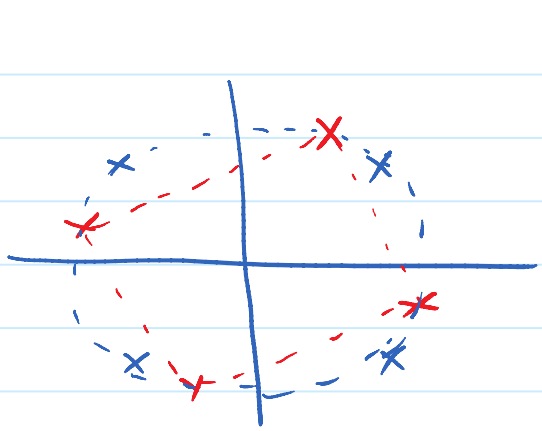
* union bound has 15 terms
* Intelligent union bound has 2 or 3 or 4 terms depending on i

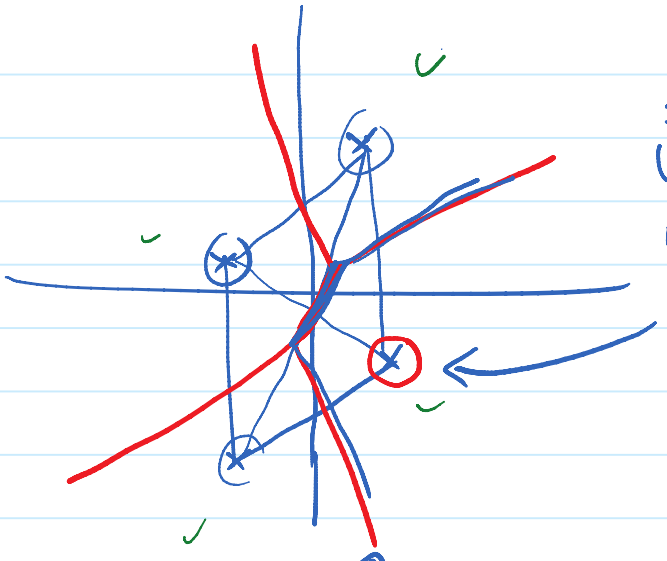
$$\frac{4}{16}\left(2Q\left(\frac{d}{2\sigma}\right)\right) + \frac{8}{16}\left(3\,Q\left(\frac{d}{2\sigma}\right)\right) + \frac{4}{16}\left(4Q\left(\frac{d}{2\sigma}\right)\right)$$

$$= 3Q\left(\frac{d}{2\sigma}\right)$$

L5

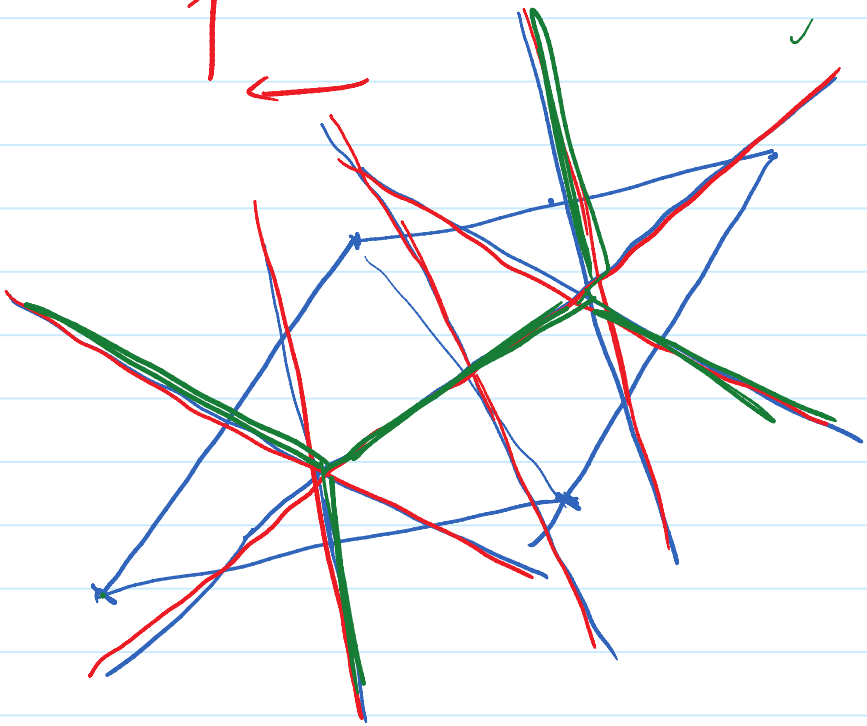
06 August 2019 12:21





Intelligent Union bound will have all 3 terms in this case

Min distance rule



Use voronoi command in MATLAB.

L5

06 August 2019 12:34

— Can use union bound / intelligent union bound / nearest neighbour approximation easily even for high dimensions. Exact analysis difficult. Here, only pairwise distances need to be computed.

— So far $\underline{Y} = \underline{U} + \underline{W}$, $\underline{W} \sim N(\underline{0}, I_n \frac{N_0}{2})$

MAP rule ↓ Nearest neighbour rule

Suppose $\underline{Y} = \underline{U} + \underline{W}$, $\underline{W} \sim N(\underline{0}, K)$

Decision rule?

MAP rule ↓

( )

$\hat{U} = \underline{u}_i$ if $P[\underline{U} = \underline{u}_i / \underline{Y} = y] \geq P[\underline{U} = \underline{u}_j / \underline{Y} = y] \quad \forall j \neq i$

(Equally likely $\underline{u}_i$) $f_{\underline{Y}/\underline{u}_i}(y) \geq f_{\underline{Y}/\underline{u}_j}(y) \quad \forall j \neq i$

L5

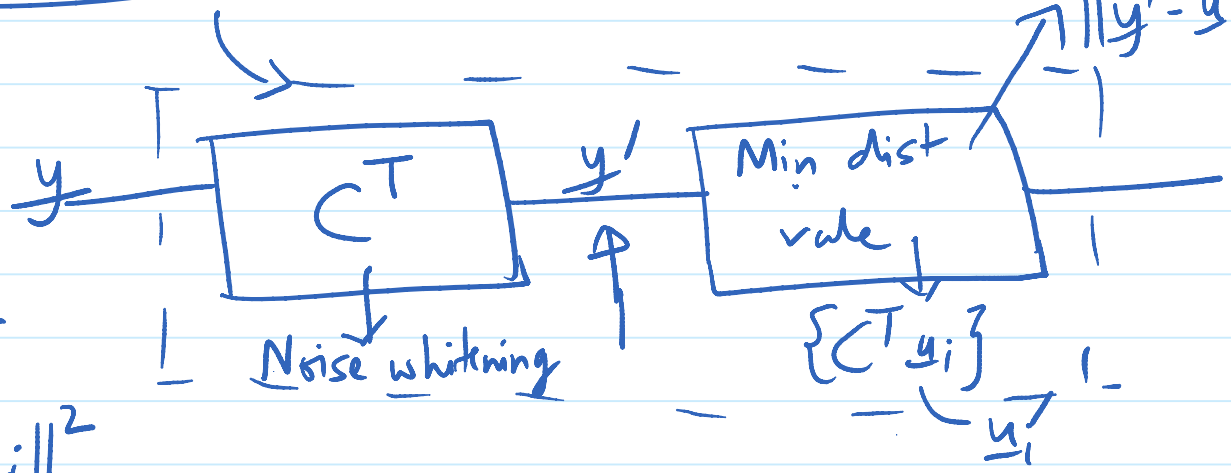
06 August 2019 12:41

$$\frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp\left\{-\frac{1}{2}(y-u_i)^T K^{-1}(y-u_i)\right\}$$

$$C^T y / u_i \sim N(C^T u_i, I_n)$$

$$\geq \frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp\left\{-\frac{1}{2}(y-u_j)^T K^{-1}(y-u_j)\right\}.$$

$\hat{u} = u_i$ if $(y-u_i)^T K^{-1}(y-u_i) \leq (y-u_j)^T K^{-1}(y-u_j) \quad \forall j \neq i$



$K^{-1} = CC^T$

p.d.

$\|C^T(y-u_j)\|^2$

$= \|C^T y - C^T u_i\|^2$

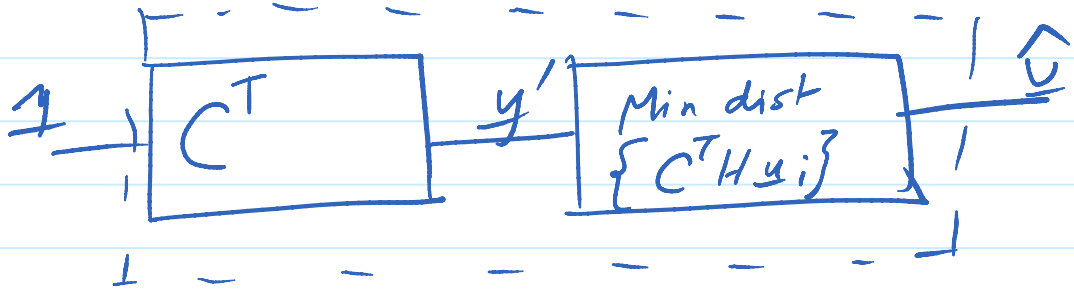
L5

06 August 2019 12:48

Suppose $\underline{Y} = H\underline{U} + \underline{W}$, $\underline{W} \sim N(\underline{0}, K)$

$\hat{\underline{U}} = \underline{u}_i$ if $(\underline{y} - H\underline{u}_i)^T K^{-1} (\underline{y} - H\underline{u}_i) \leq (\underline{y} - H\underline{u}_j)^T K^{-1} (\underline{y} - H\underline{u}_j)$

$\forall\, j \neq i$



$K^{-1} = CC^T$

Lecture 6

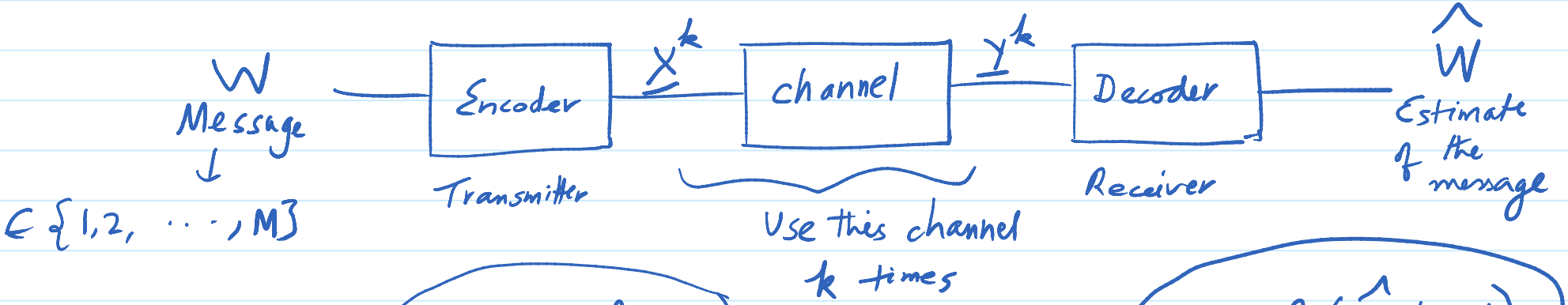
08 August 2019 11:01

For a given transmission scheme ($\underline{U}$ is either $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_M$ equally likely), we analyzed probability of error over the vector Gaussian channel ($\underline{Y} = \underline{U} + \underline{W}$).

Fixed Transmission rate $\log_2 M$ bits/ channel use, found best possible P_e

What is the best rate possible?

To answer this, consider the following communication model.



$$\text{Rate} = \frac{\log_2 M}{k}$$

(Average) per channel use

$$P_e = Pr(\hat{W} \neq W)$$

L6

08 August 2019 11:18

$$\underline{Y}_k = \underline{U}_k + \underline{W}_k \qquad \underline{W}_k \overset{i.i.d.}{\sim} N(\underline{0}, K)$$

As $k \to \infty$, what is the highest rate C (in bits/channel use) at which information can be sent with arbitrary low P_e?

Choose $P_e = \epsilon$ (any ϵ 10^{-1} or 10^{-3} or 10^{-10} ...)

$$C = \max_{p(\underline{x})} I(\underline{X}; \underline{Y})$$ (Shannon's result)

(Discrete memoryless channel)

where $I(\underline{X}; \underline{Y})$ is the mutual information between $\underline{X}$ & $\underline{Y}$.

For the vector Gaussian channel with transmit power constraint P

$$C = \max_{\substack{K_X \\ \mathrm{tr}(K_X) \le P}} \frac{1}{2} \log \frac{|K_X + K|}{|K|}$$ where $K_X = E[\underline{X}\,\underline{X}^T]$

$\underline{X} \sim N(\underline{0}, K_X)$

L6

08 August 2019 11:34

$I(\underline{X};\underline{Y}) = ?$

Let $\underline{X} \sim f_{\underline{X}}(\underline{x})$

Then, differential entropy $h(\underline{X}) \triangleq -E[\log f_{\underline{X}}(\underline{X})]$

$= -\int f_{\underline{X}}(\underline{x}) \log f_{\underline{X}}(\underline{x})\, d\underline{x}$

$X \sim N(\mu, \sigma^2)$: $h(X) = \frac{1}{2}\log_2(2\pi e\sigma^2)$ bits

$\underline{X} \sim N(\underline{\mu}, K)$: $h(\underline{X}) = \frac{1}{2}\log_2((2\pi e)^n |K|)$ bits

Conditional entropy $h(\underline{X}/\underline{Y}) = \int \underbrace{\left[-\int f_{\underline{X}/\underline{Y}=y}(\underline{x}) \log f_{\underline{X}/\underline{Y}=y}(\underline{x})\, d\underline{x}\right]}_{E_{\underline{Y}}[h(\underline{X}/\underline{Y}=y)]} f_{\underline{Y}}(y)\, dy$

L6

08 August 2019 11:43

Joint entropy of $\underline{X}$ & $\underline{Y}$

$$h(\underline{X},\underline{Y}) = -E_{\underline{X},\underline{Y}}\left[\log f_{\underline{X},\underline{Y}}(x,y)\right]$$

$f_{\underline{X}}(x) f_{\underline{Y}|\underline{X}=x}(y)$ (a)

$f_{\underline{Y}}(y) f_{\underline{X}|\underline{Y}=y}(x)$

$$h(\underline{X},\underline{Y}) = h(\underline{X}) + h(\underline{Y}|\underline{X})$$
$$= h(\underline{Y}) + h(\underline{X}|\underline{Y})$$

Defn $$I(\underline{X};\underline{Y}) \triangleq h(\underline{Y}) - h(\underline{Y}|\underline{X})$$
$$= h(\underline{X}) - h(\underline{X}|\underline{Y})$$

Can show $I(\underline{X};\underline{Y}) \geq 0$

$\Rightarrow h(\underline{Y}|\underline{X}) \leq h(\underline{Y})$

Thm Let $\underline{X}$ be a n-dim real random vector with zero mean and covariance K. $\left(E[\underline{X}] = \underline{0},\ E[\underline{X}\underline{X}^T] = K\right)$

Then $h(\underline{X}) \leq \frac{1}{2}\log_2\left((2\pi e)^n |K|\right)$ with equality if and only if $\underline{X} \sim N(\underline{0}, K)$

Lecture 7

09 August 2019 09:18

Capacity of the vector Gaussian channel under a sum power constraint

$$\underset{n\times 1}{\underline{Y}} = \underline{X} + \underline{W}$$

$\underline{W} \sim N(\underline{0}, K)$, independent of $\underline{X}$

From information theory

$$C = \max_{f_{\underline{X}}(\underline{x})} I(\underline{X};\underline{Y})$$

Avg. sum power constraint: $E\left[\underline{X}^T \underline{X}\right] \leq P$

$$\begin{aligned} I(\underline{X};\underline{Y}) &= h(\underline{Y}) - h(\underline{Y}/\underline{X}) \\ &= h(\underline{Y}) - h(\underline{X}+\underline{W}/\underline{X}) \\ &= h(\underline{Y}) - h(\underline{W}/\underline{X}) = h(\underline{Y}) - h(\underline{W}) \end{aligned}$$

L7

09 August 2019 10:06

$$= h(\underline{Y}) - \frac{1}{2}\log\left[(2\pi e)^n |K|\right]$$

Need to maximize $I(\underline{X};\underline{Y})$ over $f_{\underline{X}}(\underline{x})$ such that $E[\underline{X}^T\underline{X}] \le P$.

$$E[\underline{X}^T\underline{X}] = \mathrm{tr}\left(E[\underline{X}\underline{X}^T]\right)$$

$$\mathrm{tr}\underbrace{\left[E[\underline{X}\underline{X}^T]\right]}_{K_X} \le P$$

$$K_Y = K_X + K$$

$$h(\underline{Y}) \le \frac{1}{2}\log\left[(2\pi e)^n |K_Y|\right] \text{ for a given } K_Y$$

$$I(\underline{X};\underline{Y}) \le \frac{1}{2}\log\left((2\pi e)^n |K_Y|\right) - \frac{1}{2}\log\left((2\pi e)^n |K|\right) \text{ for a given } K_Y$$

equality if $\underline{Y} \sim N(\underline{0}, K_Y)$ → achieved if $\underline{X} \sim N(\underline{0}, K_X)$

L7

09 August 2019 10:13

$$C = \max_{\substack{K_x \\ \mathrm{tr}(K_x) \le P}} \frac{1}{2} \log \frac{|K_y|}{|K|} = \boxed{\max_{\substack{K_x \\ \mathrm{tr}(K_x) \le P}} \frac{1}{2} \log \frac{|K_x + K|}{|K|}}$$

↓

Can be solved

(We will solve the more general case $\underline{Y} = H\underline{X} + \underline{W}$)

Next: Emre Telatar, "Capacity of multi-antenna Gaussian channels," European Trans. on Telecommunications, 1999.

L7

09 August 2019 10:18

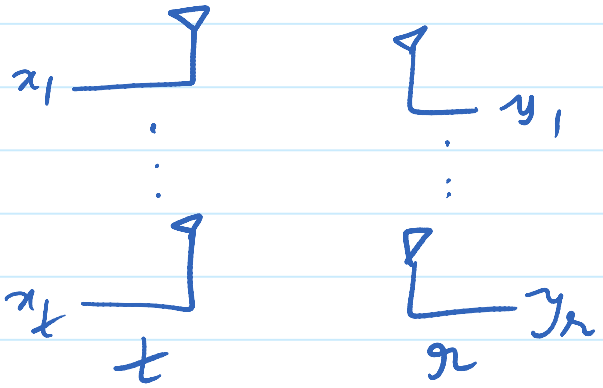
→ Capacity formulas and methods to evaluate

→ Focus : Potential gains of multi-antenna systems over single-antenna systems

→ $\underline{y} = H\underline{x} + \underline{n}$

(1) Deterministic & known

(2) H is random, known at the rx, statistics known at tx



slow fading / fast fading

$\underline{n} \sim CN(\underline{0}, I_r)$

Power constraint : $E[\underline{x}^H \underline{x}] \leq P$

i.e. $tr(E[\underline{x}\underline{x}^H]) \leq P$.

L7

09 August 2019 10:25

Preliminaries in section 2 :

(1) $\underline{X}_{n\times 1}$ ✓ $\quad \underline{\hat{X}}_{2n\times 1} = \begin{bmatrix} Re(\underline{X}) \\ Im(\underline{X}) \end{bmatrix}$ ✓

$CN(\underline{0}, Q)$ — real Gaussian

$\sim N(\underline{0}, \hat{Q})$

$$\hat{Q} = \frac{1}{2}\begin{bmatrix} Re(Q) & -Im(Q) \\ Im(Q) & Re(Q) \end{bmatrix}$$ ✓

(2) $\underline{X} \sim CN(\underline{0}, Q)$ has pdf $\dfrac{1}{|\pi \hat{Q}|^{1/2}} e^{-(\hat{\underline{x}}\hat{Q}^{-1}\hat{\underline{x}})} = \dfrac{1}{|\pi Q|} e^{-\underline{x}^{H} Q^{-1} \underline{x}}$

$h(\underline{X}) = \log|\pi e Q|$

(3) $\underline{X}$: zero mean & $E[\underline{X}\underline{X}^{H}] = Q$, then $h(\underline{X}) \le \log|\pi e Q|$

equality iff $\underline{X} \sim CN(\underline{0}, Q)$

L7

09 August 2019 10:36

(4) If $\underline{X} \sim CN(\underline{0}, K_x)$, $A\underline{X} \sim CN(\underline{0}, AK_xA^H)$

(5) If $\underline{X}$ & $\underline{Y}$ are independent CN, $\underline{X}+\underline{Y} \sim CN(\underline{0}, K_x+K_y)$

$$\underline{y} = H\underline{x} + \underline{n} \qquad K \qquad Q \triangleq E[\underline{x}\,\underline{x}^H]$$

$$C = \max_{Q,\ [tr(Q) \le P]} \log |\underbrace{HQH^H + I_n}_{K_y}|$$

$$\psi(Q, H) \triangleq \log |HQH^H + I_n|$$

$|I+AB| = |I+BA|$ using

$$\log |I_n + \underbrace{H}_{n\times t}\underbrace{Q}_{t\times t}\underbrace{H^H}_{t\times n}| = \log |I_t + \underbrace{QH^HH}_{t\times t}|$$

L7

09 August 2019 10:46

$$\log|I_t + Q H^H H| = \log|I_t + Q U^H \Lambda U|$$

$\left(\underbrace{H^H H}_{t\times t} \right.$ positive semi definite $x^H H^H H \underline{x} \geq 0$ for any $\underline{x}$ $\left.\right)$

$= \underbrace{U^H \Lambda U}_{t\times t}$ where U unitary

Λ diagonal ≥ 0.

$$= \log\left|I_t + \underbrace{Q U^H \Lambda^{1/2}}_{A} \underbrace{\Lambda^{1/2} U}_{B}\right|$$

$$= \log\left|I_t + \Lambda^{1/2} \underbrace{U Q U^H}_{\triangleq \tilde{Q}} \Lambda^{1/2}\right|$$

Observe $\tilde{Q} = U Q U^H$ is also positive semi-definite. In fact $\tilde{Q} \geq 0$ if and only if $Q \geq 0$. $\operatorname{tr}(\tilde{Q}) = \operatorname{tr}(U Q U^H) = \operatorname{tr}(Q U^H U) = \operatorname{tr}(Q)$

L 7

09 August 2019 10:52

$$\max_{\substack{Q \\ \mathrm{tr}(Q) \le P}} \psi(Q, H) = \max_{\substack{\tilde{Q} \ge 0 \\ \mathrm{tr}(\tilde{Q}) \le P}} \log \left| I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \right|$$

solve this

$$\tilde{Q} = U Q U^H \quad \longleftrightarrow \quad Q = U^H \tilde{Q} U$$

Lecture 8

13 August 2019 12:02

$$\max_{\substack{\tilde{Q} \\ \mathrm{tr}(\tilde{Q}) \le P}} \log \left| I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \right| \qquad (H^H H = U^H \Lambda U)$$

Fact: For any $A \succeq 0$, $\det(A) \le \prod_i A_{ii}$, with equality when A is diagonal.

Apply this fact to $I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2}$

$$\tilde{Q} \succeq 0 \Rightarrow \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \succeq 0 \Rightarrow I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \succeq 0$$

$$\left| I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \right| \le \prod_i \left(i^{\text{th}} \text{ diagonal entry of } I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \right)$$

L 8

13 August 2019 12:09

Say $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_t \end{bmatrix}$ $\lambda_1, \lambda_2, \cdots \lambda_t \geq 0$

$\Lambda^{1/2} \tilde{Q}$

i^{th} diagonal entry of $\Lambda^{1/2} \tilde{Q} \Lambda^{1/2} = \lambda_i \tilde{Q}_{ii}$

$$\left| I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2} \right| \leq \prod_{i=1}^{t} (1 + \lambda_i \tilde{Q}_{ii})$$

$=$

Equality achieved when $I_t + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2}$ is diagonal

$\Rightarrow \tilde{Q}$ is diagonal

$\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & 0 & \cdots & 0 \\ 0 & \tilde{Q}_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{Q}_{tt} \end{bmatrix}$

$$\max_{\substack{\tilde{Q} \\ tr(\tilde{Q}) \leq P}} \log \left(\prod_{i=1}^{t} (1 + \lambda_i \tilde{Q}_{ii}) \right)$$

L8

13 August 2019 12:14

Need to solve

$$\max_{\tilde{Q}_{11}, \tilde{Q}_{22}, \ldots, \tilde{Q}_{tt}} \sum_{i=1}^{t} \log\left(1 + \lambda_i \tilde{Q}_{ii}\right)$$

$$\sum_{i=1}^{t} \tilde{Q}_{ii} \leq P$$

- Standard convex optimization problem
- Solution satisfies KKT conditions

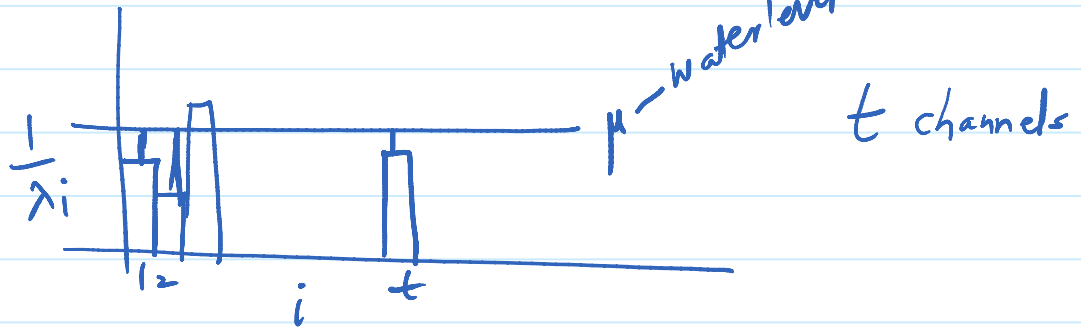
SOLUTION:

$$\tilde{Q}_{ii} = \left(\mu - \frac{1}{\lambda_i}\right)^+ \quad \text{for } i = 1, 2, \ldots, t$$

where μ is chosen such that $\sum_{i=1}^{t} \tilde{Q}_{ii} = P$

(Waterfilling soln)

More power allocated to better channels (λ larger is better)



L8

13 August 2019 12:21

$$\text{optimal} \sum_{i=1}^{t} \log\left(1+\lambda_i \tilde{Q}_{ii}\right)$$

$$= \sum_{i=1}^{t} \log\left(1+(\mu\lambda_i - 1)^+\right)$$

$$= \sum_{\substack{i: \\ \mu\lambda_i > 1}} \log(\mu\lambda_i) \quad \text{(capacity)} \quad \text{with} \quad \sum_i \left(\mu - \frac{1}{\lambda_i}\right)^+ = P \;\checkmark$$

$$\tilde{Q}_{ii} = \left(\mu - \frac{1}{\lambda_i}\right)^+$$

$$\lambda_i \tilde{Q}_{ii} = (\mu\lambda_i - 1)^+$$

optimal $Q = U^H \tilde{Q} U$ (optimal)

optimal input distribution for $\underline{X} \sim CN(\underline{0}, Q)$

$\tilde{\underline{X}} \sim CN(\underline{0}, \tilde{Q})$ — U^H — $\underline{X} \sim CN(\underline{0}, U^H \tilde{Q} U)$

$CN(\underline{0}, U^H \tilde{Q} U)$

$H^H H = U^H \Delta U$

L 8

13 August 2019 12:27

Alternative approach to find capacity (H known)

$$\underset{r\times 1}{\underline{y}} = \underset{r\times t}{H}\ \underset{t\times 1}{\underline{x}} + \underset{r\times 1}{\underline{n}}$$

$H^H(HH^H\underline{x} = \lambda\underline{x})$ ✓

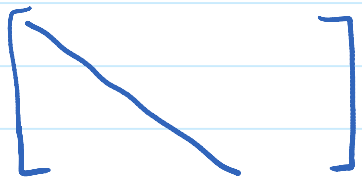
$H^H H\,\underline{x}' = \lambda\underline{x}'$ ✓

SVD of H

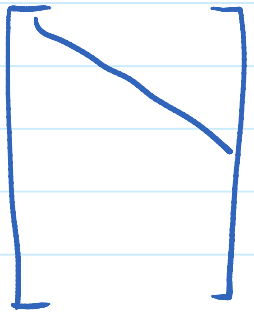
$$\underset{r\times t}{H} = \underset{r\times r}{U}\ \underset{r\times t}{D}\ \underset{t\times t}{V^H}$$

U, V are unitary

D



$r < t$ ✓



$r > t$ ✓

- Entries in D: sq. root of eigenvalues of HH^H $\gg 0$
- Columns of U: Eigenvectors of HH^H
- Columns of V: Eigenvectors of $H^H H$

L8

13 August 2019 12:38

$$\underset{r\times 1}{\underline{y}} = UDV^H \underline{x} + \underline{n} \quad \text{— } CN(\underline{0}, I)$$

$\downarrow$ information lossless

$$\underset{r\times 1}{U^H \underline{y}} = \underset{r\times t}{D}\,\underset{t\times t}{V^H}\,\underset{t\times 1}{\underline{x}} + \underbrace{U^H \underline{n}}_{CN(\underline{0},\, U^H I U = I)}$$

Let $\tilde{\underline{x}} = V^H \underline{x}$, $\tilde{\underline{y}} = U^H \underline{y}$ $\qquad \tilde{\underline{y}} = D\tilde{x} + \tilde{\underline{n}}$

$\tilde{\underline{n}} = U^H \underline{n}$

Since H is a $r\times t$ matrix, its rank is atmost $\min\{r, t\}$.
Only $\min\{r,t\}$ of the diagonal entries of D can be non-zero

$$\boxed{\tilde{y}_i = (\lambda_i^{1/2})\,\tilde{x}_i + \tilde{n}_i \quad \text{for } i = 1, 2, \ldots, \min\{r,t\} \checkmark}$$

$r<t$: r channels
$r>t$: t channels (remaining channels $\tilde{y}_i = \tilde{n}_i$) $\rightarrow$ independent of signal and noise in $\min\{r,t\}$ channels

Now, follow steps from Page 3 in the lecture

Lecture 9

16 August 2019 10:01

Multiantenna Gaussian channel with Rayleigh fading

- Model: $\underline{Y} = \mathbb{H}\underline{X} + \underline{N}$

$\mathbb{H}$: random matrix ~ each entry is $\mathcal{CN}(0,1)$ independent

Each time we use the channel we see an independent realization of $\mathbb{H}$ (fast fading)

This realization is not known at the tx
It is known at the receiver.

Tx knows the statistical model $\mathbb{H} \sim$ each entry $\mathcal{CN}(0,1)$

Capacity C (Avg transmission rate as # channel uses $\longrightarrow \infty$)

L 9

16 August 2019 10:10



$$I(\underline{x}; \underline{y}, \mathbb{H}) = I(\underline{x}; \mathbb{H}) + I(\underline{x}; \underline{y}/\mathbb{H})$$

($\mathbb{H}$ independent of the input)

$$= 0 + I(\underline{x}; \underline{y}/\mathbb{H})$$

$$= E_{\mathbb{H}}\left[I(\underline{x}; \underline{y}/\mathbb{H} = H)\right]$$

$$C = \max_{\substack{f_{\underline{x}}(\underline{x}) \\ tr(K_x) \le P}} E_{\mathbb{H}}\left[I(\underline{x}; \underline{y}/\mathbb{H} = H)\right]$$

L9

16 August 2019 10:15

Since $I(\underline{X};\underline{Y}|H=H) \le \log|I_r + HQH^H|$ ✓ for any $\underline{X}$ with $K_X = Q$,

$$E_H\left[I(\underline{X};\underline{Y}|H=H)\right] \le E_H\left[\log|I_r + HQH^H|\right]$$

(equality if $\underline{X} \sim CN(\underline{0}, Q)$)

$$C = \max_{\substack{Q \ge 0 \\ \mathrm{tr}(Q) \le P}} \underbrace{E_H\left[\log|I_r + HQH^H|\right]}_{\triangleq \psi(Q)}$$

So far we used that $\underline{N} \sim CN(\underline{0}, I_r)$ and H is independent of $\underline{X}$.

We have not yet used that each entry is $CN(0,1)$ i.i.d.

Guess: Optimal Q is $\frac{P}{t} I_t$

Proof:

$Q \ge 0 \Rightarrow Q = UDU^H$ where U is unitary, D is diagonal, ≥ 0.

L9

16 August 2019 10:25

$$\psi(Q) = \mathbb{E}_{\underline{H}}\left(\log\left|I_r + \underline{H}\,\underline{U}\,D\,\underline{U}^H\underline{H}^H\right|\right)$$

[**Lemma**: $H_{r\times t}$ i.i.d. $CN(0,1)$

Then, for any unitary $U_{r\times r}$, $V_{t\times t}$,

UHV^H has the same distribution as H]

* Since the distribution of HU is the same as H, it is enough to optimize over D.

$$E_H\left[\log\left|I_r + HUDU^HH^H\right|\right] = \underline{E_H\left[\log\left|I_r + HDH^H\right|\right]}$$

$\Rightarrow$ We can restrict ourselves to diagonal $Q \geq 0$, $\mathrm{tr}(Q) \leq P$.

L9

16 August 2019 10:30

* Consider a permutation matrix Π (unitary)

$H\Pi$ has the same distribution as H

$$\psi(Q) = E_H\left[\log\left|I_n + HQH^\dagger\right|\right] \quad \text{(diag)}$$

$$= E_H\left[\log\left|I_n + H\underbrace{\Pi Q \Pi^\dagger}_{\triangleq Q^\Pi} H^\dagger\right|\right]$$

$$\psi(Q^\Pi) = \psi(Q)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I_3$$

$$E_H[f(Q)] \overset{f \text{ concave}}{\leq} f(E_H[Q])$$

→ For any H, $Q \to I_n + HQH^\dagger$ is affine & preserved positive semi definiteness

$(Q_1 \to I_n + HQ_1H^\dagger$, $Q_2 \to I_n + HQ_2H^\dagger$, $\alpha Q_1 + \beta Q_2 \to ?)$

L9

16 August 2019 10:40

$\log\det(\,)$ is concave on the set of positive definite matrices

$\Rightarrow \quad Q \longrightarrow \log\left|I_r + H Q H^H\right|$ is concave

$\rightarrow E_H\left[\log\left|I_r + H Q H^H\right|\right]$

Evaluate for Q^π and average

$$\psi(Q) \leq E_H\left[\log\left|I_r + H\left(\frac{1}{t}\sum Q^\pi\right)H^H\right|\right]$$

$\psi(Q)$: equality if $Q = \alpha I_t$

$\left(\frac{1}{t}\sum Q^\pi\right)$: Diagonal with equal entries on the diagonal

$\rightarrow \mathrm{tr}(Q) \leq P, \Rightarrow Q = \frac{P}{t} I_t$

L 9

16 August 2019 10:50

$H \sim \overset{\text{each entry}}{\mathcal{CN}(0,1)}$ i.i.d. $\longrightarrow$ $C = E_H\left[\log\left|I_n + \frac{\rho}{t} HH^H\right|\right]$

Lecture 10

19 August 2019 08:01

Remarks

1) $H = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1t} \\ \vdots & & & \vdots \\ H_{r1} & \cdots & \cdots & H_{rt} \end{bmatrix}_{r \times t}$

$$E_H\left[\log\left|I_r + \frac{\rho}{t} HH^H\right|\right]$$

$$\frac{1}{t} HH^H = \frac{1}{t} \begin{bmatrix} \sum_{i=1}^{t} |H_{1i}|^2 & & & \\ & \sum_{i=1}^{t} |H_{2i}|^2 & & \\ & & \ddots & \\ & & & \sum_{i=1}^{t} |H_{ri}|^2 \end{bmatrix}$$

$(i,j)^{th}$ element: $\sum_{k=1}^{t} H_{ik} H_{kj}^*$

For a fixed r, as $t \to \infty$, $\frac{1}{t} \sum_{k=1}^{t} H_{ik} H_{kj}^* \xrightarrow[a.s.]{r \times r} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ Law of large numbers

L10

19 August 2019 08:08

$$C \longrightarrow \underline{n \log(1+P)}$$

$$E_H\left[\log\left|I_n + \underbrace{\frac{P}{t} HH^H}\right|\right] \downarrow P I_n$$

(2) $\underline{n=1}$ $\quad H = [H_{11}\ H_{12}\ \cdots\ H_{1t}]$

$$HH^H = \sum_{i=1}^{t} |H_{1i}|^2$$

$$\boxed{C = E\left[\log\left(1 + \frac{P}{t}\sum_{i=1}^{t}|H_{1i}|^2\right)\right]}$$

$\underline{t=1}$ $\quad H = \begin{bmatrix} H_{11} \\ H_{21} \\ \vdots \\ H_{n1} \end{bmatrix}$ $\quad HH^H = \left[\ \text{rank 1 matrix}\ \right]$

$$C = E\left[\log\left|I_n + \frac{P}{t} HH^H\right|\right] = E\left[\log\left(1 + H^H H \frac{P}{t}\right)\right]$$

$$= E\left[\log\left(1 + P\sum_{i=1}^{n}|H_{i1}|^2\right)\right] \quad (t=1)$$

$|I+AB| = |I+BA|$

L 10

19 August 2019 08:18

10×1

$$E\left[\log\left[1+\frac{P}{10}\sum_{i=1}^{10}|H_{i,1}|^2\right]\right] \qquad E\left[\log\left[1+P\sum_{i=1}^{10}|H_{i,1}|^2\right]\right]$$

1×10

Avg. received power increases with n

What is achieved by 10×1 at P is achieved by 1×10 system at $\frac{P}{10}$ ($\frac{1\times10}{10}$ dB better)

3) Evaluation of C.

$$E\left[\log\det\left(I_r+\frac{P}{t}HH^H\right)\right] \quad (\text{or}) \quad E\left[\log\det\left(I_t+\frac{P}{t}H^HH\right)\right]$$

(Use this when $r<t$) (Use this when $t<r$)

Define $W = \begin{cases} HH^H & r<t \quad (r\times r) \\ H^HH & t\le r \quad (t\times t) \end{cases}$, $n=\max(r,t)$, $m=\min(r,t)$

$W_{m\times m}$ non-negative definite random matrix: follows the Wishart distribution with parameters m, n.

L 10

19 August 2019 08:30

Eigenvalues of W : $\lambda_1, \lambda_2, \ldots, \lambda_m$ (≥ 0) .

$$E_H\left[\log\left|I_n + \frac{P}{t}\underbrace{HH^H}_{W}\right|\right]$$

$$W = U \Lambda U^H$$

(unitary)

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$$\log\left|I_m + \frac{P}{t} U\Lambda U^H\right|$$

$$= \log\left|U I_m U^H + \frac{P}{t} U\Lambda U^H\right|$$

$$= \log\left|U\left(I_m + \frac{P}{t}\Lambda\right)U^H\right|$$

$$= \log\left|I_m + \frac{P}{t}\Lambda\right| = \underbrace{\sum_{i=1}^{m} \log\left(1 + \frac{P}{t}\lambda_i\right)}$$

L10

19 August 2019 08:35



$$C = E_{\underline{\lambda}}\left[\sum_{i=1}^{m} \log\left(1+\frac{\rho}{t}\lambda_i\right)\right]$$

$$HH^H \to U \Lambda U^H$$

$$H^H H \to$$

Joint density of unordered eigenvalues

$$f_{\underline{\lambda}}(\lambda_1, \dots, \lambda_m) = \frac{1}{m!\, K_{m,n}} e^{-\sum_i \lambda_i} \prod_i \lambda_i^{n-m} \prod_{i<j} (\lambda_i - \lambda_j)^2$$

$$U^H H H^H U = \Lambda$$

for all $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^{m+}$

$\times\, m!$

Joint density of ordered eigenvalues

$$f_{\underline{\lambda},\text{ordered}}(\lambda_1, \dots, \lambda_m) = \frac{1}{K_{m,n}} e^{-\sum_i \lambda_i} \prod_i \lambda_i^{n-m} \prod_{i<j} (\lambda_i - \lambda_j)^2$$

for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$

L 10

19 August 2019 08:48

$$E_{\underline{\lambda}}\left[\sum_{i=1}^{m} \log\left(1+\frac{p}{t}\lambda_i\right)\right] = \sum_{i=1}^{m} E_{\underline{\lambda}}\left[\log\left(1+\frac{p}{t}\lambda_i\right)\right]$$

m-dim integral

$$= \sum_{i=1}^{m} E_{\lambda_i}\left[\log\left(1+\frac{p}{t}\lambda_i\right)\right]$$

$$= m\, E_{\lambda_i}\left[\log\left(1+\frac{p}{t}\lambda_i\right)\right]$$

1-dim integral

Lecture 11

20 August 2019 12:03

Density of any one of the unordered eigenvalues

$$f_{\lambda_1}(\lambda_1) = \frac{1}{m} \sum_{i=1}^{m} \varphi_i(\lambda_1)^2 \lambda_1^{n-m} e^{-\lambda_1}$$

where $\varphi_{k+1}(\lambda) = \left[\frac{k!}{(k+n-m)!}\right]^{1/2} L_k^{n-m}(\lambda)$ $k = 0, 1, \ldots, m-1$

$L_k^{n-m}(x) = \frac{1}{k!} e^x x^{m-n} \frac{d^k}{dx^k}(e^{-x} x^{n-m+k})$ is the Laguerre polynomial of order k.

$$C = \int_0^\infty \log\left(1 + \frac{\rho}{t}\lambda_1\right) \overline{f_{\lambda_1}(\lambda_1)} d\lambda_1$$

L 11

20 August 2019 12:07

Special cases

1) $t = 1$ $$C = \frac{1}{\Gamma(r)} \int_0^\infty \log(1+P\lambda)\, \lambda^{r-1} e^{-\lambda} d\lambda$$

As $r \to \infty$, $C \to \log(1+Pr)$ (Fig 2 in paper)

2) $r = 1$ $$C = \frac{1}{\Gamma(t)} \int_0^\infty \log\left(1+\frac{P}{t}\lambda\right) \lambda^{t-1} e^{-\lambda} d\lambda$$

As $t \to \infty$, $C \to \log(1+P)$ (fig 3 in paper)

3) $r = t$ $$C = \int_0^\infty \log\left(1+\frac{P\lambda}{r}\right) \sum_{k=0}^{r-1} L_k(\lambda)^2 e^{-\lambda} d\lambda$$

$$\sim r\left[\int_0^4 \log(1+P\nu)\, \frac{1}{\pi}\sqrt{\frac{1}{\nu}-\frac{1}{4}}\, d\nu\right]$$ linear in r (Fig. 4)

L 11

20 August 2019 12:15

4) $C(r, t, P)$

- r: # rx ant
- t: #tx ant
- P: Tx Power

$$C(a, b, P_b) = C(b, a, P_a)$$

eg $C(1, t, P_t) = C(t, 1, P)$

5) This capacity C is called <u>ergodic capacity</u>

We get the same C even if H is ergodic

We assumed H to be i.i.d. across time

6) See Lu Wei's paper in ISIT 2019 for a simplified expression for C.

L 11

20 August 2019 12:24

Non-ergodic channels

→ H is chosen randomly at the beginning and held fixed for all uses of the channel (models slow fading)

$H \overset{\text{each element}}{\text{i.i.d}} \mathcal{CN}(0,1)$

- statistics known at tx
- realization not known at tx
- realization known at rx.

→ **Capacity = 0.**

For any $R > 0$, there is a non-zero probability that the realized H cannot support R. This is called an outage.

Let $\psi(Q, H) \triangleq \log\det\left(I_n + HQH^H\right)$

$$\inf_{Q:\, Q\geq 0,\ \mathrm{tr}(Q)\leq P} \Pr\left(\psi(Q,H) < R\right) \triangleq P_{out}(R,P)$$

L 11

20 August 2019 12:32

* Fix $P_{out}(R, P) \leq \epsilon$.

What is the best R? **ϵ-outage capacity**

eg $t = 1$

$$\psi(Q, H) = \log\det(I_r + H Q H^H)$$

$$\left(\text{opt } \underset{(1\times 1)}{Q} = P\right) = \log(1 + P\, H^H H)$$

$H^H H \sim \chi^2$ with $2r$ deg of freedom

χ^2 with $2r$ deg of freedom

$$P_{out}(R, P) = P\left(\log(1 + P\, H^H H) < R\right)$$

$$= P\left(H^H H < \frac{e^R - 1}{P}\right)$$

$$= \frac{\gamma\left(r, \frac{e^R - 1}{P}\right)}{\Gamma(r)} \leq \epsilon$$

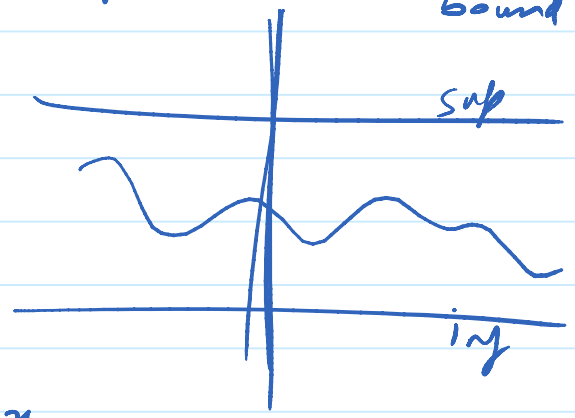
$x_1\ x_2\ x_3 \ldots x_n \ldots$

Largest lower bound

$\left(2 - \frac{1}{n}\right)$

inf: 1

Supremum : least upper bound



$$\gamma(a, x) = \int_0^x u^{a-1} e^{-u}\, du.$$

L 11

20 August 2019 12:39

→ HU has the same distribution as H for unitary U.

⇒ Q can be chosen to be diagonal without affecting $P_{out}(R,P)$

$$P_{out}(R,P) = \inf_{\substack{Q: Q \geq 0 \\ Q \text{ diag} \\ \sum_i Q_{ii} \leq P}} \Pr\left(\underbrace{\psi(Q,H)}_{\uparrow} < R\right)$$

→ Conjecture : $Q = \frac{P}{k}\,\text{diag}\{\underbrace{1,\dots,1}_{k \text{ ones}},\underbrace{0,\dots,0}_{t-k \text{ zeros}}\}$

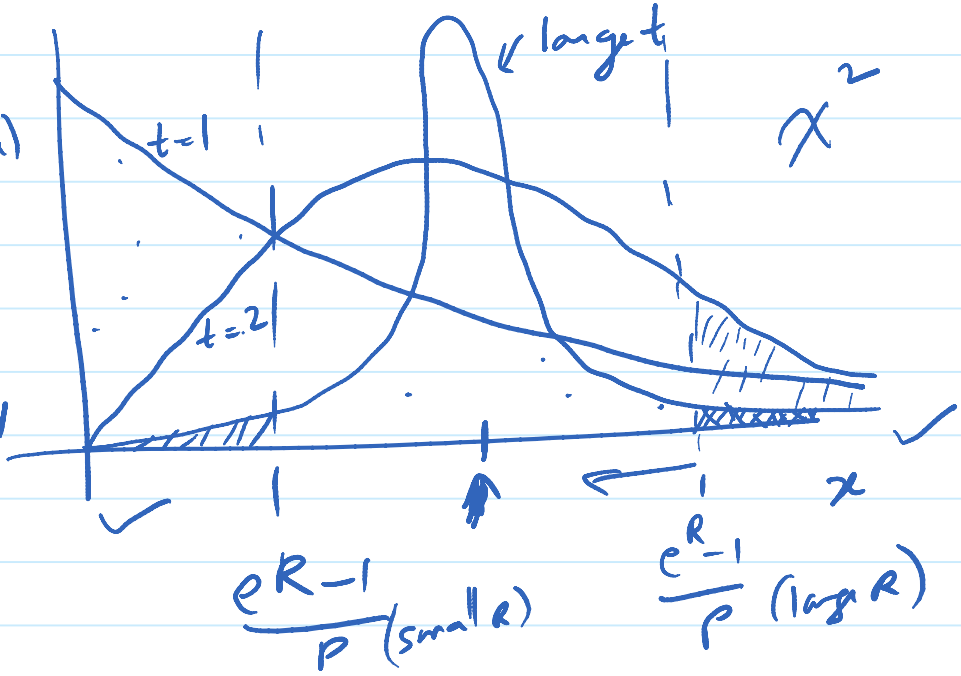
for some $1 \leq k \leq t$

Lecture 12

22 August 2019 11:05

$r = 1$ t varies $H : 1 \times t$

$$\Pr\left(\log\left(1 + \frac{P}{t} \underbrace{HH^H}_{\chi^2 \text{ dist } 2t \text{ deg of freedom}}\right) < R\right)$$



For small R
using more antennas good

For large R
using fewer antennas is good

L12

22 August 2019 11:19

Some other related papers at the same time:

1) Foschini & Gans (1998)

2) Foschini (1996)

1998

* How does outage capacity change with n, t? (linear in $n = t$)

  - since we do not know exact outage capacity (optimal Q)

    this is studied for $Q = \frac{P}{t} I_t$

    Outage capacity $\geq$ rate using $Q = \frac{P}{t} I_t$

  - $P\left(\log \left| I_n + \frac{P}{t} HH^* \right| < R \right)$ is studied for diff n, t.

    $\log \left| I_n + \frac{P}{t} HH^* \right|$ → lower bounded by eqn (12) → lin increase with $n_r = n_t$

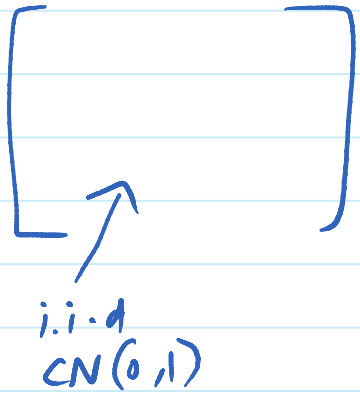
* New codes needed to achieve this

L 12

22 August 2019 11:11

For MISO, conjecture has been proved in Abbe et al (2013) Trans. IT

For MIMO, it is still a conjecture



Known H

* SVD + waterfilling

Random H (not known at tx) [CN(0,1) i.i.d.] ✓

↓ ergodic

$$C = E\left[\log\left|I_r + \frac{P}{t} HH^H\right|\right]$$

↓

linear in $\min(r,t)$

→ non-ergodic

"outage capacity"

SIMO/MISO : optimal Q known

MIMO : optimal Q not known

L 12

22 August 2019 11:35

Foschini (1996)

* Multiplexing architectures

Layered space-time architecture

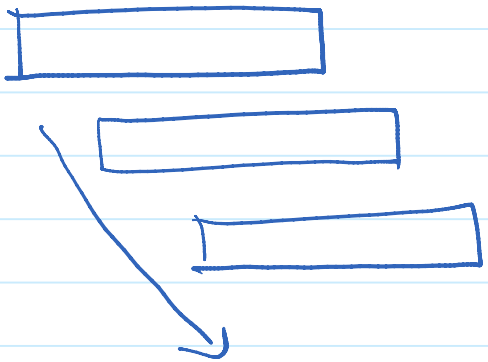
(Diagonal layering : D-BLAST)

+

SIC + ZF

↙ successive int. canc

↘ zero forcing



We will discuss Vertical layering (V-BLAST)

1) Wolniansky & others 1998

Exp. result : (Fig 24.3) $^{tx}_{8} \times ^{rx}_{12}$, 16-QAM

2) Golden & others 1999

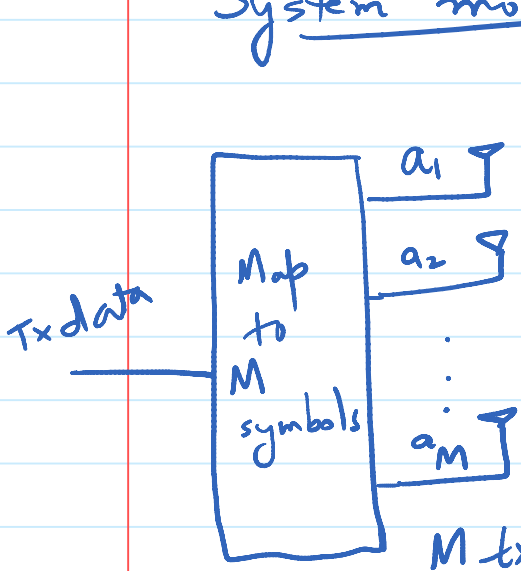
Lecture 13

23 August 2019 09:49

V-BLAST (Vertical - Bell Labs Layered Space Time) system.

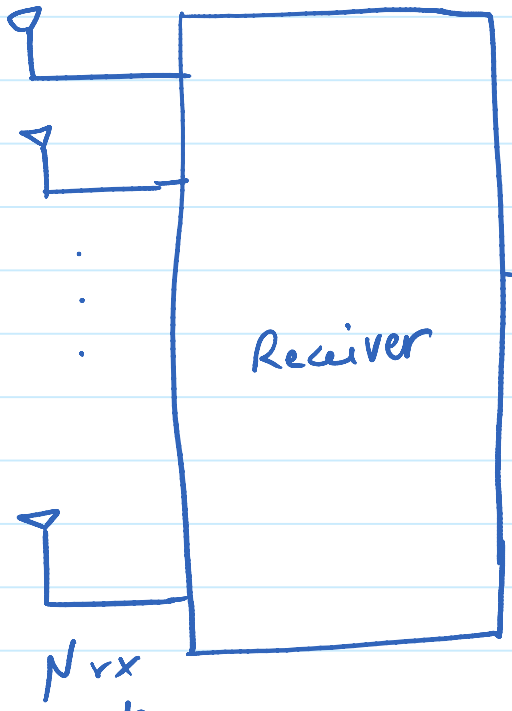
(Wolniansky & others 98, Golden & others 99)

System model



$a_i \in$ QAM constellation $\forall i$

(eg: 16-QAM
4M bits/transmission)



$(N \geq M)$

Tx power from each antenna $\propto \frac{1}{M}$

$\Rightarrow$ Total tx power constant independent of M.

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{pmatrix}$$

$\hookrightarrow$ received vector $\underline{r}_1 = \underset{N \times M}{H}\,\underline{a} + \underline{v}$ ($\underline{v}$: i.i.d. noise)

L13

23 August 2019 10:05

→ Transmissions in bursts of L symbols, channel is quasi-static in this duration.

→ H known at the receiver. (Training 20% overhead)

$$\underline{y}_1 = H\underline{a} + \underline{v}.$$

V-BLAST detection

→ choose an order in which the symbols in $\underline{a}$ are detected

Let the order be given by $S = \{k_1, k_2, \ldots, k_M\}$

↳ Permutation of $\{1, 2, \ldots, M\}$

(Will be chosen later based on H)

L13

23 August 2019 10:11

Step 1

Compute $y_{k_1} = \underline{w}_{k_1}^T \underline{r}_1$ (Linear receiver)

(zero-forcing $\underline{w}$

$\underline{w}_{k_1}$ chosen such that

$$\underline{w}_{k_1}^T (H)_{k_j} = \begin{cases} 0 & j \neq 1 \\ 1 & j = 1 \end{cases}$$

$$\underbrace{\begin{bmatrix} | & & | \\ & & \\ | & & | \end{bmatrix}}_{H} \underline{a}$$

$(H)_{k_j}$: k_j^{th} column of H

Step 2 $\hat{a}_{k_1} = Q(y_{k_1})$ (QAM decision rule)

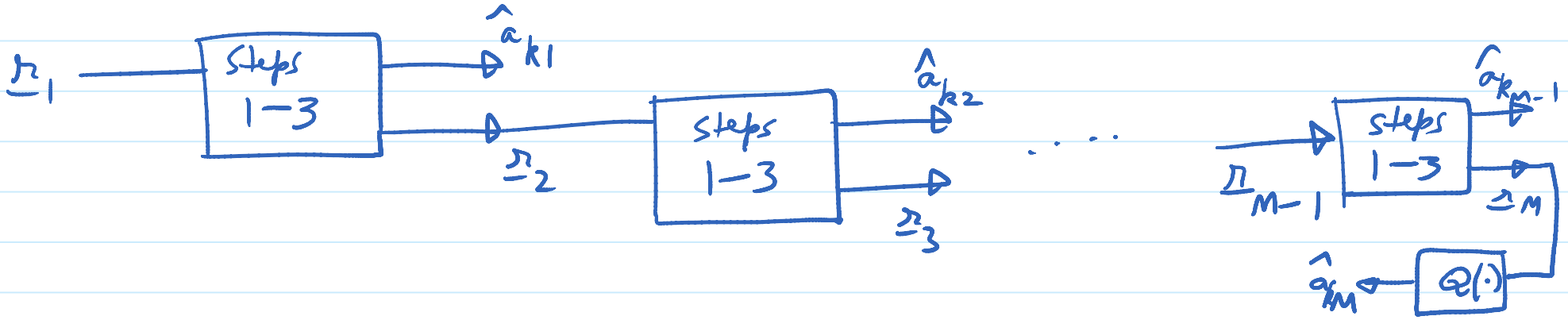
Step 3 $\underline{r}_2 = \underline{r}_1 - (H)_{k_1} \hat{a}_{k_1}$

Successive interference cancellation – Zero forcing (SIC-ZF)

L13

23 August 2019 10:16

(If $\underline{W}$ is chosen to be the linear MMSE detector, then SIC-LMMSE)



* How do we fix the order of detection? $\{k_1, k_2, \ldots, k_M\}$

$$y_{k_1} = \begin{bmatrix} & \underline{w}_{k_1}^T & \end{bmatrix} \begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix} \begin{bmatrix} & \\ & \underline{a} \\ & \end{bmatrix} + \underline{v} = a_{k_1} + \underline{w}_{k_1}^T \underline{v}$$

L13

23 August 2019 10:22

Variance of noise in $y_{k_1} = \|\underline{w}_{k_1}\|^2 \sigma^2$

SNR for decoding $a_{k_1} = \dfrac{E\left[|a_{k_1}|^2\right]}{\sigma^2 \|\underline{w}_{k_1}\|^2}$.

Choose k_1 to maximize this SNR, i.e., k_1 with the least $\|\underline{w}_{k_1}\|^2$

$\underline{w}_{k_1}$ is the k_1^{th} row of $H^\dagger$ (pseudoinverse of H)

$$\underset{M\times N}{H^\dagger}\,\underset{N\times M}{H} = I_M \qquad (N \geq M)$$

Look at $H^\dagger$: choose the row k_1 that minimizes $\|\underline{w}_{k_1}\|^2$.

How to choose k_2? Assume $\hat{a}_{k_1} = a_{k_1}$

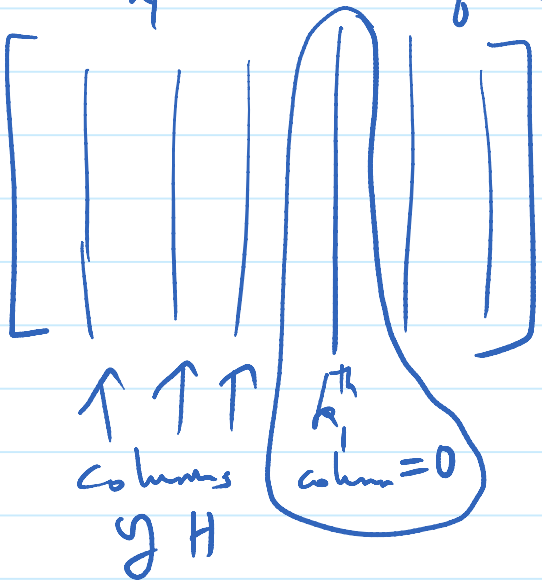
$$\underline{r}_2 = \begin{array}{c} k_1^{th} \\ \left[\begin{array}{c|c} & \\ & \end{array}\right] \end{array} \begin{bmatrix} \cdot \\ \bullet \\ \cdot \end{bmatrix} \rightarrow k_1^{th} + \underline{v}$$

L13

23 August 2019 10:28

Equivalent to setting the k_1^{th} column of H to zero.

$H_{k_1} =$



choose k_2 to be the index of the row with min. norm in $H^{+}_{k_1}$

choose k_3 " " " in $H^{+}_{k_2}$

k_j " " " in $H^{+}_{k_{j-1}}$

$H_{k_{j-1}}$: Start with H, set columns $k_1 \cdots k_{j-1}$ to zero.

L13

23 August 2019 10:36

Algorithm

Initialization:

$$G_1 = H^+$$
$$i = 1$$

Recursion (for $i = 1$ to M)

$$k_i = \arg \min_{j \notin \{k_1, \dots, k_{i-1}\}} \| (G_i)_j \|^2$$

($(G_i)_j$: j^{th} row of G_i)

$$\underline{w}_{k_i}^T = (G_i)_{k_i} \quad \leftarrow k_i^{th} \text{ row of } G_i$$

$$y_{ki} = \underline{w}_{k_i}^T \underline{r}_i$$
$$\hat{a}_{k_i} = Q(y_{k_i})$$
$$\underline{r}_{i+1} = \underline{r}_i - \hat{a}_{k_i} (H)_{k_i}$$
$$G_{i+1} = H_{\overline{k_i}}^+$$
$$i = i+1$$

L13

23 August 2019 10:46

optimal ML detector

choose $\hat{\underline{a}}$ that $\min_{\underline{a}} \|\underline{r}_1 - H\underline{a}\|^2$

↓

possible $\underline{a}$ vectors (eg 16QAM) $\boxed{(16)^M}$ ← exp in M

Suboptimal SIC-ZF detector

Each $\hat{a}_{k_i}$ separately decoded. (lower complexity)

{ Linear complexity in M
or
M^2

Lecture 14

27 August 2019 12:02

Space-time codes for high data rate wireless communication: Performance criterion and code construction

V. Tarokh, N. Seshadri, A. R. Calderbank (1998).



In SISO case, we use codewords over time that span many channel uses. Now, we have a space dimension as well. So, we have design code matrices.

L 14

27 August 2019 12:11

c_t^i : Transmit signal from the ith tx antenna during the tth slot (t^{th} channel use)

$$\begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_\ell^1 \\ c_1^2 & c_2^2 & \cdots & c_\ell^2 \\ \vdots & & & \\ c_1^n & c_2^n & \cdots & c_\ell^n \end{bmatrix} \triangleq C$$

$n \times \ell$ code matrix

(In V-BLAST, each c_t^i is an independently chosen symbol)

L 11

27 August 2019 12:14

received vector at j^{th} rx antenna over ℓ slots

$$\left[d_1^j \; d_2^j \cdots d_\ell^j\right] = \underbrace{\left(\alpha_{1,j} \; \alpha_{2,j} \cdots \alpha_{n,j}\right)}_{\triangleq \Omega_j} \underbrace{\begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_\ell^1 \\ c_1^2 & c_2^2 & \cdots & c_\ell^2 \\ \vdots & \vdots & & \\ c_1^n & c_2^n & \cdots & c_\ell^n \end{bmatrix}}_{\text{channel coefficients}} \sqrt{E_s}$$

(Assume channel is constant during the ℓ slots)
'quasi-static fading'

$$+ \left(n_1^j \; n_2^j \cdots n_\ell^j\right)$$

Noise

$$d_t^j = \sum_{i=1}^{n} \alpha_{i,j} \, c_t^i \sqrt{E_s} + n_t^j$$

$n_t^j \sim \mathcal{CN}(0, N_0)$

L14

27 August 2019 12:20

Analyze pairwise probability of error for 2 code matrices C and E

$$C = \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_\ell^1 \\ \vdots & & & \\ c_1^n & c_2^n & \cdots & c_\ell^n \end{bmatrix} \qquad E = \begin{bmatrix} e_1^1 & e_2^1 & \cdots & e_\ell^1 \\ \vdots & & & \\ e_1^n & e_2^n & \cdots & e_\ell^n \end{bmatrix}$$

ML decoding at the receiver

$$\begin{pmatrix} d_1^1 & d_2^1 & \cdots & d_\ell^1 \\ d_1^2 & d_2^2 & \cdots & d_\ell^2 \\ \vdots & & & \\ d_1^m & d_2^m & \cdots & d_\ell^m \end{pmatrix} \longrightarrow$$

observations

→ decide which code matrix sent

L14

27 August 2019 12:26

$$(d_1^j \cdots d_\ell^j) = \underbrace{\Omega_j C} + (n_1^j \cdots n_\ell^j)$$

$$\underbrace{(d_1^1 \cdots d_\ell^1 \; d_1^2 \cdots d_\ell^2 \cdots d_1^m \cdots d_\ell^m)}_{\underline{c}} = (\Omega_1 C \;\; \Omega_2 C \;\; \cdots \;\; \Omega_m C)\sqrt{E_s} + (n_1^1 \;\; n_2^1 \cdots n_\ell^1 \;\; n_1^2 \cdots n_\ell^2 \;\; \cdots \;\; n_1^m \cdots n_\ell^m)$$

Equivalent vector model

Pairwise error probability

$$P(C \to E / \{\alpha_{i,j}\}) = Q\left(\frac{d(\underline{c}, \underline{e})\sqrt{E_s}}{2\sqrt{N_0}}\right)$$

L14

27 August 2019 12:31

$$\leq \exp\left(-\frac{E_s\, d^2(\underline{c},\underline{e})}{4N_0}\right)$$

$$d^2(\underline{c},\underline{e}) = \left\| (\Omega_1 c\ \Omega_2 c \cdots \Omega_m c) - (\Omega_1 E\ \Omega_2 E \cdots \Omega_m E) \right\|^2$$

$$= \sum_{j=1}^{m} \left\| \Omega_j (c - E) \right\|^2$$

$B(\underline{c},\underline{e}) \triangleq C - E$

codeword difference matrix

$$\begin{bmatrix} c_1^1 - e_1^1 & \cdots & c_\ell^1 - e_\ell^1 \\ c_1^n - e_1^n & & c_\ell^n - e_\ell^n \end{bmatrix}$$

L14

27 August 2019 12:34

$A(\underline{c},\underline{e})$ $n \times n$
$B(\underline{c},\underline{e})$ $n \times \ell$

$$d^2(\underline{c},\underline{e}) = \sum_{j=1}^{m} \left(\Omega_j \underbrace{B(\underline{c},\underline{e})\, B(\underline{c},\underline{e})^*}_{\triangleq A(\underline{c},\underline{e})} \Omega_j^* \right)$$

$A(\underline{c},\underline{e}) \geq 0$

$$= \sum_{j=1}^{m} \Omega_j \, A(\underline{c},\underline{e}) \, \Omega_j^*$$

$$P(C \to E \,/\, \{\alpha_{i,j}\}) \leq \exp\left(-\frac{E_s}{4N_0} \sum_{j=1}^{m} \Omega_j \underbrace{A(\underline{c},\underline{e})}_{V^* D V} \Omega_j^* \right)$$

where V unitary
D diagonal
$D \geq 0$.

$$= \exp\left(-\frac{E_s}{4N_0} \sum_{j=1}^{m} \Omega_j V^* D V \Omega_j^* \right)$$

L14

27 August 2019 12:38

$$\underbrace{\Omega_j}\,\underbrace{V^*} \triangleq \underline{\beta}_j \xrightarrow{1\times n} = (\beta_{1,j}\ \beta_{2,j}\ \cdots\ \beta_{n,j})$$

$$P(C \to E / \{\alpha_{i,j}\}) \le \exp\left(-\frac{E_s}{4N_0}\sum_{j=1}^{m} \underline{\beta}_j D \underline{\beta}_j^*\right)$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= \prod_{j=1}^{m} \exp\left(-\frac{E_s}{4N_0}\beta_j D \underline{\beta}_j^*\right)$$

$$\boxed{= \prod_{j=1}^{m} \exp\left(-\frac{E_s}{4N_0}\sum_{i=1}^{n} \underbrace{|\beta_{i,j}|^2}\lambda_i\right)}$$

✓

L14

27 August 2019 12:42

We have derived the pairwise error probability bound for a given channel realization.

Now, for a given fading model, find the average pairwise error probability.

$$P(\underline{c} \to \underline{e} \mid \{\alpha_{i,j}\}) \xrightarrow[\{\alpha_{i,j}\}]{\text{avg over}} P(\underline{c} \to \underline{e})$$

(1) Rayleigh fading $\alpha_{i,j} \sim CN(0,1)$ i.i.d.

(2) Rician fading $\alpha_{i,j}$ independent $\sim CN(E[\alpha_{i,j}], 1)$

L 14

27 August 2019 12:46

(1) $\beta_j \overset{i.i.d.}{\sim} CN(0,1)$ elements $\quad P(\underline{c} \to \underline{e}) \le \left[\dfrac{1}{\prod_{i=1}^{n} \left(1 + \lambda_i \frac{E_s}{4N_0}\right)} \right]^m$ ✓

(2) $\beta_j \sim CN(\quad , 1)$ elements

independent elements

$$\beta_{i,j} \sim CN\Big(\underbrace{\big[\underbrace{E[\alpha_{1,i}] E[\alpha_{2,i}] \cdots E[\alpha_{n,i}]}_{\triangleq K^j} \big] V^*_{j,i}}_{K_{i,j}}, 1 \Big)$$

$$P(\underline{c} \to \underline{e}) \le \prod_{j=1}^{m} \left[\prod_{i=1}^{n} \frac{1}{\left(1 + \lambda_i \frac{E_s}{4N_0}\right)} \exp\left(- \frac{K_{i,j} \frac{E_s}{4N_0} \lambda_i}{1 + \frac{E_s}{4N_0} \lambda_i} \right) \right]$$

$\to 1$ as $SNR \to \infty$

Lecture 15

29 August 2019 11:06

Diversity advantage

→ Exponent of SNR in the denominator of $P_e(\underline{c} \to \underline{e})$

Rayleigh fading: $P_e(\underline{c} \to \underline{e}) \leq \left[\dfrac{1}{\prod_{i=1}^{n} \left(1 + \lambda_i \frac{E_s}{4N_0}\right)} \right]^m$

$\dfrac{E_s}{N_0} \triangleq$ SNR

$$\left(\begin{array}{c}\text{SNR high} \\ \lambda_i\text{'s} > 0\end{array}\right) \approx \frac{1}{\left(\prod_{i=1}^{n} \lambda_i\right)^m \left(\frac{E_s}{4N_0}\right)^{nm}}$$

$$\left[A(\underline{c}, \underline{e}) \text{ has rank } r\right] \Leftarrow \left(\begin{array}{c}\text{SNR high} \\ \text{General case} \\ r \text{ +ve } \lambda_i\text{'s}\end{array}\right) \approx \frac{1}{\left(\prod_{i=1}^{r} \lambda_i\right)^m \left(\frac{E_s}{4N_0}\right)^{rm}}$$

L15

29 August 2019 11:13

$A(\underline{c},\underline{e}) = B(\underline{c},\underline{e})$

$B(\underline{c},\underline{e})^*$

$A \to n \times n$ ✓

$B \to n \times \ell$

Div. gain $= rm$

$r \to$ rank of $A(\underline{c},\underline{e})$

$m \to$ # rx antennas

In order to maximize div. gain, we have to maximize the rank of $A(\underline{c},\underline{e})$

Overall prob. error will be dominated by the pair $\underline{c},\underline{e}$, with min rank for $A(\underline{c},\underline{e})$.

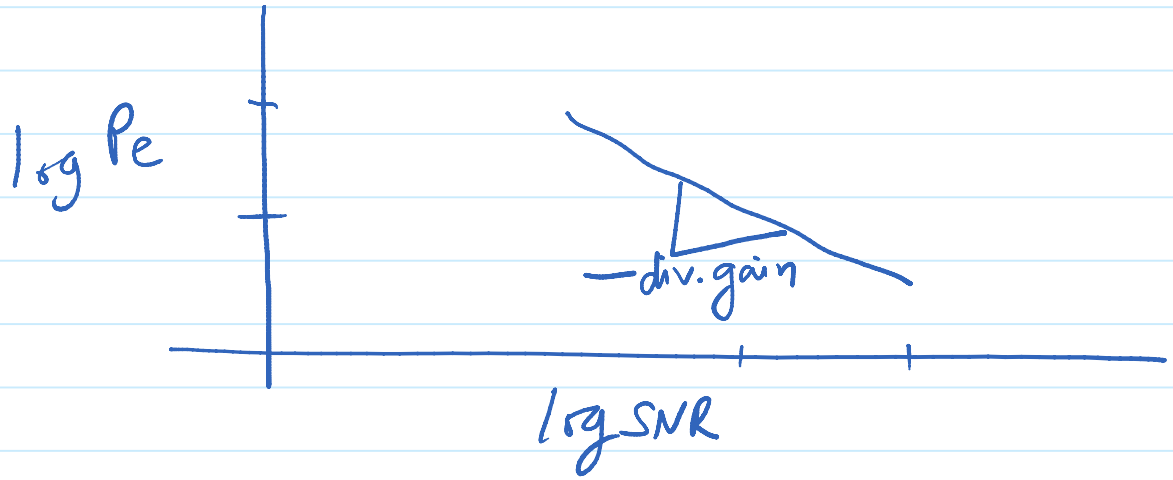
$$\text{Div. gain of code} = \left[\min_{\substack{\underline{c},\underline{e} \\ \underline{c}\neq\underline{e}}} \text{rank}\left(A(\underline{c},\underline{e})\right)\right] m \leq nm$$

Design criterion for space-time code: maximize $\left[\min_{\substack{\underline{c},\underline{e} \\ \underline{c}\neq\underline{e}}} \text{rank}\left(A(\underline{c},\underline{e})\right)\right]$

(Rank criterion)

L15

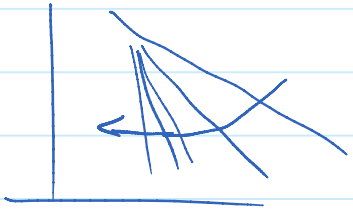
29 August 2019 11:18



$$P_e = (K)(SNR)^{-d}$$

$$\log P_e = \log K - d \log SNR$$

slope -1 $10^{-2} \rightarrow 10^{-3}$ 10dB increase in SNR



-2 1 order of mag red. in P_e 5dB —— " ——

-4 2.5dB —— " ——

L15

29 August 2019 11:23

Let us look at the constant term now.

$$\left(\prod_{i=1}^{r} \lambda_i\right)$$

If $r = n$, then this is $\det(A(\underline{c}, \underline{e}))$

Design criterion

(Determinant criterion)

$$\max_{\substack{\underline{c}, \underline{e} \\ \underline{c} \neq \underline{e}}} \left[\min \left(\prod_{i=1}^{r} \lambda_i \right) \right]$$

$$\left[P_e \leq \sum_{\underline{c}} \sum_{\underline{e}} P(\underline{c} \to \underline{e}) P(\underline{c}) \right]$$

(Intelligent) Union bound

L15

29 August 2019 11:30

Rician fading :

Rank criterion for div. gain → same as Rayleigh fading

Det. criterion for coding gain → same as Rayleigh fading

(Gain from improving the constant will be called coding gain)

$$\left(\prod_{i=1}^{r}\lambda_i\right)^{m}\prod_{j=1}^{m}\prod_{i=1}^{r}\exp\left(+K_{i,j}\right)$$

← constant term at high SNR

↑ depends on the code

L 15

29 August 2019 11:38

Correlated fading coefficients

$$d^2(\underline{c}, \underline{e}) = \sum_{j=1}^{m} \Omega_j A(\underline{c}, \underline{e}) \Omega_j^* = \Omega Y(\underline{c}, \underline{e}) \Omega^*$$

$$= \underbrace{[\Omega_1 \; \Omega_2 \cdots \Omega_m]}_{\triangleq \Omega} \overbrace{\begin{bmatrix} A(\underline{c}, \underline{e}) & & \\ & \ddots & \\ & & A(\underline{c}, \underline{e}) \end{bmatrix}}^{\triangleq Y(\underline{c}, \underline{e})} \begin{bmatrix} \Omega_1^* \\ \vdots \\ \Omega_m^* \end{bmatrix}$$

$$P(\underline{c} \to \underline{e} / \Omega) \leq \exp\left(-\Omega Y(\underline{c}, \underline{e}) \Omega^* \frac{E_s}{4N_0}\right)$$

Lecture 16

30 August 2019 10:02

$\underline{\Omega}^* \sim CN(\underline{0}, \underline{\Theta})$ $\qquad$ $\underline{\Omega}^* = C\underline{\nu}^*$, $\underline{\nu}^* \sim CN(\underline{0}, \underline{I})$ $\qquad$ $\underline{C}\underline{C}^* = \underline{\Theta}$

$$d^2(\underline{c}, \underline{e}) = \underline{\Omega}\, Y(\underline{c}, \underline{e})\, \underline{\Omega}^*$$

$$= \underline{\nu}\, C^*\, Y(\underline{c}, \underline{e})\, C\, \underline{\nu}^*$$

$$Pr(\underline{c} \rightarrow \underline{e} / \underline{\Omega}) \leq \exp\left(-\underline{\nu}\, \underbrace{C^* Y(\underline{c}, \underline{c}) C}\, \underline{\nu}^* \frac{E_s}{4N_0}\right)$$

$$\downarrow$$

$$V^* D V$$

✓ Rank criterion and determinant criterion : for matrix $C^* Y(\underline{c}, \underline{e}) C$ (instead of $A(\underline{c}, \underline{e})$ earlier)

Special case: Suppose C is full rank.

✓ maximizing rank $(\underbrace{C^* Y(\underline{c}, \underline{e}) C}) \equiv$ maximizing rank $(Y(\underline{c}, \underline{e}))$

L16

30 August 2019 10:13

$$\text{rank}\left(Y(\underline{c},\underline{e})\right) = \text{rank}\left(\begin{bmatrix} A(\underline{c},\underline{e}) & & 0 \\ & A(\underline{c},\underline{e}) & \\ 0 & & \ddots \\ & & & A(\underline{c},\underline{e}) \end{bmatrix}\right)$$

$$= m \,\text{rank}\left(A(\underline{c},\underline{e})\right)$$

Design Criterion:

$$\max_{\substack{\underline{c},\underline{e} \\ \underline{c}\neq\underline{e}}} \text{rank}\left(A(\underline{c},\underline{e})\right)$$

Same as the i.i.d. case

Determinant of $C^* Y(\underline{c},\underline{e}) C$ (C full rank)

$$= \det(C^*)\det(Y(\underline{c},\underline{e}))\det(C)$$

$$= \det(\Theta)\det(Y(\underline{c},\underline{e}))$$

Determinant criterion also the same as in the i.i.d. case.

L16

30 August 2019 10:20

Rapid fading case

Model $$d_t^j = \sum_{i=1}^{n} \alpha_{i,j}(t) c_t^i \sqrt{E_s} + n_t^j$$

- Quasistatic
  - Constant for all t
  - $\alpha_{i,j}$ i.i.d. $\forall\, i,j$ $CN(0,1)$

  Earlier model

- Rapid fading
  - i.i.d. $\forall\, i,j,t$ $CN(0,1)$ ✓

$$P(\underline{c} \to \underline{e} \mid \{\alpha_{i,j}(t)\}) \leq \exp\left(-\frac{d^2(\underline{c},\underline{e}) \cdot E_s}{4N_0}\right)$$

$d^2(\underline{c},\underline{e})$: sq. distance b/w received vectors corresponding to $\underline{c}$ & $\underline{e}$.

L16

30 August 2019 10:24

$$\left(d_1^1\ d_1^2 \cdots d_1^m\ d_2^1\ d_2^2 \cdots d_2^m \cdots \quad d_\ell^1\ d_\ell^2 \cdots d_\ell^m\right)$$

$$=$$

$$\left(\qquad\qquad\qquad\qquad \right)$$

$$+$$

$$\left(\quad \text{noise} \quad \right)$$

$$d_2^m = \overbrace{\left(\alpha_{1,m}(2)\ \alpha_{2,m}(2) \cdots \alpha_{n,m}(2)\right)}^{\triangleq\, \Omega_m(2)} \begin{pmatrix} c_2^1 \\ c_2^2 \\ \vdots \\ c_2^n \end{pmatrix} \triangleq \underline{c}(2)$$

$$d_t^j = \underbrace{\Omega_j(t)\, \underline{c}(t)}$$

L16

30 August 2019 10:29

$$\sum_{j=1}^{m}\sum_{t=1}^{l}\left|\Omega_j(t)\left(\underline{c}(t)-\underline{e}(t)\right)\right|^2 = d^2(\underline{c},\underline{e})$$

$$= \sum_{j=1}^{m}\sum_{t=1}^{l}\Omega_j(t)\underbrace{\left[\left(\underline{c}(t)-e(t)\right)\left(\underline{c}(t)-e(t)\right)^*\right]}_{V^* D V}\Omega_j^*(t)$$

$$= \sum_{U=1}^{m}\sum_{t=1}^{l}\left(\sum_{i=1}^{n}\underbrace{|\beta_{i,j}(t)|^2}_{\sim \exp\ \text{i.i.d}}\underbrace{D_{ii}(t)}_{\text{eigenvalues of }[\,]}\right)$$

$[\,]$ Hermitian rank 1 or 0

$\downarrow$ if $\underline{c}(t)=\underline{e}(t)$ code matrix column corresponds to t

only one non-zero eigenvalue at most

L 16

30 August 2019 10:39

$$D_{ii}(t) = \begin{cases} 0 & \text{for } (n-1) \text{ elements} \\ \|\underline{c}(t) - \underline{e}(t)\|^2 & 1 \text{ element} \end{cases}$$

$$P(\underline{c} \to \underline{e}) \leq \prod_{j=1}^{m} \prod_{t=1}^{l} \prod_{i=1}^{n} \left[\left(1 + D_{ii}(t) \frac{E_s}{4N_0} \right)^{-1} \right]$$

After $E[\,]$ over $\beta_{ii}(t)$'s

$$= \prod_{j=1}^{m} \prod_{t=1}^{l} \left(1 + \|\underline{c}(t) - \underline{e}(t)\|^2 \frac{E_s}{4N_0} \right)^{-1}$$

Diversity gain = $m \times$ no. of time instances for which $\underline{c}(t) \neq \underline{e}(t)$.

$\triangleq |\nu(\underline{c}, \underline{e})| \leq ml$

L16

30 August 2019 10:46

Coding gain criterion

$$\max_{\underline{c},\underline{e}} \left[\min_{\substack{\underline{c}\neq\underline{e} \\ t\in V(\underline{c},\underline{e})}} \left(\prod \|\underline{c}(t) - \underline{e}(t)\|^2 \right) \right]$$

Lecture 17

03 September 2019 12:01

$$\underset{n\times n}{A(\underline{c},\underline{e})} = \underbrace{B(\underline{c},\underline{e})}_{n\times l} B(\underline{c},\underline{e})^*$$

* code matrix $\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{n\times l}$

Div. $m.\left[\min_{\substack{\underline{c},\underline{e}\\ \underline{c}\neq\underline{e}}} \operatorname{rank}(A(\underline{c},\underline{e}))\right]$

if we want n, need $l \geq n$.

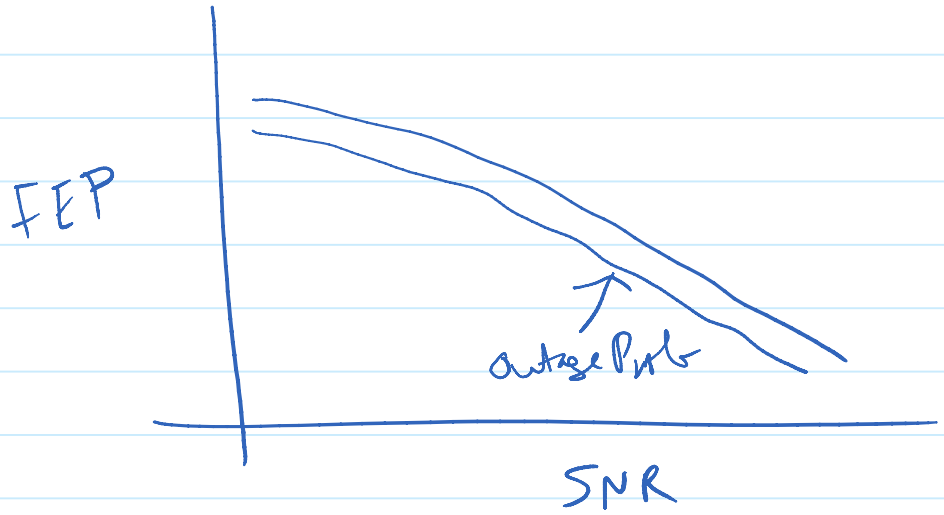
$$\Rightarrow n \leq \min(n, l)$$

L17

03 September 2019 12:06

Rest of this paper: Code designs and comparison with outage capacity

- Space-time trellis codes
- Space-time block codes (in a different paper)



L17

03 September 2019 12:36

* How to read the trellis?
* n_T-spacetime trellis code has div. r
* Trade-off between rate, diversity, constellation size, trellis complexity
* Comparison with outage capacity / outage prob.
* Smart-greedy codes
  * Achieves max. div. gain for quasi-static fading
  * Higher div. gain for rapid fading

L17

03 September 2019 12:43

"A simple transmit diversity technique for Wireless Communications," S. M. Alamouti, JSAC 1998.

→ Space-time block code for 2 × 1 case

- 2 → # tx ant
- 1 → # rx ant.

can be generalized to 2 × M case

Div. gain achieved = 2M

→ Decoding is simple
  ↳ linear in # of symbols

L17

03 September 2019 12:48

ML in general can be exponentially complex in # symbols.

$$\min_{\underline{x}} \|\underline{y} - H\underline{x}\|^2 \quad \checkmark$$

$\underline{x}$ → (# constellation pts)$^{\#\text{symbols}}$ possibilities

$$\min_{X} \|Y - HX\|^2$$

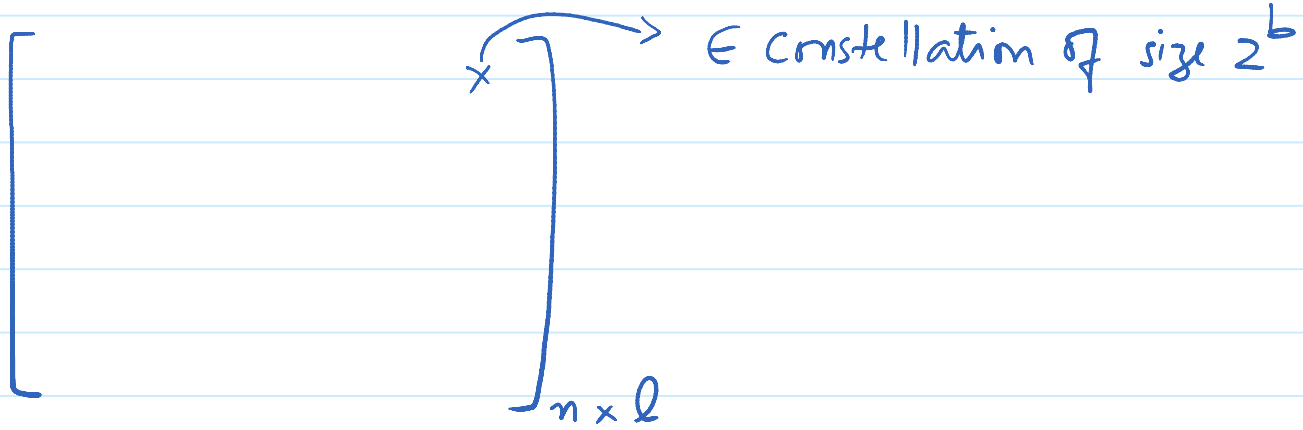
Need structure in H to reduce complexity

Lecture 18

05 September 2019 10:58

Lecture 19

06 September 2019 09:57



$$\frac{\log_2 |2^{bl}|}{l} = b.$$

Then, for diversity nm (full diversity), $R \leq b$ $(l \overset{\text{also}}{\geq} n)$

(R: one symbol/time slot)

$R = b$ (full rate)

Alamouti scheme 2×1 : full rate, full diversity.

Now Orthogonal designs

L19

06 September 2019 10:05

Real orthogonal designs

- A real orthogonal design of size n is an $n \times n$ orthogonal matrix with entries $\pm x_1, \pm x_2, \ldots, \pm x_n$
- An orthogonal design exists if and only if $n = 2, 4$ or 8.

W.l.o.g. we can assume first row to be $(x_1 \; x_2 \; \ldots \; x_n)$

$n = 2$

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$$

$n = 4$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{pmatrix}$$

$n = 8$

$$\begin{pmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

L 19

06 September 2019 10:12

nb bits $\longrightarrow (s_1\ s_2 \cdots s_n) \longrightarrow \mathcal{O}(s_1, s_2, \ldots, s_n)$

code matrix $\mathcal{C}$ (set $x_i = s_i$)

Thm Div. order $= mn$

Pf $\mathcal{O}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) - \mathcal{O}(s_1, s_2, \ldots, s_n)$ should be full rank

for any $(\tilde{s}_1, \ldots, \tilde{s}_n) \neq (s_1, s_2 \ldots, s_n)$

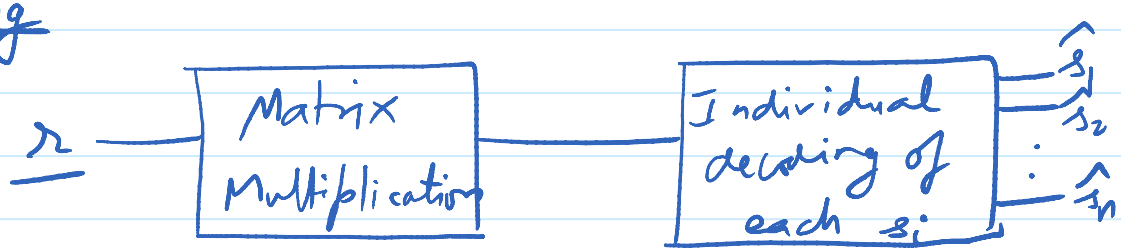
$= \mathcal{O}(\tilde{s}_1 - s_1, \tilde{s}_2 - s_2, \ldots, \tilde{s}_n - s_n)$

$\det\left(\mathcal{O}(\tilde{s}_1 - s_1, \tilde{s}_2 - s_2, \ldots, \tilde{s}_n - s_n)\right) = \left(\sum_i |\tilde{s}_i - s_i|^2\right)^{n/2} \neq 0$

L19

06 September 2019 10:18

Decoding



$$r_t^j = \sum_{i=1}^{n} \alpha_{i,j}\, c_t^i + n_t^j$$

n individual problems

$$\min_{s_1,\dots,s_n} \sum_{t=1}^{l=n} \sum_{j=1}^{m} \left| r_t^j - \sum_{i=1}^{n} \alpha_{i,j}\, c_t^i \right|^2 \iff \min_{s_1,\dots,s_n} \sum_{i=1}^{n} S_i$$

where $S_i = \underbrace{\left| \sum_{t=1}^{n} \sum_{j=1}^{m} r_t^j \alpha^*_{\epsilon_t^{(i)},j}\, \delta_t^{(i)} \right|}_{\triangleq R_i} - s_i \Big|^2 + \left(-1 + \sum_{k,l} |\alpha_{k,l}|^2\right) |s_i|^2$

L19

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Important properties → Max. possible rate at full diversity

→ Simple ML decoding

These 2 properties are preserved even if linear processing is allowed at transmitter.

Linear processing orthogonal design

An $n \times n$ matrix $\mathcal{E}$ in variables $x_1, x_2, \ldots, x_n$ such that

(1) entries are real linear combinations of $x_1, x_2, \ldots, x_n$

(2) $\mathcal{E}^T\mathcal{E} = D$ where D is diagonal

→ enough to consider $(x_1^2 + \cdots + x_n^2)\underline{I}$

$$\mathcal{E} = x_1 A_1 + \cdots + x_n A_n$$

A_i: $n \times n$

Design $A_1, A_2, \ldots, A_n$

L19

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$$\left(\sum_{i=1}^{n} x_i A_i\right)^T \left(\sum_{j=1}^{n} x_j A_j\right) = (x_1^2 + \cdots + x_n^2) I$$

$$\left\{\begin{aligned} A_i^T A_i &= I \quad \forall i \\ A_i^T A_j &= -A_j^T A_i \quad (i \neq j) \end{aligned}\right\} \text{ needed}$$

Exists only for $n = 2, 4, 8$

Generalized Real Orthogonal design G of size n is a $p \times n$ matrix with entries $0, \pm x_1, \pm x_2, \cdots, \pm x_k$ such that $G^T G = D = (x_1^2 + \cdots + x_n^2) I_n \quad (p \geq n)$

L19

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→ constellation A has 2^b symbols

→ Rate

$\frac{kb}{p}$ bits/slot $\qquad$ $\frac{k}{p}$ symbols/slot

↳ (if $= 1$, then full rate)

Given R, n : minimize p

Lecture 20

10 September 2019 12:00

For a given R, n, define $A(R,n)$ to be the minimum p such that there exists a $p \times n$ generalized orthogonal design with rate at least R.

* If we attain $A(R,n)$, the design is delay-optimal.

$A(1,n)$: full rate $R=1$ $\quad A(1,n) < \infty$ $\quad G_3^H G_3 = I$

$A(1,2) = 2$ (Alamouti code)
$A(1,3) = 4$
$A(1,4) = 4$
$A(1,5) = 8$
$A(1,6) = 8$
$A(1,7) = 8$
$A(1,8) = 8$
$\vdots$

$$G_3 = \begin{pmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & -x_4 \\ -x_3 & x_4 & x_1 \\ -x_4 & -x_3 & x_2 \end{pmatrix}$$

first 3 columns of design for 4

L 20

10 September 2019 12:18

- Proof of separate decodability of symbols of a STBC from general orthogonal designs (Jafarkhani 2005 book p75-76)

- Complex orthogonal designs : Summary of results

$n=2$ $\begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix}$ Alamouti

Complex design of size n $\longrightarrow$ Real design of size $2n$

↓

orthogonal $n \times n$ exists only for 2,4,8

$\longleftarrow$

only ~~2,4~~ possible

L 20

10 September 2019 12:23

— General complex orthogonal designs.

$G_c : p \times n$ (p: time slots, n: # tx ant)

$0, \pm x_1, \pm x_1^*, \ldots, \pm x_k, \pm x_k^*$

$R = \frac{k}{p}$

— $A_c(R, n)$: min p such that there exists a generalized complex linear processing orthogonal design of size $p \times n$ with rate $\geq R$.

* For any R

$$A(R, 2n) \leq 2 A_c(R, n) \quad \text{(Lower bound)}$$

* For $R \leq 0.5$ $\quad G_c = \begin{pmatrix} G \\ G^* \end{pmatrix}$

$$A_c(R, n) \leq 2 A(2R, n) \quad \text{(Upper bound)}$$

L 20

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$$A(1,n) < \infty \Rightarrow A_c\left(\frac{1}{2},n\right) < \infty$$

For $n=3,4$ $R=\frac{3}{4}$ designs are proposed in Sec V.F of Tarokh 1999

Quasi-orthogonal designs

Orthogonal design

- full diversity
- single symbol decodability

$R \leq 1$ (real)

$R \leq \frac{1}{2}, \frac{3}{4}$ (complex)

→ dec. complexity ↑ (eg. double symbol decoding)

→ R ↑

Quasi-orthogonal

diversity ↓

L20

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One such quasi-orthogonal code for $n = 4$, $R = 1$ (complex)

(Jafarkhani 2001)

$$A_{12} \triangleq \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} \longrightarrow A = \begin{pmatrix} A_{12} & A_{34} \\ -A_{34}^* & A_{12}^* \end{pmatrix}$$

Alamouti

$$= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ -x_3^* & -x_4^* & x_1^* & x_2^* \\ x_4 & -x_3 & -x_2 & +x_1 \end{pmatrix}$$

Diversity ? = 2

Single symbol Decodability ? → double symbol decodability

Lecture 21

12 September 2019 11:02

$\alpha_{i,j}$ constant over the block of 4 symbols

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ -x_3^* & -x_4^* & x_1^* & x_2^* \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}$$

time ↓

$$r_1^j = \alpha_{1,j} x_1 + \alpha_{2,j} x_2 + \alpha_{3,j} x_3 + \alpha_{4,j} x_4 + n_1^j$$

$$r_2^j = \alpha_{1,j}(-x_2^*) + \alpha_{2,j}(x_1^*) + \alpha_{3,j}(-x_4^*) + \alpha_{4,j}(x_3^*) + n_2^j)^*$$

out ⟶

$$\begin{pmatrix} r_1^j \\ r_2^{j*} \\ r_3^{j*} \\ r_4^j \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{1,j} & \alpha_{2,j} & \alpha_{3,j} & \alpha_{4,j} \\ \alpha_{2,j}^* & -\alpha_{1,j}^* & \alpha_{4,j}^* & -\alpha_{3,j}^* \\ \alpha_{3,j}^* & \alpha_{4,j}^* & -\alpha_{1,j}^* & -\alpha_{2,j}^* \\ \alpha_{4,j} & -\alpha_{3,j} & -\alpha_{2,j} & \alpha_{1,j} \end{pmatrix}}_{H} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} n_1^j \\ n_2^{j*} \\ n_3^{j*} \\ n_4^j \end{pmatrix}$$

$h_1 \quad h_2 \quad h_3 \quad h_4$

L 21

12 September 2019 11:16

$$\underline{h}_i^H \underline{h}_i = \sum_{t=1}^{4} |\alpha_{t,j}|^2$$

H

$$\begin{pmatrix} \alpha_{1,j} & \alpha_{2,j} & \alpha_{3,j} & \alpha_{4,j} \\ \alpha_{2,j}^* & -\alpha_{1,j}^* & \alpha_{4,j}^* & -\alpha_{3,j}^* \\ \alpha_{3,j}^* & \alpha_{4,j}^* & -\alpha_{1,j}^* & -\alpha_{2,j}^* \\ \alpha_{4,j} & -\alpha_{3,j} & -\alpha_{2,j} & \alpha_{1,j} \end{pmatrix}$$

$\underline{h}_1 \quad \underline{h}_2 \quad \underline{h}_3 \quad \underline{h}_4$

$$\underline{h}_1^H \underline{h}_2 = \alpha_{1,j}^* \alpha_{2,j} + \alpha_{2,j}(-\alpha_{1,j}^*) + \alpha_{3,j}\alpha_{4,j}^* + \alpha_{4,j}^*(-\alpha_{3,j}) = 0$$

$$\underline{h}_1^H \underline{h}_3 = \alpha_{1,j}^* \alpha_{3,j} \checkmark + \alpha_{2,j}\alpha_{4,j}^* \checkmark - \alpha_{3,j}\alpha_{1,j}^* \checkmark - \alpha_{4,j}^*\alpha_{2,j} \checkmark = 0$$

$$\underline{h}_1^H \underline{h}_4 = \alpha_{1,j}^* \alpha_{4,j} - \alpha_{2,j}\alpha_{3,j}^* - \alpha_{3,j}\alpha_{2,j}^* + \alpha_{4,j}^*\alpha_{1,j} \neq 0$$

L21

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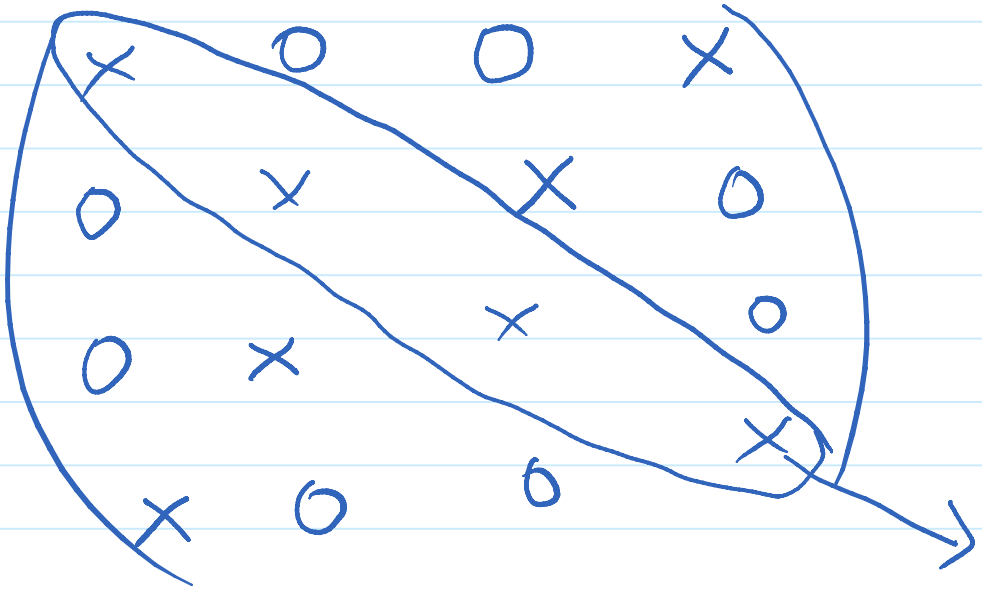
$H^H H =$



$$\sum_{t=1}^{4} |\alpha_{t,j}|^2$$

$(\underline{h}_1, \underline{h}_4)$ $(\underline{h}_2, \underline{h}_3)$

$\Rightarrow$ x_1, x_4 should be decoded jointly } Why?

x_2, x_3 ——"—— }

L 21

12 September 2019 11:22

$$\underline{r} = H\underline{x} + \underline{n}$$

$$\min_{\underline{x}'} \| \underline{r} - H\underline{x}' \|^2$$

$$= \min_{\underline{x}'} \| H\underline{x} + \underline{n} - H\underline{x}' \|^2$$

$$= \min_{\underline{x}'} \| H(\underline{x} - \underline{x}') + \underline{n} \|^2$$

$$= \min_{\underline{x}'} \| \underbrace{H_1(\underline{x}_1 - \underline{x}_1')}_{\text{orthogonal vectors}} + \underbrace{H_2(\underline{x}_2 - \underline{x}_2')}_{\text{orthogonal vectors}} + \underline{n} \|^2$$

$$H\underline{x} = H_1\underline{x}_1 + H_2\underline{x}_2$$

$$H = \left[\, H_1 \,\middle|\, H_2 \,\right]$$

$$H_1^H H_2 = 0$$

$\underline{x}$: $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$ / $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$

$\underline{x}_1$ $\underline{x}_2$

↓

symbols corresponding to columns of H_1

$\underline{n} = \underline{n}_1 + \underline{n}_2 + \underline{n}'$

L21

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$$= \min_{\underline{x}'} \left\| \underline{H_1 (\underline{x}_1 - \underline{x}_1') + \underline{n}_1} + \underline{H_2 (\underline{x}_2 - \underline{x}_2') + \underline{n}_2} + \underline{\underline{n}'} \right\|^2$$

orthogonal

$$= \min_{\underline{x}'} \left(\underline{\| H_1 (\underline{x}_1 - \underline{x}_1') + \underline{n}_1 \|^2} + \underline{\| H_2 (\underline{x}_2 - \underline{x}_2') + \underline{n}_2 \|^2} + \underline{\| \underline{n}' \|^2} \right)$$

choose $\underline{x}_1'$ to min

choose $\underline{x}_2'$ to min

$$\| \underline{(H_1 \underline{x}_1 + \underline{n}_1)} - H_1 \underline{x}_1' \|^2$$

$\underline{y}$ → Project onto column space of H_1

L 21

12 September 2019 11:35

$\underline{r}$ → Projection on $\mathcal{C}(H_1)$ → Decode x_1, x_4 jointly

$\underline{r}$ → Projection on $\mathcal{C}(H_2)$ → Decode x_2, x_3 jointly

Double symbol decodability

Diversity gain $\left(\min_{\underline{c},\underline{e}\ \underline{c}\neq\underline{e}} \operatorname{rank}(A(\underline{c},\underline{e})) \right) m$

m : # rx ant

find?

$= 2$

L 21

12 September 2019 11:37

$$\begin{pmatrix} s_1-\tilde{s}_1 & s_2-\tilde{s}_2 & s_3-\tilde{s}_3 & s_4-\tilde{s}_4 \\ -(s_2-\tilde{s}_2)^* & (s_1-\tilde{s}_1)^* & -(s_4-\tilde{s}_4)^* & (s_3-\tilde{s}_3)^* \\ -(s_3-\tilde{s}_3)^* & -(s_4-\tilde{s}_4)^* & (s_1-\tilde{s}_1)^* & (s_2-\tilde{s}_2)^* \\ (s_4-\tilde{s}_4) & -(s_3-\tilde{s}_3) & -(s_2-\tilde{s}_2) & (s_1-\tilde{s}_1) \end{pmatrix}$$

$A(s_1, s_2, s_3, s_4)$

$-A(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)$

rank > 1

Let $s_1-\tilde{s}_1 = s_4-\tilde{s}_4 = a$

$s_2-\tilde{s}_2 = s_3-\tilde{s}_3 = 0$

$$\begin{pmatrix} a & 0 & 0 & a \\ 0 & a^* & -a^* & 0 \\ 0 & -a^* & a^* & 0 \\ a & 0 & 0 & a \end{pmatrix}$$

$\times(-1)$

rank = 2

$\min_{\underline{c}, \underline{e}, \underline{c} \neq \underline{e}}$ rank $(A(\underline{c}, \underline{e})) = 2$

Div. gain = 2m

Lecture 22

13 September 2019 09:45

So far

MIMO capacity

- Known at both tx & Rx & fixed H
  - SVD + waterfilling (sum power constraint)
- Fading: Ergodic
  - channel known at Rx (CSIR)
  - channel statistics known at tx
  - Ergodic capacity $\propto \min(t, r)$ (i.i.d. Rayleigh fading)
  - Space-time coding (design criterion)
- Fading: Quasi-static
  - CSIR
  - statistics known at tx
  - Outage capacity opt Q (MISO) $\propto \min(t, r)$
  - Space-time coding (Rate, diversity, decoding complexity)
  - VBLAST: Multiplexing

L 22

13 September 2019 10:09

"Diversity and Multiplexing: A fundamental trade-off in multiple antenna channels"

(Zheng & Tse 2003)

Model

$$Y_{n\times l} = \sqrt{\frac{SNR}{m}}\, \underbrace{H}_{n\times m} X_{m\times l} + W_{n\times l}$$

W: i.i.d. $\mathcal{CN}(0,1)$

tx ant m
rx ant n
slots l

SNR: Average SNR at each rx antenna

Codebook $\mathcal{C}$: Collection of matrices of size $m \times l$

matrices $= |\mathcal{C}|$ $\quad \{X(1), X(2), \cdots, X(|\mathcal{C}|)\}$

$$\text{Rate } R = \frac{\log_2 |\mathcal{C}|}{l} \text{ bits/slot} \qquad |\mathcal{C}| = \lfloor 2^{Rl} \rfloor$$

L 22

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Power constraint: $$\frac{1}{|\mathcal{C}|}\sum_{i=1}^{|\mathcal{C}|} \|X(i)\|_F^2 \leq nl$$

$$\|A\|_F^2 = \sum_i \sum_j |A_{ij}|^2$$

→ Analyse a family of codes

scheme $\{\mathcal{C}(SNR)\}$ — codes for each SNR

Pick code from this family depending on SNR.

Rate ↑ as a fn. of SNR

A scheme achieves spatial multiplexing gain r if

$$\lim_{SNR\to\infty} \frac{R(SNR)}{\log SNR} = r$$

L 22

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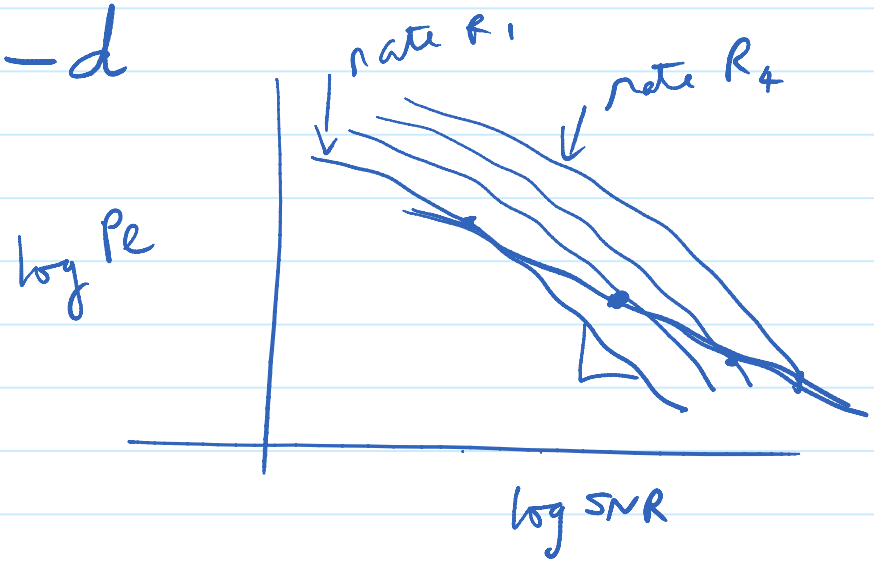
A scheme achieves diversity gain d if

$$\lim_{SNR \to \infty} \frac{\log P_e(SNR)}{\log SNR} = -d$$



$d^*(r)$: Supremum of the diversity gain achieved over all schemes

Main result: $d^*(k) = (m-k)(n-k)$ for $k = 0, 1, 2, \ldots, \min(m,n)$

$d^*(r)$ for other r : piecewise linear fn connecting $(k, d^*(k))$

$d^*_{max} = mn$, $r^*_{max} = \min(m,n)$

L 22

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Figs. 1, 2, 3

Alamouti family of codes

$$\begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$$

1 sym/slot

choose constellation size based on SNR

Repetition scheme

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

1/2 sym/slot

Lecture 23

16 September 2019 07:50

Outage probability

$$I(\underline{x}; \underline{y} \mid \mathbb{H} = H) = \log \det\left(I + \frac{SNR}{m} H Q H^H\right)$$

$$P_{out}(R) = \inf_{\substack{Q \geq 0 \\ \mathrm{tr}(Q) \leq m}} P\left[\underbrace{\log \det\left(I + \frac{SNR}{m} \mathbb{H} Q \mathbb{H}^H\right)}_{f(Q)} < R\right]$$

$Q = I$

$$P\left[\log \det\left(I + \frac{SNR}{m} \mathbb{H}(mI)\mathbb{H}^H\right) < R\right] \leq P_{out}(R) \leq P\left[\log \det\left(I + \frac{SNR}{m} \mathbb{H}\mathbb{H}^H\right) < R\right]$$

$Q = mI$

[For any $Q \geq 0$ with $\mathrm{tr}(Q) \leq m$, $mI - Q \geq 0$;

$f(Q_1) \geq f(Q_2)$ if $Q_1 - Q_2 \geq 0$]

L23

16 September 2019 08:12

Both the upper bound and the lower bound lead to the same diversity.

(value of lowerbound at $\frac{SNR}{m}$ is value of upperbound at SNR
$\Rightarrow$ only a shift in logPe vs log SNR plot)

* Now, analyze the lower bound $P\left[\log\det\left(I + SNR\, HH^H\right) < R\right]$ to get diversity & DMT.

$$P_{out}(R) \doteq P\left[\log\det\left(I + SNR\, HH^H\right) < R\right]$$

$\doteq$: exponential equality

HH^H : eig values $\lambda_1, \dots, \lambda_n$ $(m \geq n)$

$$= P\left[\log\left(\prod_{i=1}^{n}(1+SNR\,\lambda_i)\right) < R\right]$$

$\log\left[(SNR)^r\right]$

$R = r(\log SNR)$
$= \log(SNR^r)$

L23

16 September 2019 08:24

$$= P\left[\prod_{i=1}^{n}(1+SNR\,\lambda_i) < SNR^{r}\right]$$

[→ Joint PDF of λ_i's for Rayleigh fading model from Telatar's paper]

Define $\lambda_i = SNR^{-\alpha_i}$ $\left(\alpha_i = -\frac{\log \lambda_i}{\log SNR}\right)$

$$P_{out}(R) \doteq P\left[\prod_{i=1}^{n}(1+SNR.SNR^{-\alpha_i}) < SNR^{r}\right]$$

$$= P\left[\prod_{i=1}^{n}(1+SNR^{(1-\alpha_i)}) < SNR^{r}\right]$$

L23

16 September 2019 08:28

$$1 + SNR^{(1-\alpha_i)}$$

if $1-\alpha_i > 0$, $\longrightarrow SNR^{(1-\alpha_i)}$ as $SNR \to \infty$.

$1-\alpha_i < 0$, $\longrightarrow 1$ as $SNR \to \infty$.

$$P\left[\prod_{i=1}^{n}\left(1+SNR^{(1-\alpha_i)}\right) < SNR^{r}\right]$$

$$x^+ = \begin{cases} 1 & x>0 \\ 0 & \text{else} \end{cases}$$

$$= P\left[\prod_{i=1}^{n} SNR^{(1-\alpha_i)^+} < SNR^{r}\right]$$

$$= P\left(\sum_{i=1}^{n}(1-\alpha_i)^+ < r\right)$$

L23

16 September 2019 08:32

$$P_{out}(R) \doteq P(\mathcal{A}) \qquad \mathcal{A} = \left\{ \underline{\alpha} : \sum_{i=1}^{m} (1-\alpha_i)^+ < r \right\}$$

$$= \int_{\mathcal{A}} f_{\underline{\alpha}}(\underline{\alpha})\, d\underline{\alpha}$$

Rayleigh fading

$$f_{\underline{\lambda}}(\lambda_1, \ldots, \lambda_n) = K_{m,n}^{-1} \prod_{i=1}^{n} \lambda_i^{m-n} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\sum_i \lambda_i}$$

ordered eig values

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

L23

16 September 2019 08:37

want $f_{\underline{\alpha}}(\underline{\alpha})$ $\qquad \alpha_i = -\dfrac{\log \lambda_i}{\log SNR}$ (or) $\lambda_i = SNR^{-\alpha_i}$

$$f_{\underline{\alpha}}(\underline{\alpha}) = f_{\underline{\lambda}}(\underline{\lambda})\Big|_{\underline{\lambda} = \begin{pmatrix} SNR^{-\alpha_1} \\ \vdots \\ SNR^{-\alpha_n} \end{pmatrix}} \cdot \left| J(\underline{\alpha}, \underline{\lambda}) \right|$$

$$\frac{\partial \lambda_i}{\partial \alpha_i} = (-SNR^{-\alpha_i}) \log SNR$$

$$J(\underline{\alpha}, \underline{\lambda}) = \begin{bmatrix} & & 0 \\ & \iddots & \\ 0 & & \end{bmatrix}$$

$$\left| J(\underline{\alpha}, \underline{\lambda}) \right| = (\log SNR)^n \prod_{i=1}^{n} SNR^{-\alpha_i}$$

L 23

16 September 2019 08:44

$$f_{\underline{\alpha}}(\underline{\alpha}) = K_{m,n}^{-1} \prod_{i=1}^{n} SNR^{-\alpha_i (m-n)} \prod_{i<j} \left(SNR^{-\alpha_i} - SNR^{-\alpha_j}\right)^2 e^{-\sum_i SNR^{-\alpha_i}}$$

$$\cdot \underbrace{(\log(SNR))^n \prod_{i=1}^{n} SNR^{-\alpha_i}}$$

$$= K_{m,n}^{-1} \prod_{i=1}^{n} SNR^{-\alpha_i (m-n+1)} \prod_{i<j} \left(SNR^{-\alpha_i} - SNR^{-\alpha_j}\right)^2 e^{-\sum_i SNR^{-\alpha_i}}$$

$$P_{out}(R) \doteq \int_A f_{\underline{\alpha}}(\underline{\alpha})\, d\underline{\alpha} \qquad (\log SNR)^n$$

$$\frac{\log P_{out}(R)}{\log SNR} = \frac{\log\left(\int_A f_{\underline{\alpha}}(\underline{\alpha})\, d\underline{\alpha}\right)}{\log SNR}$$

L23

16 September 2019 08:48

$$\frac{\log\left(\int_{\mathcal{A}} f_{\underline{\alpha}}(\underline{\alpha})\, d\underline{\alpha}\right)}{\log SNR} = \frac{\log\left(K_{m,n}^{-1}(\log SNR)^n\right)}{\log SNR} + \frac{\log\left(\int_{\mathcal{A}} \prod_{i=1}^{n} SNR^{-\alpha_i(m-n+1)} \prod_{i<j}\left(SNR^{-\alpha_i} - SNR^{-\alpha_j}\right)^2 e^{-\sum_{1} SNR^{-\alpha_i}}\, d\underline{\alpha}\right)}{\log SNR}$$

$\to 0$ as $SNR \to \infty$

Lecture 24

17 September 2019 12:01

$$\left(\int_{\mathcal{A}} \prod_{i=1}^{n} SNR^{-\alpha_i (m-n+1)} \prod_{i<j} \left(SNR^{-\alpha_i} - SNR^{-\alpha_j} \right)^2 e^{-\sum_i SNR^{-\alpha_i}} d\underline{\alpha} \right)$$

$$\mathcal{A} = \left\{ \underline{\alpha} : \sum_i (1-\alpha_i)^+ < r \right\}$$

$$\left\{ \underline{\alpha} : \sum_i (1-\alpha_i)^+ < r, \right\} \cup \{ \quad \}$$

$\mathcal{A}'$ $\alpha_i \geq 0$

integral over this set $\to 0$ as $SNR \to \infty$

$\alpha_i < 0$, $e^{-SNR^{-\alpha_i}}$ decays exponentially with SNR

$\alpha_i = 0$ $\quad e^{-SNR^{-\alpha_i}} = e^{-1}$

$\alpha_i > 0$ $\quad e^{-SNR^{-\alpha_i}} \to 1$

$e^{-\sum_i SNR^{-\alpha_i}} \to$ constant

L24

17 September 2019 12:11

Now, we will consider

$$\int_{\mathcal{A}'} \prod_{i=1}^{n} SNR^{-\alpha_i (m-n+1)} \prod_{i<j} \left(SNR^{-\alpha_i} - SNR^{-\alpha_j}\right)^2 \, d\underline{\alpha}$$

λ_i's are ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$

$$\prod_{i=1}^{n} SNR^{-\alpha_i (m-n+1)} \prod_{i<j} \left(SNR^{-\alpha_i} - SNR^{-\alpha_j}\right)^2 \qquad \left(\text{consider } \alpha_1 < \alpha_2 < \cdots < \alpha_n\right)$$

$$\doteq \prod_{i=1}^{n} SNR^{-\alpha_i (m-n+1)} \prod_{i<j} SNR^{-2\alpha_j} = \prod_{i=1}^{n} \left[SNR^{-\alpha_i (m-n+1)} \left(SNR^{-2\alpha_i}\right)^{(i-1)} \right]$$

L24

17 September 2019 12:23

$$= \prod_{i=1}^{n} SNR^{\left(-\alpha_i(m-n+1) + 2(i-1)(-\alpha_i)\right)}$$

$$= \prod_{i=1}^{n} SNR^{-\alpha_i(m-n+1+2(i-1))}$$

$$= \prod_{i=1}^{n} SNR^{-\alpha_i(m-n+2i-1)}$$

$$= SNR^{-\sum_{i=1}^{n}\alpha_i(m-n+2i-1)}$$

$$\lim_{SNR\to\infty} \int_{A'} SNR^{-\underbrace{\sum_{i=1}^{n}\alpha_i(m-n+2i-1)}} d\underline{\alpha}$$

L 24

17 September 2019 12:26

$$\doteq SNR^{\sup_{\underline{\alpha} \in \mathcal{A}'} \left(-\sum_i \alpha_i (m-n+2i-1) \right)}$$

$$P_{out}(R) \doteq SNR^{-\left[\inf_{\underline{\alpha} \in \mathcal{A}'} \sum_i \alpha_i (m-n+2i-1) \right]} \qquad (m \geq n)$$

$$\left[\inf_{\underline{\alpha} \in \mathcal{A}'} \sum_i \alpha_i (m-n+2i-1) \right] \rightarrow d_{out}(r)$$

For general m, n

$$\boxed{d_{out}(r) = \inf_{\underline{\alpha} \in \mathcal{A}'} \sum_i \alpha_i (|m-n|+2i-1)}$$

$$\mathcal{A}' = \left\{ \underline{\alpha} : \sum_i (1-\alpha_i)^+ < r, \ \alpha_i \geq 0 \right\}$$

L24

17 September 2019 12:33

First consider r to be an integer k $\quad 0 \le k \le n$

$$\inf_{\underline{\alpha} \in \mathcal{A}'} \underbrace{\sum_i \alpha_i (|m-n| + 2i - 1)}_{\downarrow}$$

$$\mathcal{A}' = \{ \underline{\alpha} : \underbrace{\sum_i (1-\alpha_i)^+ < r}_{\alpha_i \ge 0} \}$$

$$\alpha_1 (|m-n| + 1) \cdot$$
$$+ \alpha_2 (|m-n| + 3) \cdot$$
$$\vdots$$
$$+ \alpha_n (|m-n| + 2n - 1)$$

$$0 \le (1-\alpha_i)^+ \le 1$$

$$\underbrace{\alpha_1^* = \alpha_2^* = \cdots = \alpha_{n-k}^* = 1 \quad \alpha_{n-k+1}^* \cdots = \alpha_n^* = 0}_{\in \mathcal{A}'}$$

$\underline{\alpha}$ corresponding to inf

L24

17 September 2019 12:46

$$d_{ow}(k) = \sum_{i=1}^{n-k} (m-n+2i-1) = \frac{(n-k)(m-k)}{}$$

$$(m-n-1)(n-k) + \not{2}\,\frac{(n-k)(n-k+1)}{\not{2}}$$

Non-integer n : Soln. is the linear interpolation of the soln. for the 2 nearby integer n.

Lecture 25

19 September 2019 10:57

Typical outage event

(for $r = k$) $(n-k)$ smallest eigenvalues $\doteq SNR^{-1}$

k largest eigenvalues $\doteq 1$

Relating error probability to outage probability

Lemma 5 For any coding scheme, prob. of detection error is lower bounded by

$$P_e(SNR) \dot{\geq} SNR^{-d_{out}(r)}$$

[Show using (1) Fano's inequality
(2) conditioned on outage event, it is very likely that detection error occurs]

L 25

19 September 2019 11:07

Thm 2 $l \geq m+n-1$ is sufficient to achieve $d_{out}(r)$

$$P_e(SNR) = \underbrace{P_{out}(R)\, P(\text{error}/\text{outage}) + P(\text{error, no outage})}$$

$P_{out}(R) \leq$ ↓ $d_{out}(r)$

check div. order of this term

$\leq SNR^{-d_G(r)}$

$d_G(r) = d_{out}(r)$ if $l \geq m+n-1$

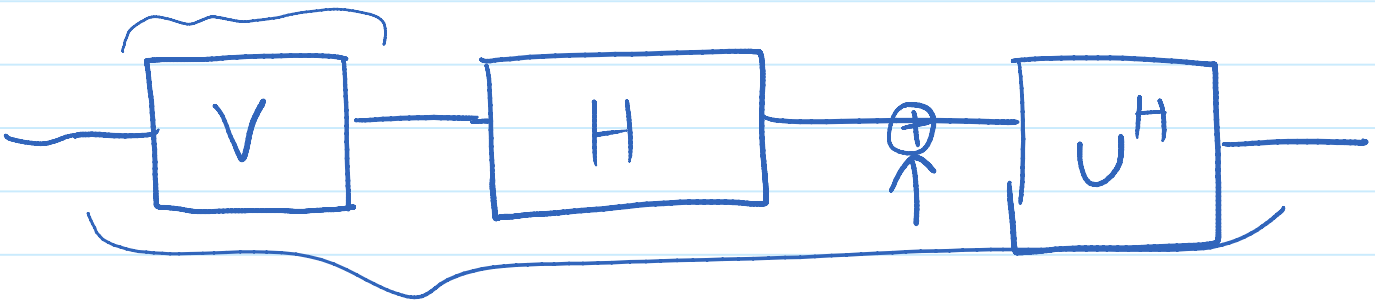
L25

19 September 2019 11:11

Self Study — Space Time Coding: An Overview

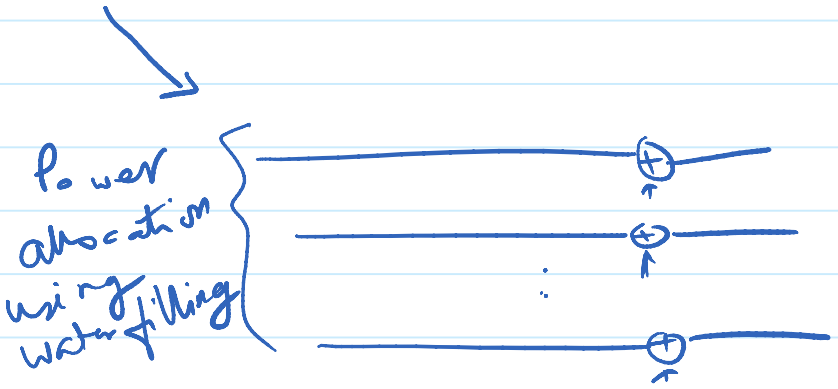
Caire, Elia, Raj Kumar

Go back setting where H is deterministic & known at tx & rx.



$$H = U \Sigma V^H$$

Σ



L 25

19 September 2019 11:18

tx rx

1 × n

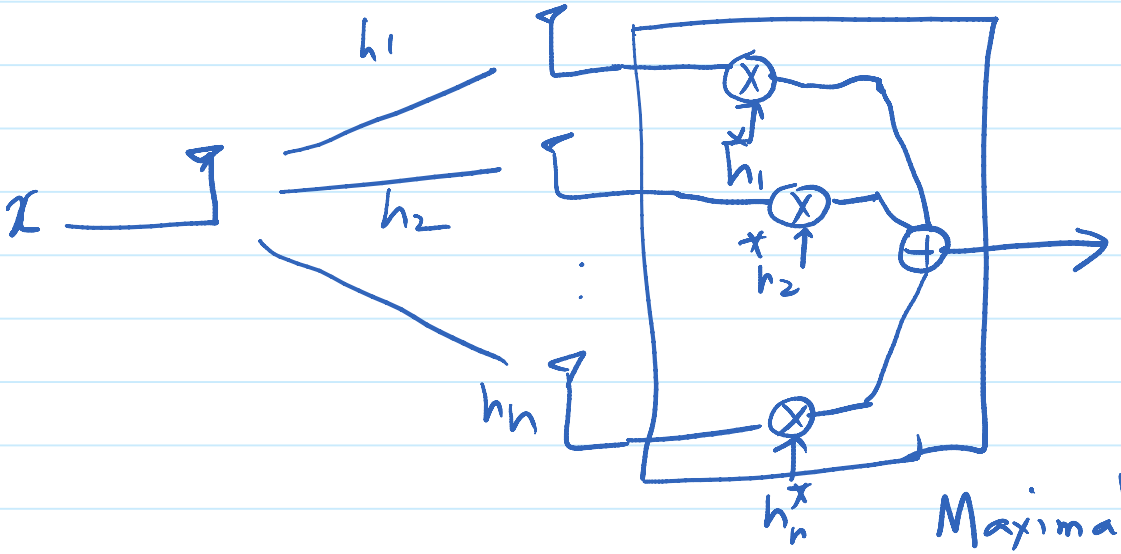
(SIMO)

$$H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}_{n \times 1}$$

$$H = U \Sigma V^H$$

U: $n \times 1$, Σ: $n \times n$, V^H: 1×1

Only one non-zero singular value



Maximal ratio combining (MRC) is optimal in terms of capacity

L25

19 September 2019 11:24

tx rx

m × 1

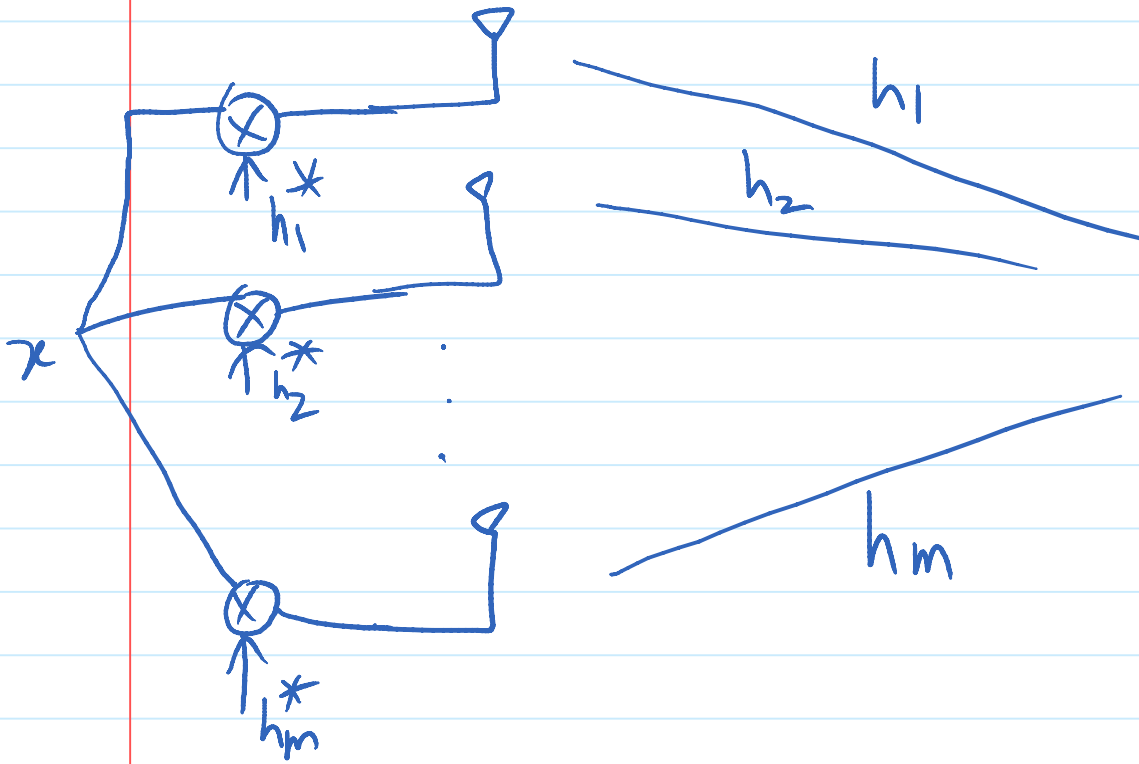
(MISO)

$H_{1\times m} = U \Sigma V^{H}$

1×1 $m\times m$

only one non-zero singular value

$(h_1\ h_2 \cdots h_m)$



Maximal ratio transmission (MRT) is capacity optimal

Tx Power in ant i

$$= \left(\frac{|h_i|^2}{\sum_i |h_i|^2}\right) P$$

sum power constraint

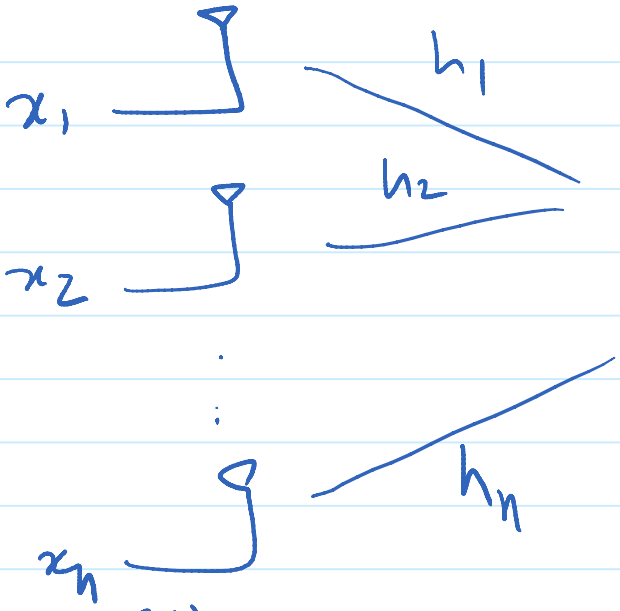
$\propto |h_i|^2$

L25

19 September 2019 11:29

MISO $n \times 1$ with Per antenna power constraint

(Mai Vu 2011 TCOM)



$$y = \underline{h}^T \underline{x} + n$$

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

L25

19 September 2019 11:36

Sum Power constraint: soln

$\left(tr(Q) \le P \right)$

$$\underline{x} = \begin{pmatrix} h_1^* \\ h_2^* \\ \vdots \\ h_n^* \end{pmatrix} \cdot \frac{\sqrt{P}}{\sqrt{\sum_i |h_i|^2}} \; ; \; x \sim CN(0,1)$$

Per Antenna Power Constraint soln:

$$\left(Q = E\left[\underline{x}\,\underline{x}^H\right] \quad Q_{ii} \le P_i \;\; \forall i = 1, 2, \cdots, n \right)$$

$$\underline{x} = \underbrace{\begin{pmatrix} \frac{h_1^*}{|h_1|}\sqrt{P_1} \\ \frac{h_2^*}{|h_2|}\sqrt{P_2} \\ \vdots \\ \frac{h_n^*}{|h_n|}\sqrt{P_n} \end{pmatrix}}_{q} x \qquad Q = q\,q^H$$

Use full power P_i at i^{th} antenna

Phase of x_i = — Phase of h_i

L 25

19 September 2019 11:43

$$\max_{\substack{Q \\ (Q \geq 0 \\ Q_{ii} \leq P_i \forall i)}} \log\left(1 + \frac{\underline{h}^T Q \underline{h}^*}{\sigma^2}\right) \qquad \left(\underline{h}^T \underline{q} \underline{q}^H \underline{h}^*\right)$$

→ First show rank (optimal Q) = 1 $\left(\text{rank}(Q) \leq \text{rank}(H) \atop 1\right)$

→ Write $Q = \underline{q}\underline{q}^H$

Show phase (q_i) = − phase (h_i)

→ Now it remains to find $|q_i|$

Best soln is to use all available power

L25

19 September 2019 11:49

MIMO PAPC

$$\max_{Q} \left[\log \left| I + \frac{HQH^H}{\sigma^2} \right| \right]$$

$$\{Q \succeq 0,\ Q_{ii} \le P_i \ \forall i\}$$

Semi-definite program (convex opt)

(Can be solved numerically

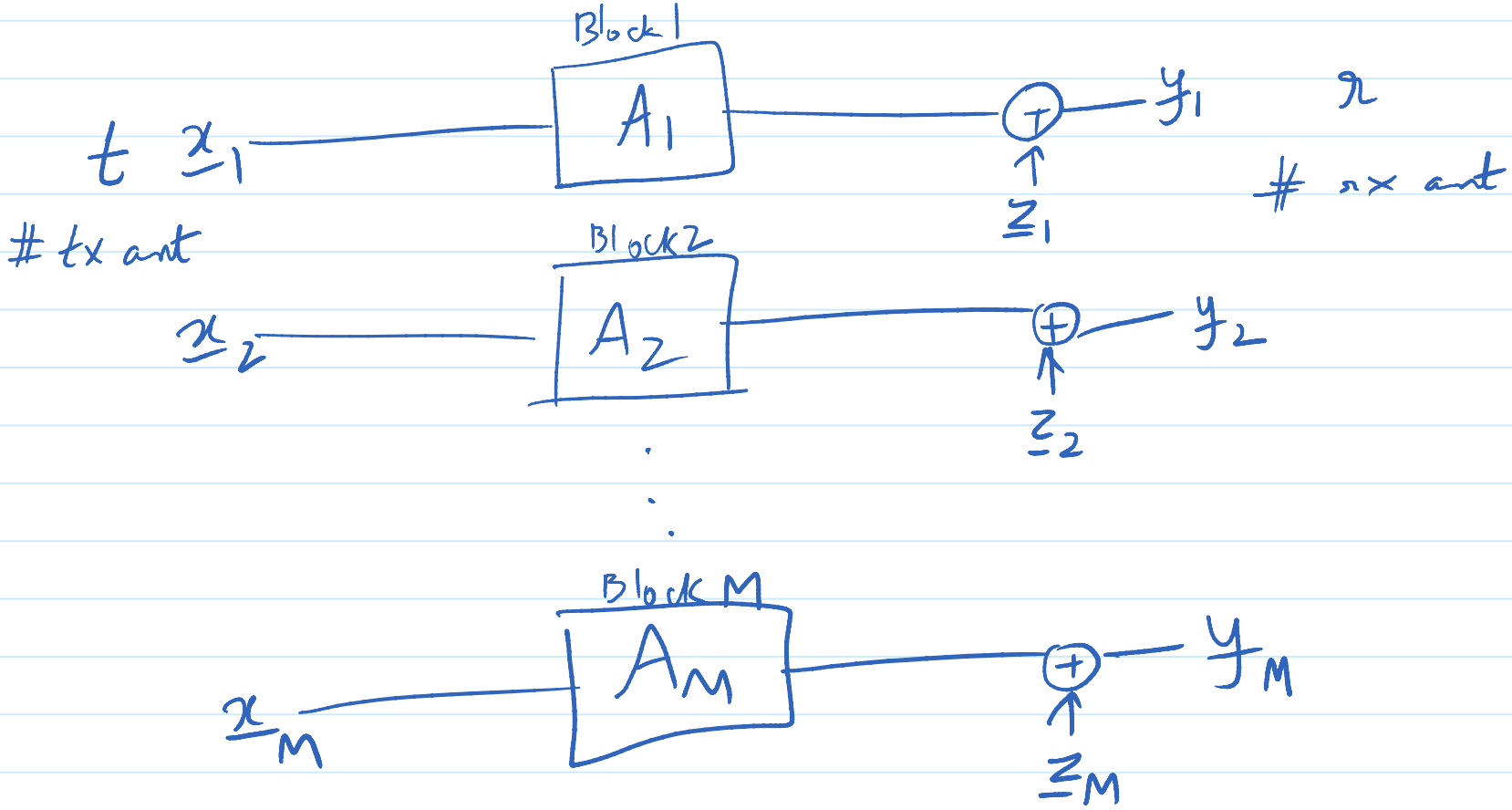
Analytical soln available if (1) MISO (Vu 2011)

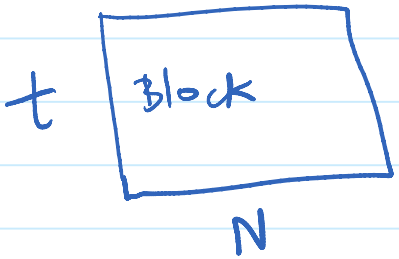
(2) H is full rank & opt Q is also full rank (Tuninetti 2014)

(3) $2 \times n_r$

Lecture 26

20 September 2019 09:57



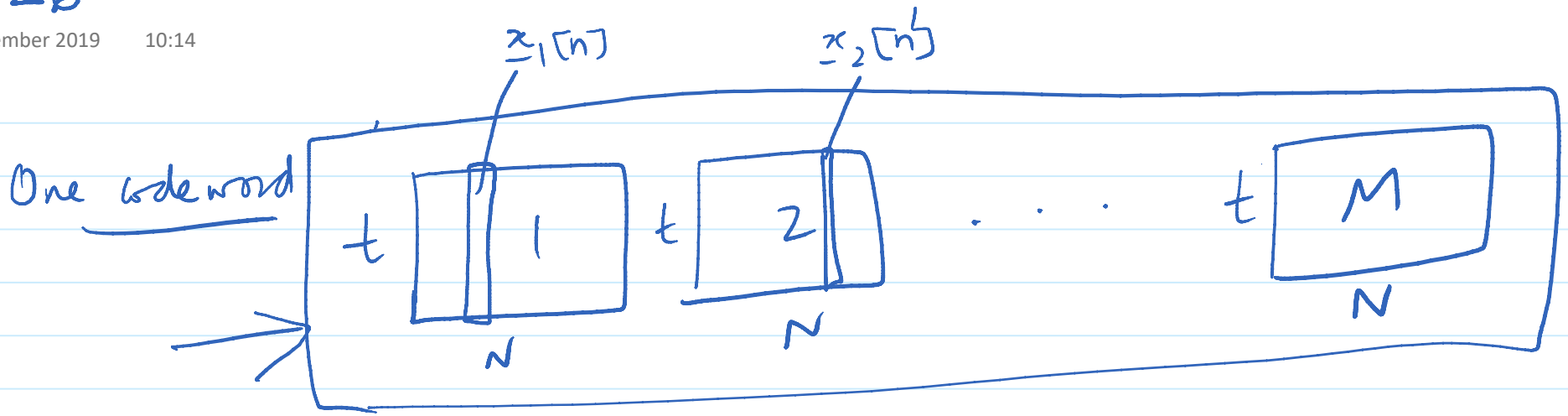


channel constant in each block of N sym

i.i.d. b/w blocks

L26

20 September 2019 10:14



Different power constraints

tr. Cov. matrix for each block $E\left[\underline{x}_k[n]\,\underline{x}_k^H[n]\right] \triangleq Q_k$

Power constraint over M blocks $\frac{1}{M}\sum_{k=1}^{M} \mathrm{tr}(Q_k) \leq \gamma$

L26

20 September 2019 10:25

→ $M \times m$ parallel channels

($m = \min(t, r)$)

→ Power allocation between blocks is now possible based on CSIT, (Everything else remains same as fixed H SVD based parallel channels)

For no CSIT case, $Q = \alpha I$

Lecture 27

23 September 2019 07:53

$$\gamma_{k,i} = \left(\xi - \frac{1}{\lambda_{k,i}}\right)^+$$

$$\frac{1}{M}\sum_{i=1}^{m}\sum_{k=1}^{M}\gamma_{k,i} = \gamma$$

Ergodic case (No delay constraints)

As $M \to \infty$

$$\frac{1}{M}\sum_{k=1}^{M}\left(\xi - \frac{1}{\lambda_{k,i}}\right)^+ \to E\left[\left(\xi - \frac{1}{\lambda_i}\right)^+\right]$$

(Ergodic channel)

$$\sum_{i=1}^{m}\left(\frac{1}{M}\sum_{k=1}^{M}\left(\xi - \frac{1}{\lambda_{k,i}}\right)^+\right) = \gamma$$

$$\sum_{i=1}^{m} E\left[\left(\xi - \frac{1}{\lambda_i}\right)^+\right] = \gamma.$$

L27

23 September 2019 08:06

$$I_M(\Lambda, \Gamma) \triangleq \frac{1}{M} \sum_{k=1}^{M} \sum_{i=1}^{m} \log_2(1 + \lambda_{k,i} \gamma_{k,i}) \qquad \text{(rate per block)}$$

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{M,1} \\ \vdots & & \vdots \\ \lambda_{1,m} & \cdots & \lambda_{M,m} \end{pmatrix} \qquad n = \min(t, r)$$

$\lambda_{1,m}$ → block 1 ; $\lambda_{M,m}$ → block M

$$\Gamma = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{M,1} \\ \vdots & & \vdots \\ \gamma_{1,m} & \cdots & \gamma_{n,m} \end{pmatrix}$$

Power allocation

L27

23 September 2019 08:10

Short term power constraint: $P_M(\Gamma)$

$$\frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m} \gamma_{k,i} \leq \gamma$$

Long term power constraint:

$$E\left[P_M(\Gamma) \right] \leq \gamma$$

$$P_{out}(R,\gamma) \triangleq P_r\left(I_M(\Lambda,\Gamma) < R \right)$$

ϵ-outage capacity

$$C_\epsilon(\gamma) \triangleq \sup_{\Gamma(\Lambda)} \sup \left\{ R : P_{out}(R,\gamma) \leq \epsilon \right\}$$

Delay limited capacity

$$C_{delay}(\gamma) = \lim_{\epsilon \to 0} C_\epsilon(\gamma).$$

L 27

23 September 2019 08:14

Results

(1) optimal scheme (short-term)

$$\Gamma(\Lambda) = \begin{cases} \Gamma^{st}(\Lambda, \gamma) & \text{if } \Lambda \in \mathcal{R}_{in}(R, \gamma) \\ 0 & \text{if } \Lambda \in \mathcal{R}_{out}(R, \gamma) \end{cases}$$

Do not transmit for this set of channel realizations

$$\gamma_{k,i}^{st} = \left(\xi^{st}(\Lambda, \gamma) - \frac{1}{\lambda_{k,i}} \right)^+$$

$$I_M(\Lambda, \Gamma^{st}(\Lambda, \gamma)) \geq R$$

$$\downarrow$$

$$I_M(\Lambda, \Gamma^{st}) < R$$

L27

23 September 2019 08:20

2) optimal scheme (long term)

$$\Pi(\Lambda) = \begin{cases} \Pi^{lt}(\Lambda, R) & \text{if } \Lambda \in R^{*}_{on}(R, \gamma^{*}) \\ 0 & \text{if } \Lambda \in R^{*}_{off}(R, \gamma^{*}) \end{cases}$$

`$R_{off}$ in st. case' $\equiv$ `R_{off} in l.t. case' with a diff γ

L27

23 September 2019 08:27

Regular fading $\quad E\left[\frac{1}{\overline{\lambda}_M}\right] < \infty$

$\overline{\lambda}_M \triangleq$ geometric mean of $\lambda_{1,k,i}$'s

$$= \left(\prod_{k,i} \lambda_{k,i}\right)^{\frac{1}{MM}}$$

$\rightarrow C_{delay}(\gamma) > 0$ for every $\gamma > 0$ for a regular BF-AWGN channel.

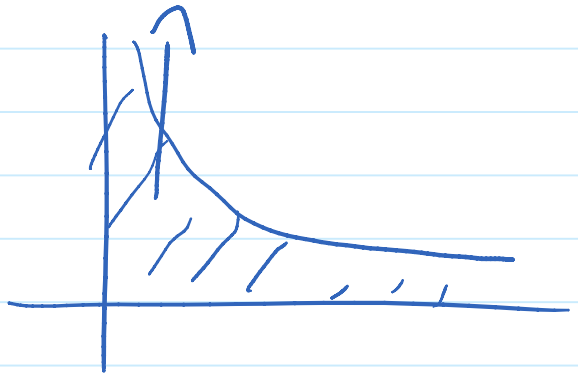
L27

23 September 2019 08:29

If $M = m = 1$, (Rayleigh fading) (1-block SISO)

$$C_{delay}(\gamma) = 0$$

$$E\left[\frac{1}{\lambda_1}\right] = \int_0^\infty \frac{1}{x} e^{-x} dx \quad \text{diverges}$$



If $Mm > 1$ Rayleigh $E\left[\frac{1}{\lambda_m}\right] < \infty$

$$C_{delay}(\gamma) \sim m \left[\log \frac{\gamma}{mE\left[\frac{1}{\lambda_m}\right]}\right] \quad \text{as } \gamma \to \infty$$

L27

23 September 2019 08:38

→ A suboptimal scheme based on 'channel inversion'

$$\overline{\gamma_{k,i}} = \frac{2^{R/m}}{\overline{\lambda_M}} \quad \forall\, k,i$$

close to optimal as $R \to \infty$.

→ Propositions 7 & 8 for 'large no. of antennas' case

$$\frac{C_{delay}(\gamma)}{m} \longrightarrow \frac{C_{erg}(\gamma)}{m} \quad \text{(No delay penalty)}$$

→ See Figs. 2 – 7

Lecture 28

24 September 2019 12:04

Lattice-reduction-aided detectors for MIMO Communication Systems (Yao & Wornell 2002)

$$\hat{\underline{x}} = \arg\min_{\underline{x}} \|\underline{y} - H\underline{x}\|^2 \quad \text{(ML detection)}$$

Suboptimal methods

— V-BLAST : ZF-SIC, ZF

$$\underline{y} = H\underline{x} + \underline{n}$$

(ZF) $H^{-1}\underline{y} = \underline{x} + H^{-1}\underline{n}$ ⟶ detect each element of $\underline{x}$ individually

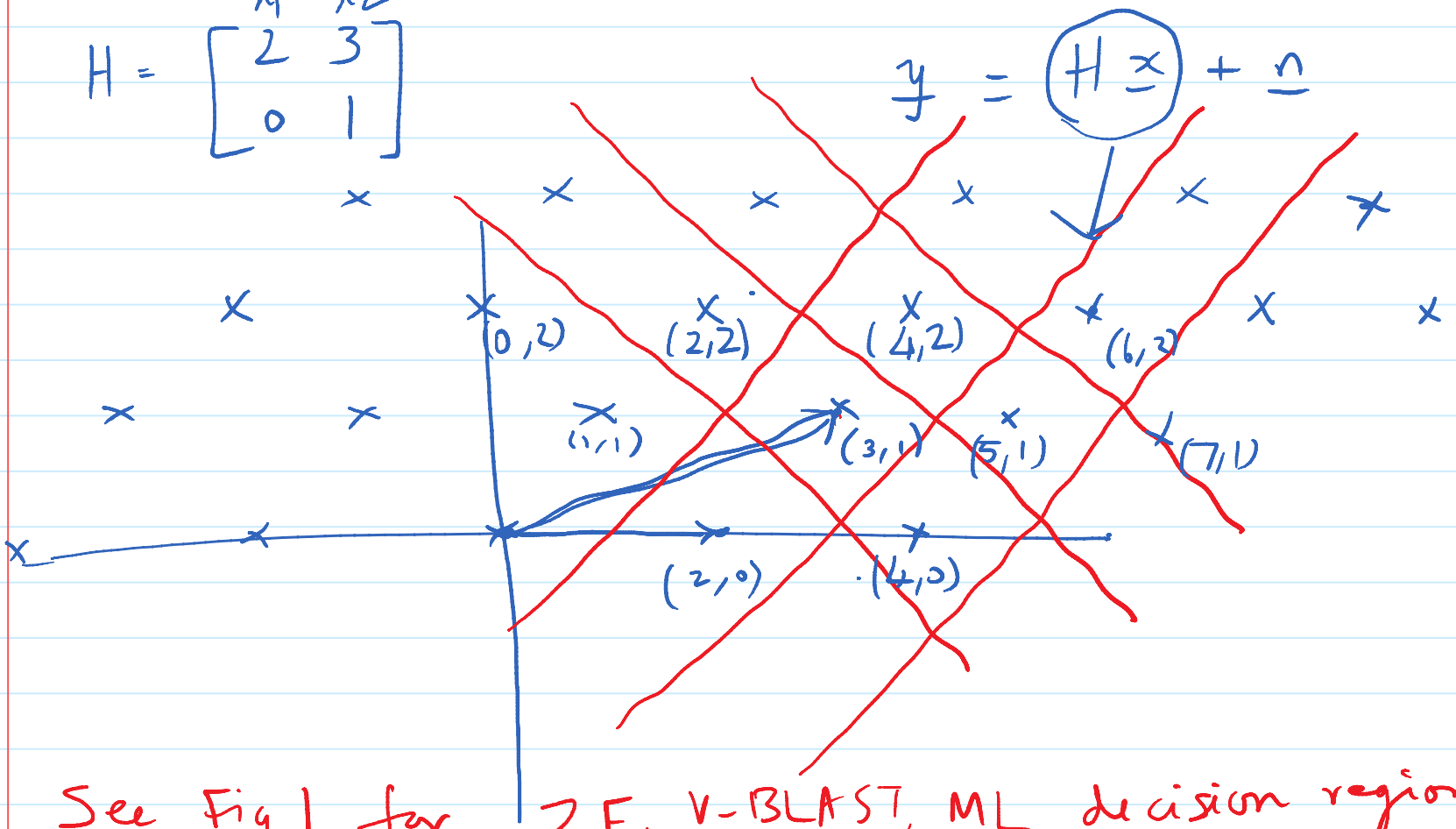
L28

24 September 2019 12:10

x_1, x_2 integers $-N$ to N, N large

$$H = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{matrix} x_1 & x_2 \end{matrix}$$

$$\underline{y} = H\underline{x} + \underline{n}$$



- ML decision boundaries

Lattice

See Fig 1 for ZF, V-BLAST, ML decision regions

L28

24 September 2019 12:30

If we were to consider the lattice basis vectors to be $[1 \ 1]^T$ & $[1 \ -1]^T$ instead of $[2 \ 0]^T$ and $[3 \ 1]^T$, ZF ≡ MLD for this example.

Finding an optimal basis : Lattice Reduction problem.

$$\underline{y} = H\underline{x} + \underline{n}$$

$$= (HP)(P^{-1}\underline{x}) + \underline{n}$$

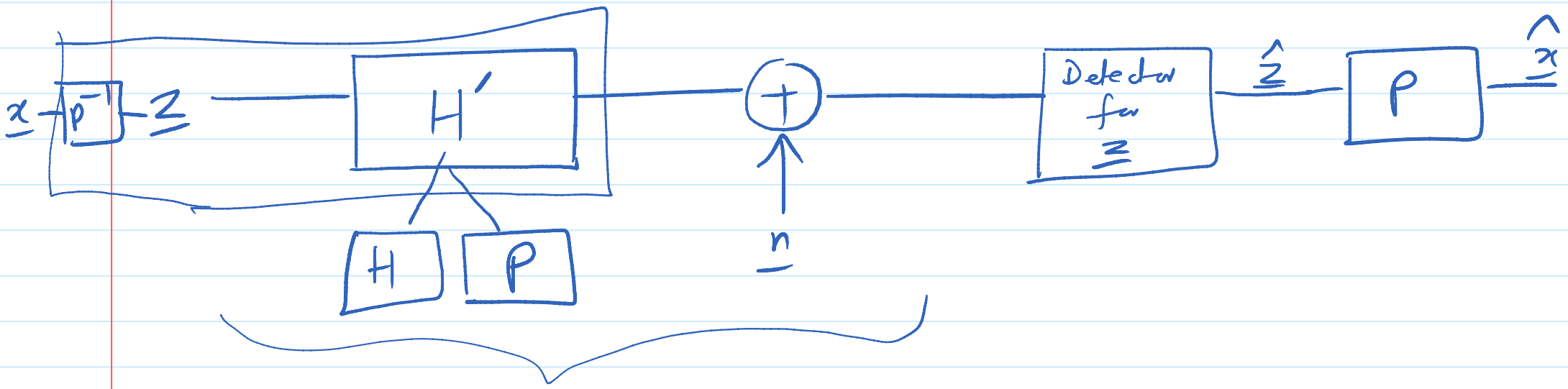
$$= (HP)\underline{z} + \underline{n}$$

where P, P^{-1} have integer entries

Decode $\underline{z}$ first using ZF $((HP)^{-1}\underline{y})$, $\underline{\hat{x}} = P\underline{\hat{z}}$.

L28

24 September 2019 12:34



How to find this P?

Lecture 29

26 September 2019 11:02

* A lattice in n complex dimensions

$$\mathcal{L} = \{ \underline{s} \mid \underline{s} = B\underline{\lambda} \}$$

where $B = [\underline{b}_1 \; \underline{b}_2 \; \cdots \; \underline{b}_n]$ Basis vectors of the lattice

& $\underline{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ vector of complex integer weights

$\lambda_i \in \mathbb{Z} + \mathbb{Z}j$

* For any lattice $\mathcal{L}$, there are many possible bases.

If B is a basis, and B' is also a basis, then

$B' = B(P)$ where P has integer entries

L 29

26 September 2019 11:11

$$\underline{\underline{B}} = \underline{\underline{B'}}\,\underline{\underline{P}}$$ ($\underline{\underline{P}}$: integer entries)

$\underline{b}_1\ \underline{b}_2\ \ldots\ \underline{b}_n$

$\underline{b}_1'\ \underline{b}_2'\ \ldots\ \underline{b}_n'$

$$\underline{b}_1 = \sum_i \alpha_i \underline{b}_i'$$ (α_i: integers)

If P and P^{-1} have integer entries, then

$$\det(P) = \pm 1 \qquad (P \text{ is unimodular})$$

[$\det(P)$ is an integer

$\det(P^{-1})$ is also an integer } $\Rightarrow \det(P) = \pm 1$]

$$\det(P^{-1}) = \frac{1}{\det(P)}$$

L 29

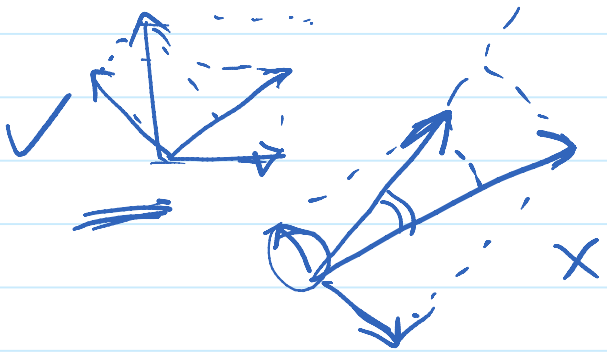
26 September 2019 11:16

* Can we find $B' = BP$ that is better for a given detector?

* $\underline{y} = H\underline{x} + \underline{n}$

* 2 x 2 case

$$H = [\underline{b}_1 \; \underline{b}_2]$$



Let $\underline{\tilde{b}}_1$ be the component of $\underline{b}_1$ orthogonal to $\underline{b}_2$

Let $\underline{\tilde{b}}_2$ —"— $\underline{b}_2$ —"— $\underline{b}_1$

ZF: Effective SNR for $x_1 \propto \|\underline{\tilde{b}}_1\|^2$

—"— $x_2 \propto \|\underline{\tilde{b}}_2\|^2$

$\min\left(\|\underline{\tilde{b}}_1\|^2, \|\underline{\tilde{b}}_2\|^2\right)$ will determine performance

Try to maximize $\min\left(\|\underline{\tilde{b}}_1\|^2, \|\underline{\tilde{b}}_2\|^2\right)$

L 29

26 September 2019 11:22

For BLAST: (order x_2, x_1)

Eff. SNR for x_2: $\propto \|\tilde{b}_2\|^2$

Eff. SNR for x_1: $\propto \|b_1\|^2$ (assuming no error propagation)

Maximize $\min\left(\|\tilde{b}_2\|^2, \|b_1\|^2\right)$

Result: Let $H = [h_1\ h_2]$. Let Λ denote the lattice generated by H. Let u be the shortest vector in Λ. Let v be the shortest vector that is not proportional to u in Λ. Then

$$\min\left\{\|u\|, \|\tilde{v}\|\right\} \geq \min\left\{\|g_1\|, \|g_2\|\right\}$$

$\tilde{v}$ is component of v orthogonal to u $\quad G = [g_1\ g_2]$ any other basis for Λ

L 29

26 September 2019 11:29

Find $\underline{u}, \underline{v}$ to find the best basis for BLAST.

Result: (Gives a property of $\underline{u}, \underline{v}$)

Given a 2-D lattice with basis vectors $\underline{u}, \underline{v}$:

$$\begin{cases} \text{if } \|\underline{u}\| \leq \|\underline{v}\|, \\ |\mathrm{Re}\langle \underline{u}, \underline{v} \rangle| \leq \frac{1}{2}\|\underline{u}\|^2, \\ |\mathrm{Im}\langle \underline{u}, \underline{v} \rangle| \leq \frac{1}{2}\|\underline{u}\|^2, \end{cases}$$

An algorithm to satisfy these constraints

then $\underline{u}$ is the shortest (non-zero) vector in Λ

& $\underline{v}$ is the shortest vector that is not a multiple of $\underline{u}$.

L29

26 September 2019 11:36

Algorithm

Consider $B = [\underline{b}_1, \underline{b}_2]$ with $\|\underline{b}_1\| \leq \|\underline{b}_2\|$

$$\underline{v}_1 = \underline{b}_1$$

$$\underline{v}_2 = \underline{b}_2 - t\,\underline{b}_1 \qquad \hat{t} = \left\lceil \frac{\langle \underline{b}_2, \underline{b}_1 \rangle}{\|\underline{b}_1\|^2} \right\rfloor$$

round to nearest integer

$$[\underline{v}_1 \;\; \underline{v}_2] = [\underline{b}_1 \;\; \underline{b}_2] \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

$t = \hat{t}$ minimizes $\|\underline{v}_2\|^2 = \|\underline{b}_2 - t\,\underline{b}_1\|^2$

L29

26 September 2019 11:41

Rounding error is $\leq \frac{1}{2}$ for both Re( ) & Im( ) of t.

$$|\mathrm{Re}\langle \underline{v}_1, \underline{v}_2\rangle| \leq \frac{1}{2}\|\underline{v}_1\|^2$$

$$|\mathrm{Im}\langle \underline{v}_1, \underline{v}_2\rangle| \leq \frac{1}{2}\|\underline{v}_1\|^2$$

Final algorithm: Inputs $\underline{b}_1, \underline{b}_2$ with $\|\underline{b}_1\| \leq \|\underline{b}_2\|$ (H)

1) Check correlation. If $|\mathrm{Re}\langle \underline{b}_1, \underline{b}_2\rangle| \leq \frac{1}{2}\|\underline{b}_1\|^2$

$|\mathrm{Im}\langle \underline{b}_1, \underline{b}_2\rangle| \leq \frac{1}{2}\|\underline{b}_1\|^2$, stop

else $\underline{b}_2 = \underline{b}_2 - \left\lceil \frac{\langle \underline{b}_2, \underline{b}_1\rangle}{\|\underline{b}_1\|^2} \right\rfloor \underline{b}_1$

2) check $\|\underline{b}_2\| > \|\underline{b}_1\|$. If so, stop. else swap $\underline{b}_1, \underline{b}_2$. Go to step 1.

L29

26 September 2019 11:47

$$H = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$\underline{b}_1 \quad \underline{b}_2$

$\|\underline{b}_1\|^2 = 4$

$\|\underline{b}_2\|^2 = 10$

$$\langle \underline{b}_2, \underline{b}_1 \rangle = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 6$$

$$\frac{\langle \underline{b}_2, \underline{b}_1 \rangle}{\|\underline{b}_1\|^2} = \frac{3}{2}$$

$\underline{v}_1 = \underline{b}_1$

$$\underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

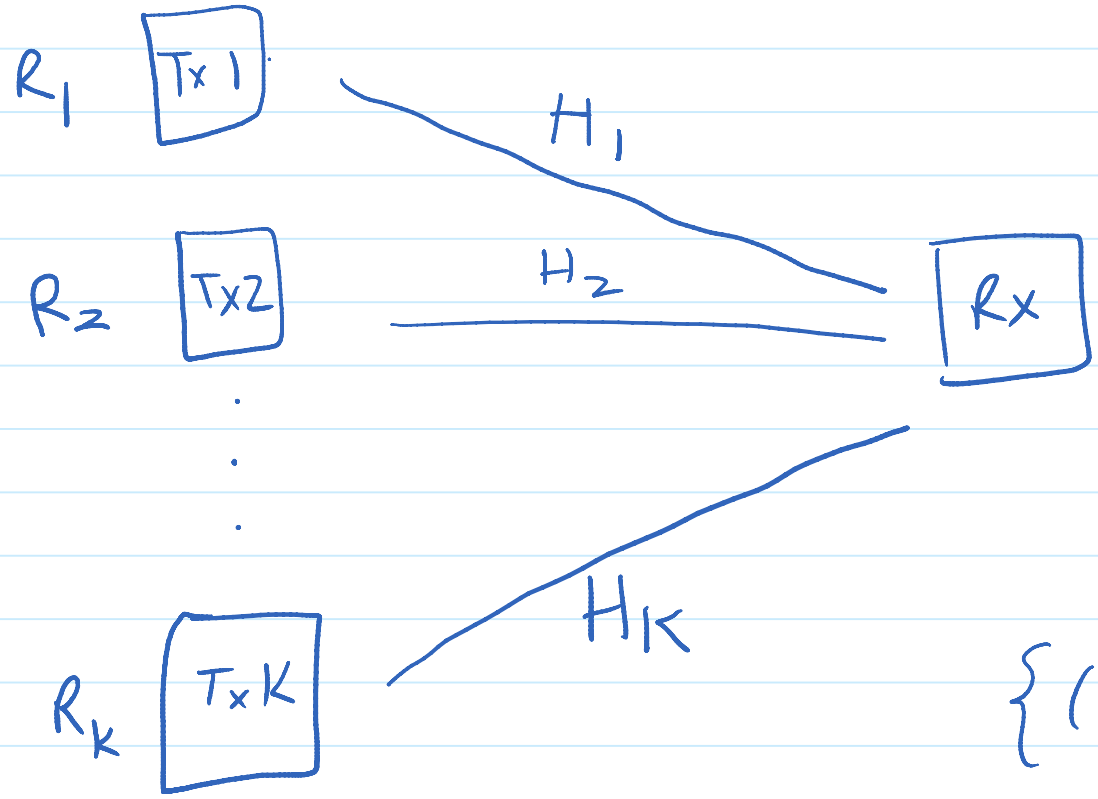
$\|\underline{v}_2\|^2 = 2$ sur $\underline{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\underline{b}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\langle \underline{b}_2, \underline{b}_1 \rangle = -2$

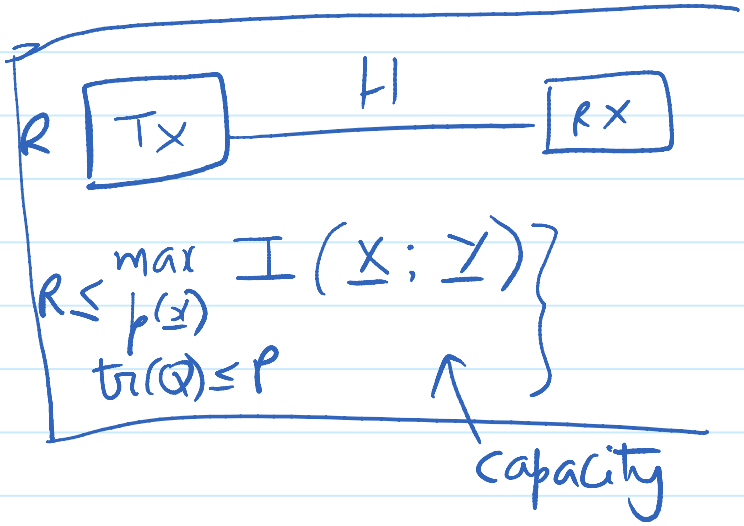
$$\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Lecture 30

27 September 2019 09:55

Multiple Access Channel





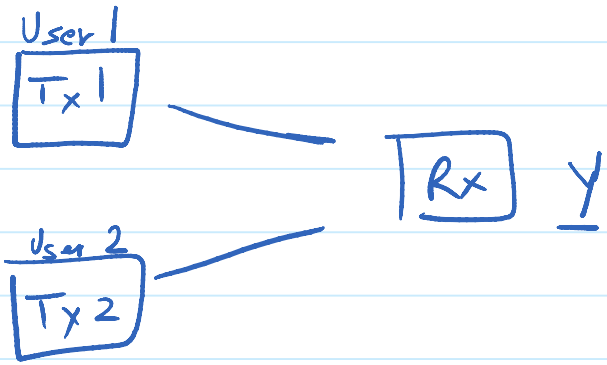
Capacity region

$$\{(R_1, R_2, \ldots, R_K) : \text{conditions}\}$$

L30

27 September 2019 10:07

$K=2$



Three types of inequalities

$R_1 < (\quad)$

$R_2 < (\quad)$

$R_1 + R_2 < (\quad)$

$$\underline{Y} = H_1 \underline{X}_1 + H_2 \underline{X}_2 + \underline{Z}$$

(MIMO Gaussian MAC)

2-user

K-user

$$\underline{y} = \sum_{i=1}^{K} H_i \underline{X}_i + \underline{Z}$$

Per-user Power constraints

$$E[\underline{X}_i \underline{X}_i^H] \triangleq S_i \qquad \mathrm{tr}(S_i) \le P_i \qquad i = 1, 2, \ldots, K$$

Encoders are independently encoding / not cooperating ($\underline{X}_i$'s independent)

L30

27 September 2019 10:12

For a given set of transmit covariance matrices $S_1, S_2, \ldots, S_K$, Gaussian distributed inputs are optimal.

K=2

$$\left.\begin{aligned} R_1 &< I(\underline{X}_1; \underline{Y}/\underline{X}_2) \\ R_2 &< I(\underline{X}_2; \underline{Y}/\underline{X}_1) \\ R_1+R_2 &< I(\underline{X}_1, \underline{X}_2; \underline{Y}) \end{aligned}\right\}$$

$\underline{X}_i$'s Gaussian, independent

$$\underline{Y} = H_1\underline{X}_1 + H_2\underline{X}_2 + \underline{Z}$$

$$\left(\begin{array}{l} \underline{X}_i \sim N(\underline{0}, S_i) \quad \underline{X}_1, \underline{X}_2, \underline{Z} \text{ indep} \\ \underline{Z} \sim N(\underline{0}, S_Z) \end{array}\right)$$

$$I(\underline{X}_1; \underline{Y}/\underline{X}_2)$$

$$= I(\underline{X}_1; H_1\underline{X}_1 + \underline{Z})$$

$$= \frac{1}{2}\log\frac{|H_1 S_1 H_1^T + S_Z|}{|S_Z|}$$

L 30

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$$I(\underline{X}_1, \underline{X}_2 ; \underline{Y}) = h(\underline{Y}) - h(\underline{Y}/\underline{X}_1, \underline{X}_2)$$

$$= h(\underline{Y}) - h(\underline{Z})$$

$$= \frac{1}{2} \log \left((2\pi e)^n \left| H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_Z \right| \right)$$

$$- \frac{1}{2} \log \left((2\pi e)^n \left| S_Z \right| \right)$$

$$= \frac{1}{2} \log \frac{\left| H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_Z \right|}{\left| S_Z \right|}$$

L30

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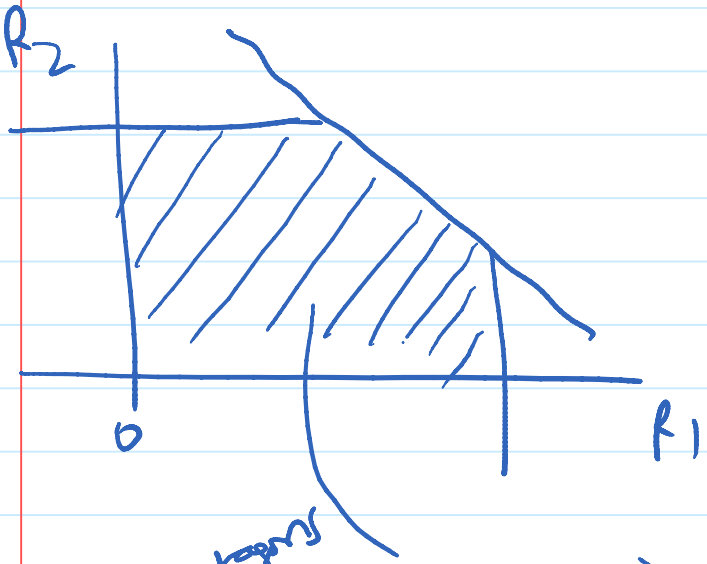
Let $B(S_1, S_2) = \Big\{ (R_1, R_2) : R_1 < \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + S_2|}{|S_2|}$

$R_2 < \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_2|}{|S_2|}$

$R_1 + R_2 < \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_2|}{|S_2|} \Big\}$



Using pentagons

$B(S_1, S_2)$ is a pentagon.

$A(q_1, q_2) = \bigcup_{S_1, S_2 : \mathrm{tr}(S_1) \le q_1, \mathrm{tr}(S_2) \le q_2, S_1, S_2 \succcurlyeq 0} B(S_1, S_2)$

L 30

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Capacity Region

$$\mathcal{C}(P_1, P_2) = \underline{\text{closure}} \left\{ (R_1, R_2) : (R_1, R_2) \in \underline{\text{convex hull}} \left(\bigcup_{q_1, q_2} A(q_1, q_2) \right) \right\}$$

(closure: redundant)

$$= \underline{A(P_1, P_2)} = \bigcup_{S_1, S_2} B(S_1, S_2)$$

$$\left\{ \begin{array}{l} tr(S_1) \le P_1 \\ tr(S_2) \le P_2 \\ S_1, S_2 \ge 0 \end{array} \right\}$$

L30

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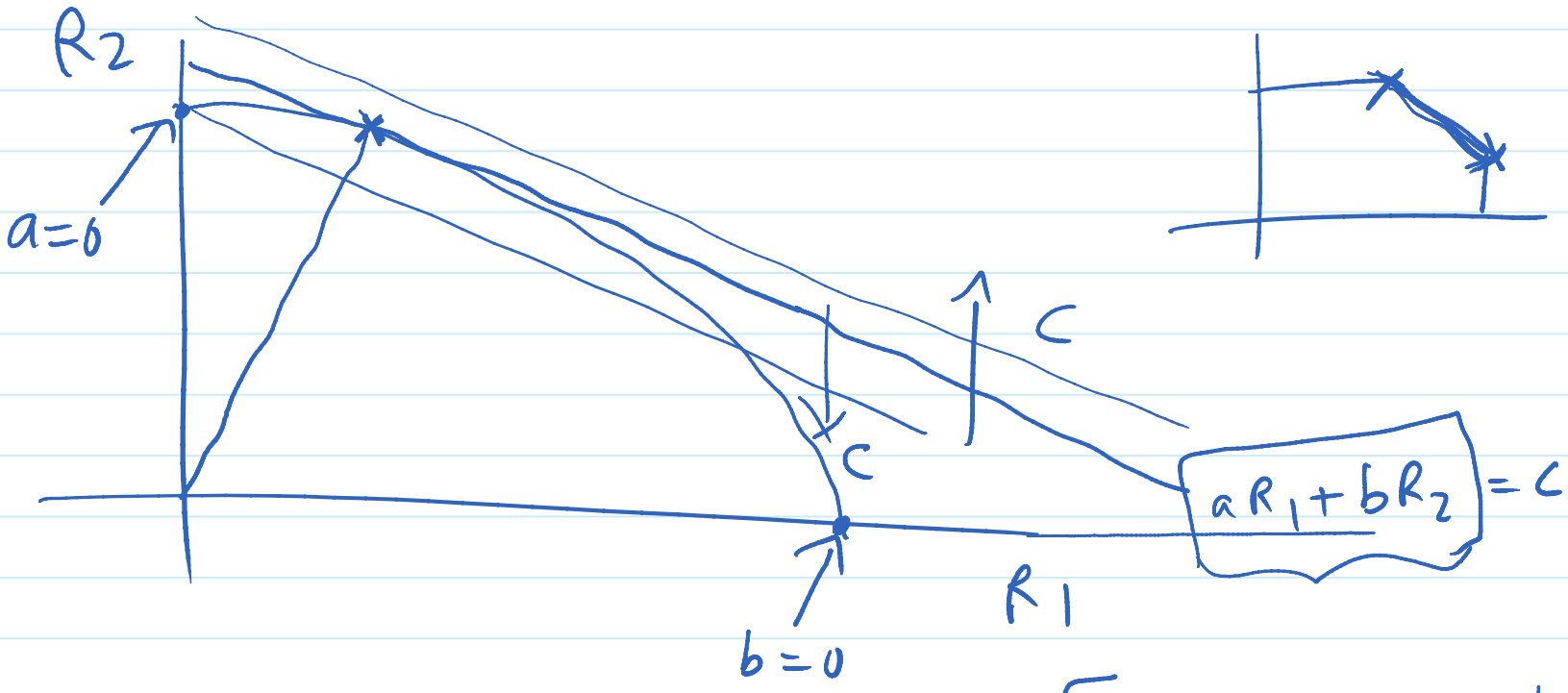


$$\left[\begin{array}{l}\max \; aR_1+bR_2 \\ (R_1,R_2)\in A(P_1,P_2)\end{array}\right]$$

Simplify using "$A(P_1,P_2)$ is a union of pentagons"

$$\max_{S_1,S_2\geq 0} \left[\quad\right]$$

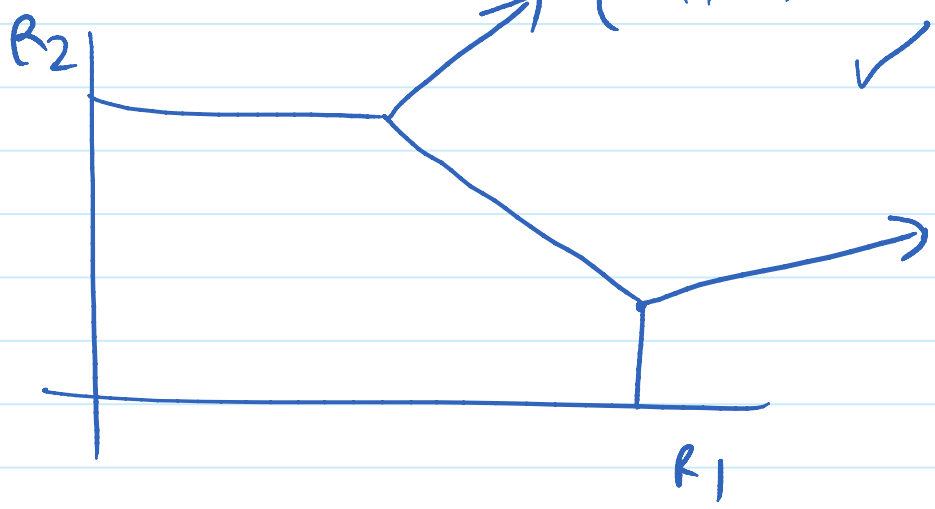
$$tr(S_1)\leq P_1$$
$$tr(S_2)\leq P_2$$

$a=b$: Sum rate maximization

$a\neq b$: Weighted sum rate maximization

L30

27 September 2019 10:47



$$(R_1, R_2) = \left(\frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_2|}{|S_2|} - R_2, \right.$$
$$\left. \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_2|}{|S_2|} \right)$$

$$(R_1, R_2) = \left(\frac{1}{2} \log \frac{|H_1 S_1 H_1^T + S_2|}{|S_2|}, \right.$$
$$\left. \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_2|}{|S_2|} - R_1 \right)$$

Lecture 31

01 October 2019 12:03

Weighted sum rate optimization problem:

$$\max_{(R_1,\dots,R_K)\in A(P_1,P_2,\dots,P_K)} \sum_{i=1}^{K} \mu_i R_i$$

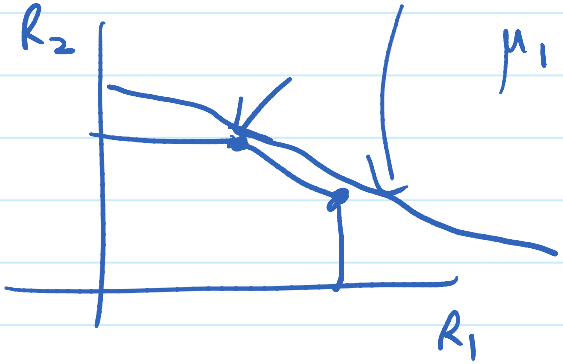
Users are indexed such that

$\mu_i \geq 0$

$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$

$\mu_1 R_1 + \mu_2 R_2 = c$

$\mu_1 \leq \mu_2$



$K=2$

$\mu_1 \leq \mu_2$ optimal (R_1, R_2) for a given pentagon specified by S_1, S_2.

$$\left(\frac{1}{2}\log\frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_Z|}{|S_Z|} - \frac{1}{2}\log\frac{|H_2 S_2 H_2^T + S_Z|}{|S_Z|}, \frac{1}{2}\log\frac{|H_2 S_2 H_2^T + S_Z|}{|S_Z|} \right)$$

L31

01 October 2019 12:15

$$\max_{S_1, S_2 \succeq 0,\ \mathrm{tr}(S_1) \le P_1,\ \mathrm{tr}(S_2) \le P_2} \mu_1 \left(\frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_2|}{|S_2|} - \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_2|}{|S_2|} \right) + \mu_2 \left(\frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_2|}{|S_2|} \right)$$

$$\max_{S_1, S_2 \succeq 0,\ \mathrm{tr}(S_1) \le P_1,\ \mathrm{tr}(S_2) \le P_2} \mu_1 \left(\frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_2|}{|S_2|} \right) + (\mu_2 - \mu_1) \left(\frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_2|}{|S_2|} \right)$$

2-user problem (2 corner points, once we fix $\mu_1 \le \mu_2$, we know which corner pt is the optimal)

L31

01 October 2019 12:17

K-user problem: $K!$ corner points → also corresponds to a decoding order

For a given $\mu_1, \cdots, \mu_K$ one of them is optimal

$$\max_{S_1, \cdots S_K \geq 0, \ tr(S_1) \leq P_1, \ \cdots, \ tr(S_K) \leq P_K} \mu_1 \left(\frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_Z \right| \right) - \mu_K \frac{1}{2} \log |S_Z| + \sum_{j=2}^{K} (\mu_j - \mu_{j-1}) \frac{1}{2} \log \left| \sum_{i=j}^{K} H_i S_i H_i^T + S_Z \right|$$

Sum rate problem ($\mu_1 = \mu_2 = \cdots = \mu_K$)

$$\max_{S_1, \ldots, S_K \geq 0, \ tr(S_i) \leq P_i \ \forall i} \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_Z \right| - \frac{1}{2} \log |S_Z|$$

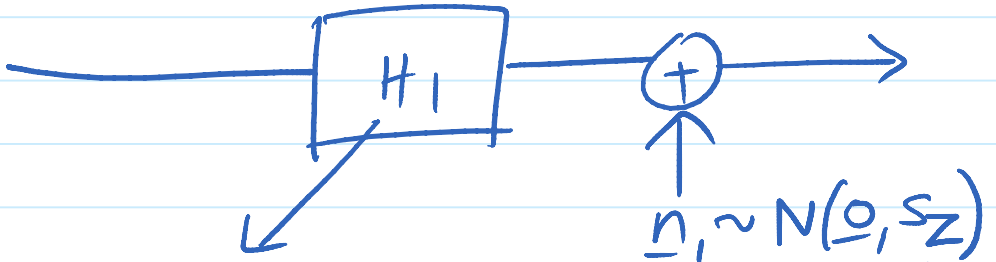
} Iterative waterfilling solution

L 3|

01 October 2019 12:30

Single user case

$$\max_{S_1 \succeq 0,\ \mathrm{tr}(S_1) \le P_1} \frac{1}{2} \log \left| H_1 S_1 H_1^T + S_2 \right| - \frac{1}{2} \log |S_2|$$



$H_1 =$

$Q \Delta Q^T$

$$\frac{1}{2} \log \frac{\left| H_1 S_1 H_1^T + Q \Delta Q^T \right|}{\left| Q \Delta Q^T \right|} = \frac{1}{2} \log \frac{\left| Q^T H_1 S_1 H_1^T Q + \Delta \right|}{|\Delta|} \quad Q Q^T = \underline{I}$$

L31

01 October 2019 12:34

$$= \frac{1}{2} \log \left| \underbrace{\Delta^{-1/2} Q^T H_1 S_1}_{\triangleq \hat{H}} \underbrace{H_1^T Q \Delta^{-1/2}}_{\hat{H}^T} + I \right|$$

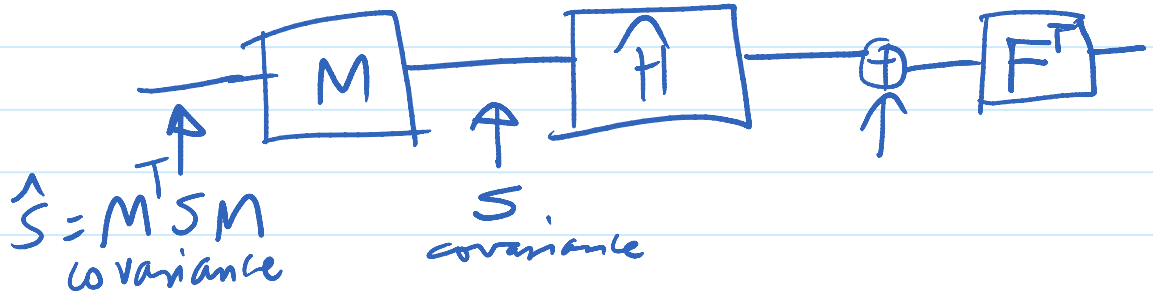
$$= \frac{1}{2} \log \left| \hat{H} S_1 \hat{H}^T + I \right|$$

[Solution for colored noise S_Z & channel H_1

$\equiv$ Solution for white noise I & channel $\hat{H} = \Delta^{-1/2} Q^T H$]

SVD of $\hat{H} = F \Sigma M^T$

$\hat{S} \triangleq M^T S M$



L 31

01 October 2019 12:39

$$\max_{\hat{S} \succeq 0,\ \mathrm{tr}(\hat{S}) \le P_1} \frac{1}{2} \log \left| \Sigma \hat{S} \Sigma^T + I \right| \longrightarrow \hat{S} \text{ diagonal}$$

$$\begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_r \end{bmatrix}$$

$p_1, \ldots, p_r$ found using waterfilling.

Main result (Thm 1)

Lecture 32

03 October 2019 10:59

Thm 1: In a K-user MAC, $\{S_i\}$ is an optimal solution to the sum-rate maximization problem

$$\max_{\{S_i\}} \ \frac{1}{2}\log\left|\sum_{i=1}^{K} H_i S_i H_i^T + S_Z\right| - \frac{1}{2}\log|S_Z| \quad \longleftarrow$$

$$S_i \geq 0$$

$$\mathrm{tr}(S_i) \leq P_i \qquad i = 1, 2, \cdots, K$$

if and only if S_i is the single-user waterfilling solution for channel H_i with $\sum_{\substack{j=1 \\ j \neq i}}^{K} H_j S_j H_j^T + S_Z$ as the noise covariance matrix for each i.

Proof: ('only if' part) sum-rate optimal $\{S_i\} \Rightarrow$ single-user waterfilling optimal $\{S_i\}$

Proof by contradiction: $\{S_i\}$ is sum-rate optimal

Let there be an S_i which is not single-user waterfilling optimal.

Change this S_i to the waterfilling optimal covariance

[Argue that this increases the sum rate]

$\Rightarrow$ Contradiction

L32

03 October 2019 11:11

$$\max_{S_i} \quad \frac{1}{2} \log \frac{\left| H_i S_i H_i^T + \left(\sum_{\substack{j=1 \\ j \neq i}}^{K} H_j S_j H_j^T + S_Z \right) \right|}{\left| \sum_{\substack{j=1 \\ j \neq i}}^{K} H_j S_j H_j^T + S_Z \right|}$$

If this objective can be improved, sum rate has been improved ✓

$$= \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_Z \right| - \frac{1}{2} \log |S_Z|$$

Sum rate

$$+ \frac{1}{2} \log |S_Z| - \frac{1}{2} \log \left| \sum_{\substack{j=1 \\ j \neq i}}^{K} H_j S_j H_j^T + S_Z \right|$$

↳ does not depend on S_i

L 32

03 October 2019 11:17

"if" part (outline)

✓ → KKT conditions are necessary and sufficient for this problem.

✓ → Write down the KKT conditions

✓ → check that if we set each S_i to be the single-user waterfilling soln, all KKT conditions are satisfied.

$\Rightarrow$ We have the optimal soln for sum rate.

L 32

03 October 2019 11:26

Proposed iterative water-filling algorithm

Initialize $S_i = 0, \quad i = 1, 2, \cdots, K$

Repeat

for $i = 1$ to K

$$S_{\Sigma}' = \sum_{\substack{j=1 \\ j \neq i}}^{K} H_j S_j H_j^T + S_z$$

end $S_i = \underset{S}{\arg\max} \frac{1}{2} \log \left| H_i S_i H_i^T + S_{\Sigma}' \right|$ (single-user Waterfilling)

until sum rate converges

Thm 2 Iterative waterfilling finds the optimal sum rate solution

L32

03 October 2019 11:40

Thm 3 :

After one iteration of the iterative waterfilling algorithm, $\{s_i\}$ achieves a total sum rate $\sum_{i=1}^{K} R_i$ that is at most $(K-1)\frac{m}{2}$ nats from sum capacity:

$$\left| \text{Sum capacity} - \sum_{i=1}^{K} R_i \right| \leq (K-1)\frac{m}{2}$$

(K: # users; m: # ant per user)

(irrespective of $H_1, \ldots, H_K$)

Lecture 33

04 October 2019 10:00

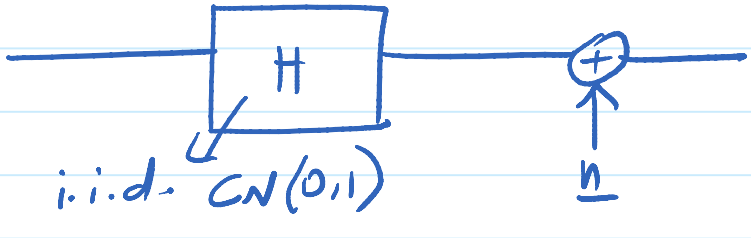
Next: MAC channel with fading

slow → outage capacity

fast → Ergodic capacity

"On the capacity of multiuser wireless channels with multiple antennas" Rhee & Cioffi (T-IT 2003)

Recap: (Tel 99)



each entry i.i.d. $\mathcal{CN}(0,1)$

$$\text{Ergodic Cap} = E\left[\log\left|I + \frac{P}{t} H H^H\right|\right] \quad \left(\text{optimal } Q = \frac{P}{t} I\right)$$

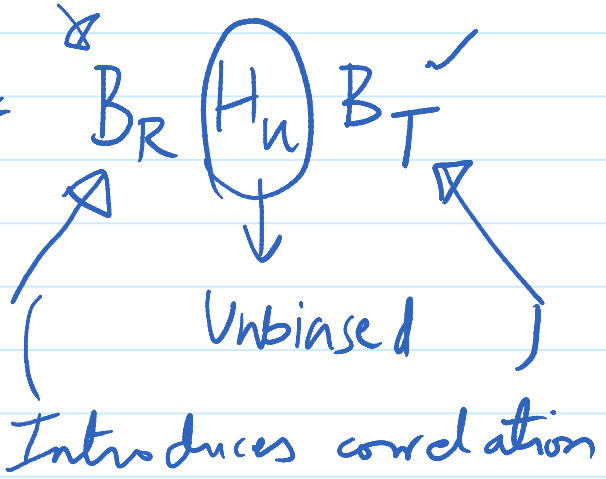
L33

04 October 2019 10:07

Two fading models

1) Unbiased $H_{r \times t}$ each entry i.i.d. & zero-mean

2) Biased $H_{r \times t} = B_R H_u B_T$



$B_R \neq I$ Biased on the receiver side
$r \times r$

$B_T \neq I$ Biased on the transmitter side
$t \times t$

L33

04 October 2019 10:17

Thm 1 (Generalization of Telatar's result)

For any H_u (unbiased) $\quad C_{erg} = E\left[\log\left|I + \frac{P}{t} H_u H_u^H\right|\right]$

Thm 2

$$H_b = H_u B_T$$

$$C_{erg} = E\left[\log\left|I_r + H_u \Lambda D \Lambda H_u^H\right|\right]$$

where $B_T = U \Lambda V^H$ (SVD of B_T)

optimal input $\underline{x} \sim$ Gaussian with zero mean & cov. S

$S = \underline{\underline{V}} D \underline{\underline{V}}^H$ $\quad D$ is diagonal $\mathrm{tr}(D) = P$

L 33

04 October 2019 10:23

$$\underline{x} = V \tilde{\underline{x}} \xrightarrow{\text{cov}} D$$

Transmit directions are the right singular vectors of B_T.

Optimal power for each direction is in D. No general expression.

Depends on $\underline{\Lambda}$, P, distribution of H_u.

MAC channels

Thm 3 (CSIR only, Tx knows distribution)

K-user MAC $\quad t_1 \; t_2 \; \dots \; t_K$

$H_1, H_2, \dots, H_K$ unbiased

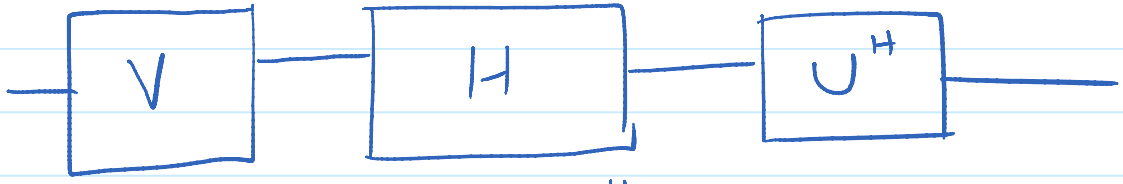
n_r ant at rx

Optimal $S_1 = \frac{P_1}{t_1} I_{t_1} \quad \dots \quad S_K = \frac{P_K}{t_K} I_{t_K}$

Ergodic capacity region is a polytope.

Lecture 34

09 October 2019 13:57



$U \Sigma V^H$

optimal precoder

Implementation constraints?

in the context of mmWave systems

- mmWave vs microwave

higher freq of operation ⇒ more pathloss ✓

⇒ smaller wavelength ⇒ antenna size smaller

⇒ more antennas in the same space

L34

10 October 2019 11:04

Beamforming gain from multiple antennas can compensate for the higher path loss.

"Spatially sparse precoding in millimeter wave MIMO systems"

El Ayach & others 2014 (TWC)

— Traditional MIMO precoder (V) : Fully digital precoder

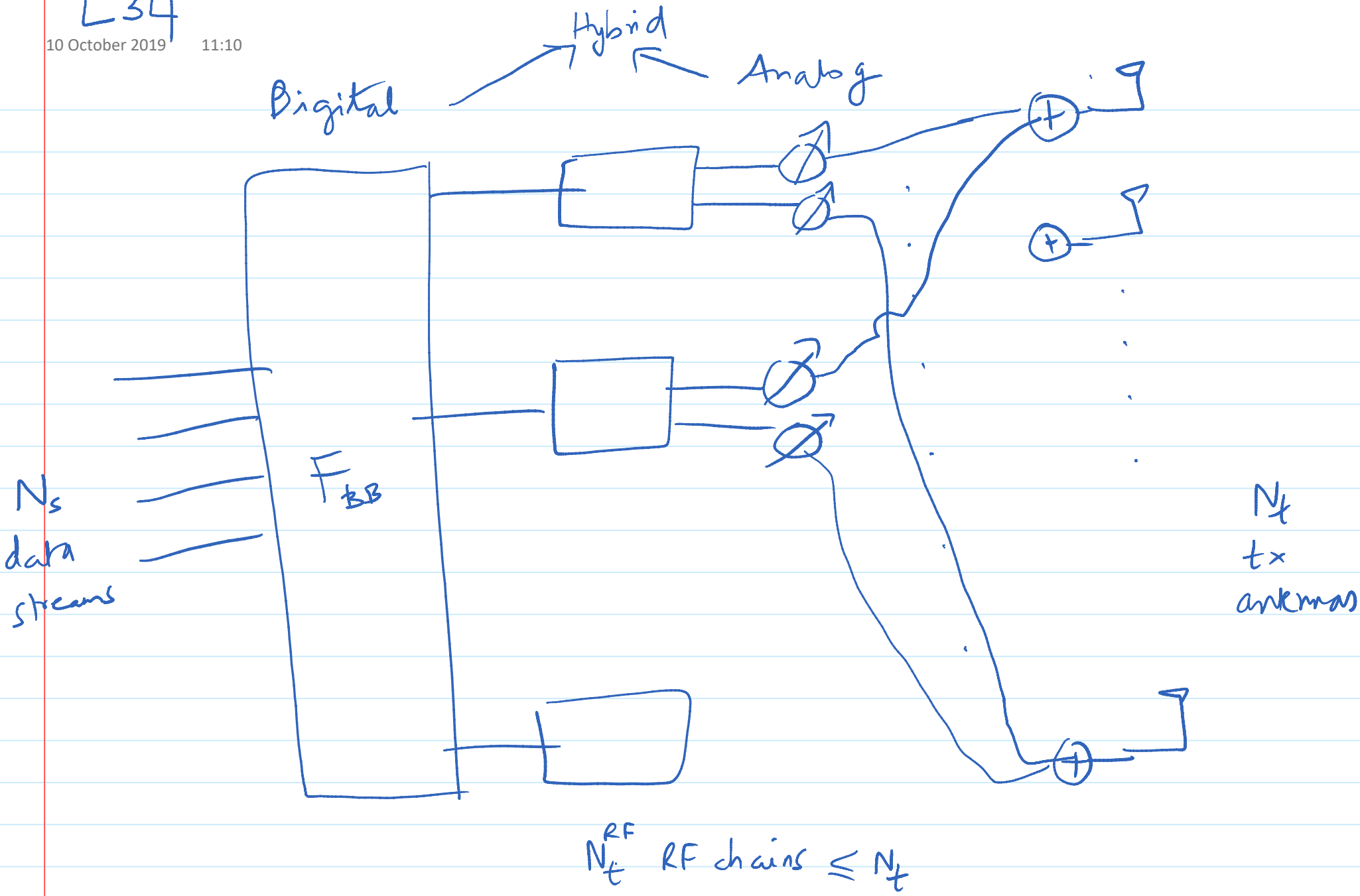
F_{opt}

High implementation cost
High power consumption
one RF chain for each antenna

$$F_{opt} = F_{BB} F_{RF}$$

L34

10 October 2019 11:10



L34

10 October 2019 11:14

$$F_{RF} = \begin{bmatrix} \times & \times & \cdots & \times \\ \times & & & \\ \vdots & & & \\ \times & \times & \cdots & \times \end{bmatrix}_{N_t \times N_t^{RF}}$$

$$\underline{x} = \underbrace{F_{RF} F_{BB}}\, \underline{s}$$

$$x \rightarrow e^{j\theta}$$

→ same magnitude for each entry

$$F_{BB} = \begin{bmatrix} & \\ & \end{bmatrix}_{N_t^{RF} \times N_s}$$

L 34

10 October 2019 11:17

Given the contraints on precoding, what is the best precoder?

* Formulate this problem as a sparse recovery problem & use existing solutions
  - Approximations
* Can approach unconstrained precoder performance under some conditions

- Constraints on the receiver too : Hybrid combiner

L34

10 October 2019 11:22

$$\underline{y} = \sqrt{\rho}\, H \underbrace{F_{RF} F_{BB} \underline{s}}_{\underline{x}} + \underline{n}$$

H : $N_r \times N_t$ channel matrix (known at tx and rx)

$\underline{n}$ i.i.d. $\mathcal{CN}(0, \sigma_n^2)$

$$E\left[\|H\|_F^2\right] = N_t N_r$$

$$E\left[\underline{s}\,\underline{s}^H\right] = \frac{1}{N_s} I_{N_s}$$

ρ : Avg. received power

Precoder constraints

$$\|F_{RF} F_{BB}\|_F^2 = N_s$$

each element of F_{RF} has unit magnitude

i^{th} col of F_{RF} $\leftarrow$ $\left(F_{RF}^{(i)} F_{RF}^{(i)*}\right)_{l,l} = \frac{1}{N_t}$

L 34

10 October 2019 11:30

$$\tilde{y} = \sqrt{\rho}\, W_{BB}^{H} W_{RF}^{H} H F_{RF} F_{BB} \underline{s} + \underbrace{W_{BB}^{H} W_{RF}^{H}}_{\tilde{n}} \frac{n}{\phi}$$

RF combining using phase shifters

Achievable rate $R = \log_2 \frac{|R_{\tilde{y}}|}{|R_{\tilde{n}}|} = \log \left| I + \frac{\rho R_{\tilde{n}}^{-1}}{N_s} (\quad) \right|$

$$R_{\tilde{n}} = \sigma_n^2 W_{BB}^{H} W_{RF}^{H} W_{RF} W_{BB}$$

$$R_{\tilde{y}} = \frac{\rho}{N_s} \left(W_{BB}^{H} W_{RF}^{H} H F_{RF} F_{BB} F_{BB}^{H} F_{RF}^{H} H^{H} W_{RF} W_{BB} \right) + R_{\tilde{n}}$$

$$= R_{\tilde{n}} \left(I + \frac{\rho R_{\tilde{n}}^{-1}}{N_s} (\quad) \right)$$

L34

10 October 2019 11:36

Problem 1

$$\max_{F_{RF}, F_{BB}, W_{RF}, W_{BB}} \quad R$$

s.t. constraints on $F_{RF}, W_{RF}, F_{BB}, W_{BB}$

Joint optimization of 4 matrices

Simpler problem (without receiver constraints)

$$\max_{F_{RF}, F_{BB}} \quad \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} H F_{RF} F_{BB} F_{BB}^H F_{RF}^H H^H \right|$$

$$\text{s.t.} \begin{cases} \left(F_{RF}^{(i)} F_{RF}^{(i)*}\right)_{l,l} = \frac{1}{N_t} \\ \|F_{RF} F_{BB}\|_F^2 = N_s \end{cases} \rightarrow F_{RF} \in \mathcal{F}_{RF}$$

$\mathcal{F}_{RF}$: Set of all $N_t \times N_t^{RF}$ matrices with constant magnitude ($\frac{1}{N_t}$) entries

L34

10 October 2019 11:42

Focus on the simpler problem

Approximate ✓

unconstrained opt precoder

$$\left(F_{RF}^{opt}, F_{BB}^{opt}\right) = \arg\min_{F_{RF}, F_{BB}} \|F_{opt} - F_{RF} F_{BB}\|_F$$

$$\text{s.t. } F_{RF} \in \mathcal{F}_{RF}$$

$$\|F_{RF} F_{BB}\|_F^2 = N_s$$

$$\|X\|_F = \sqrt{\sum_{i,j} |X_{ij}|^2}$$

✓ mmWave channel model

sparse recovery problem

↓ Use OMP

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$$I(F_{RF}, F_{BB}) = \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} H F_{RF} F_{BB} F_{BB}^H F_{RF}^H H^H \right|$$

Let the SVD of $H = U \Sigma V^*$

V^* : $\text{rank}(H) \times N_t$

$N_r \times N_t$

$\text{rank}(H) \times \text{rank}(H)$ with singular values arranged in descending order

V : $N_t \times \text{rank}(H)$

$N_r \times \text{rank}(H)$

N_s data streams $\quad 1 \le N_s \le \text{rank}(H)$

F_{opt}: Unconstrained optimal precoder for a given N_s : V_1 $(N_t \times N_s)$

First N_s columns of V

$$V = [V_1 \quad V_2]$$

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Rate achieved by $F_{opt} = V_1$ $\quad \log_2 \left| I_{N_s} + \frac{\rho}{N_s \sigma_n^2} \Sigma_1^2 \right|$

where $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$

$$I(F_{RF}, F_{BB}) = \log_2 \left| \underbrace{I_{N_r}}_{UIU^*} + \frac{\rho}{N_s \sigma_n^2} \underbrace{U \Sigma V^*}_{H} F_{RF} F_{BB} F_{BB}^* F_{RF}^* \underbrace{V \Sigma U^*}_{H^*} \right|$$

$$= \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} \Sigma V^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V \Sigma \right|$$

(I: $\operatorname{rank}(H) \times \operatorname{rank}(H)$)

$$= \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} \Sigma^2 V^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V \right|$$

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$$= \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2^2 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} F_{RF} F_{BB} F_{BD}^* F_{RF}^* \begin{bmatrix} V_1 & V_2 \end{bmatrix} \right|$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

$$= \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} \Sigma_1^2 Q_{11} \right|$$

$$+ \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} Q_{22} - \frac{\rho^2}{N_s^2 \sigma_n^4} \Sigma_2^2 Q_{21} \left(I_{N_s} + \frac{\rho}{N_s \sigma_n^2} \Sigma_1^2 Q_{11} \right)^{-1} \Sigma_1^2 Q_{12} \right|$$

(where $Q_{11} = V_1^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V_1$)

$$\approx \log_2 \left| I + \frac{\rho}{N_s \sigma_n^2} \Sigma_1^2 Q_{11} \right|$$

(Approximation 1 in the paper)

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$$= \log_2 \left| I_{N_S} + \frac{\rho}{N_S \sigma_n^2} \xi_1^2 \underbrace{V_1^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V_1}_{= I \text{ if no hybrid constraint}} \right|$$

$$= \log_2 \left| \underbrace{I_{N_S} + \frac{\rho}{N_S \sigma_n^2} \xi_1^2}_{\text{X}} \right| + \underbrace{?}_{\text{X}}$$

$$\log_2 \left| I_{N_S} - \underbrace{\left(I_{N_S} + \frac{\rho}{\sigma_n^2 N_S} \xi_1^2 \right)^{-1} \frac{\rho}{N_S \sigma_n^2} \xi_1^2 \left(I_{N_S} - V_1^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V_1 \right)}_{\text{X}} \right|$$

$$\left(I + BA = (I+B)\left(I - (I+B)^{-1} B (I-A)\right) \right)$$

$$\hookrightarrow I + B - B + BA = (I+B)\left(I - (I+B)^{-1}(BA - B)\right)$$

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- If eigenvalues of X are small

$$\log_2 |I_{N_s} - X| \approx \log_2 (1 - \mathrm{tr}(X)) \approx -\mathrm{tr}(X) \cdot \log_2 e$$

- High SNR approximation

$$\left(I_{N_s} + \frac{\rho}{\sigma_n^2 N_s} \xi_1^2\right)^{-1} \frac{\rho}{N_s \sigma_n^2} \xi_1^2 \approx I_{N_s}$$

$$I(F_{RF}, F_{BB}) \approx \log_2 \left| I_{N_s} + \frac{\rho}{N_s \sigma_n^2} \xi_1^2 \right| - \mathrm{tr}\left(I_{N_s} - V_1^* F_{RF} F_{BB} F_{BB}^* F_{RF}^* V_1\right)$$

$$= N_s - \| V_1^* F_{RF} F_{BB} \|_F^2$$

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$$\min_{F_{RF}, F_{BB}} \underbrace{N_s - \| V_1^* F_{RF} F_{BB} \|_F^2}$$

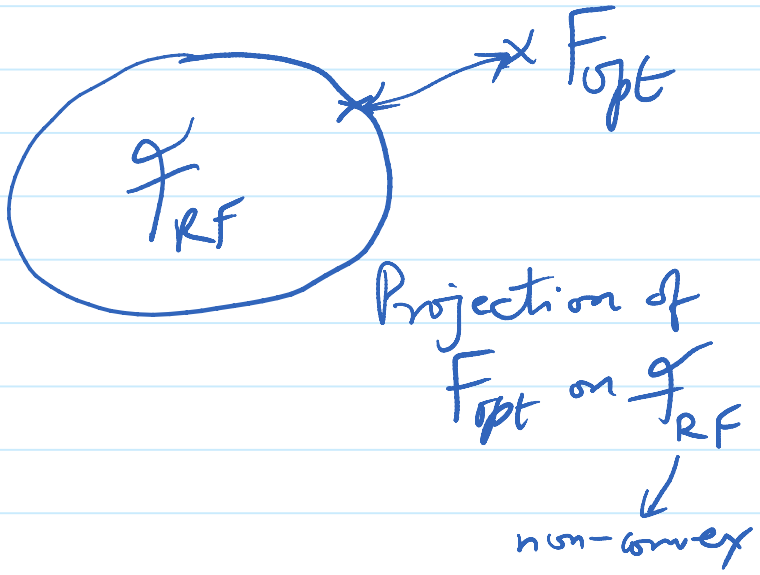
$$\text{s.t. } F_{RF} \in \mathcal{F}_{RF}$$

$$\| F_{RF} F_{BB} \|_F^2 = N_s$$

$$\rightarrow (\text{if small}) \approx \| F_{opt} - F_{RF} F_{BB} \|_F^2$$

(V_1)

$$\left\{ \begin{array}{l} \min_{F_{RF}, F_{BB}} \underbrace{\| F_{opt} - F_{RF} F_{BB} \|_F^2} \\ \text{s.t. } F_{RF} \in \mathcal{F}_{RF} \\ \| F_{RF} F_{BB} \|_F^2 = N_s \end{array} \right\} \leftarrow$$



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Now, use the mmWave channel model to reformulate this problem.

Extended Saleh-Valenzuela model

$$H = \gamma \sum_{i,l} \alpha_{il} \Lambda_r\left(\phi_{il}^r, \theta_{il}^r\right) \Lambda_t\left(\phi_{il}^t, \theta_{il}^t\right) \underbrace{a_r\left(\phi_{il}^r, \theta_{il}^r\right)}_{\text{Planar array response}} \underbrace{a_t\left(\phi_{il}^t, \theta_{il}^t\right)^*}_{\text{Planar array response}}$$