Lecture 1

17 January 2019 09:51

EE5141 Intro to Wireless and Cellular Communication

Text book : Fundamentals of Wireless Communication
David Tse & Pramod Viswanath, Cambridge Univ Press, 2005.

http://www.ee.iitm.ac.in/~skrishna/ee5141/

Moodle : Lecture notes, problem sheets & solutions.

Lecture 1

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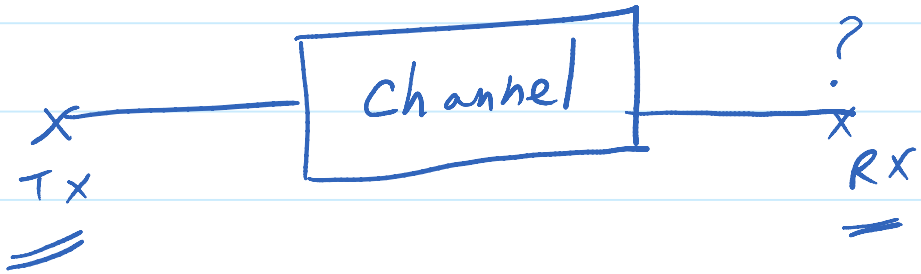
We will understand wireless communication at 3 levels

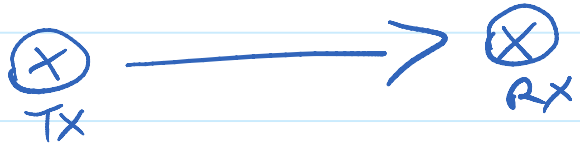
1) Channel propagation characteristics & modeling



How is the received signal related to the transmit signal?

2) 'Link design' over this channel

Point-to-point communication



L1

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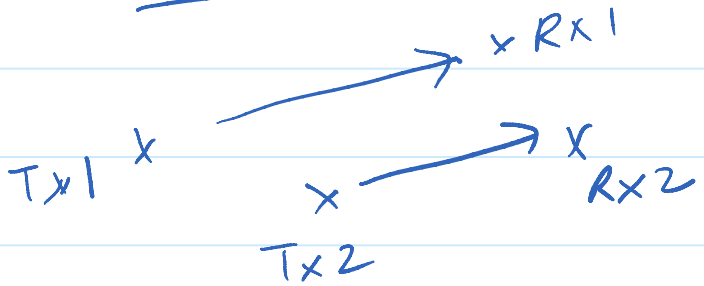
3) System design with multiple terminals and links

`Focus on cellular'



Key aspects of the wireless channel

1) Time-variations / Fading

2) Interference

How we deal with these two aspects is central to the design.

L1

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Usually we model 2 types of time variations.

'Large-scale' fading

'Small scale' fading

Simple calculation

- Electromagnetic waves $c = 3 \times 10^8$ m/s
- Frequency bands 900 MHz, 1.9 GHz

Wavelength $\lambda = \frac{c}{f}$

$f = 900$ MHz, $\lambda = \frac{1}{3}$ m

$f = 1900$ MHz, $\lambda = \frac{3}{19}$ m

'Large scale' & 'small scale' are w.r.t. λ.

L1

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Understanding large-scale variations helps in deciding

- Power level required for transmission
- Coverage range of base-stations

Understanding small-scale variations helps in designing

- 'link': modulation, coding, etc.

Channel models

- Modeling only using wave propagation is too difficult (too detailed)
- Experiments

Statistical models

Lecture 2

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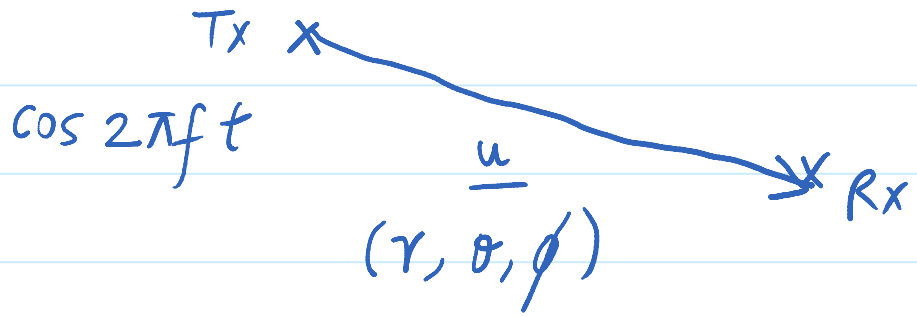
Idealized channel examples

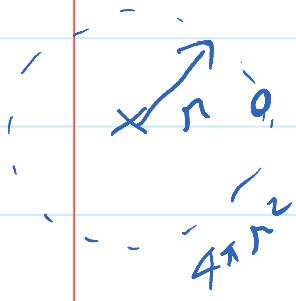
1) Free space, fixed transmit and receive antennas



r is large enough (far field assumption)

$\frac{r}{c}$

$$E(f, t, \underline{u}) = \frac{\alpha_s(\theta, \phi, f) \cos 2\pi f (t - \frac{r}{c})}{r}$$



$\alpha_s(\theta, \phi, f)$: Radiating pattern of the tx antenna at freq f in the direction (θ, ϕ)

L2

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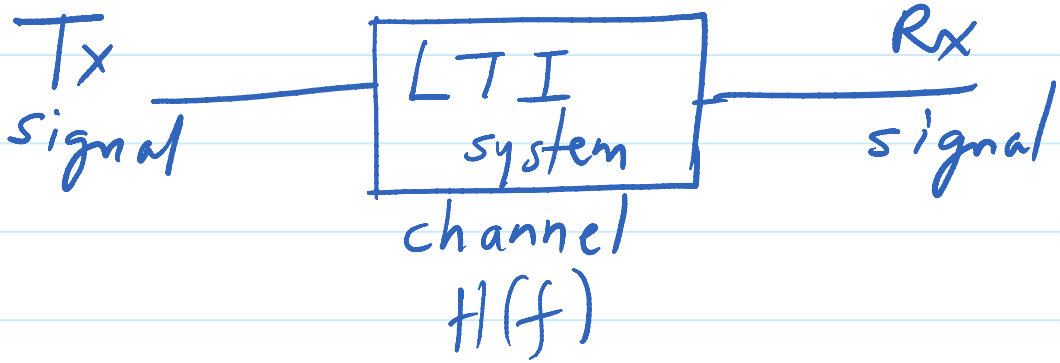
Response at a fixed rx antenna at $\underline{u}$

$$E_r(f, t, \underline{u}) = \frac{\alpha(\theta, \phi, f)}{r} \cos 2\pi f \left(t - \frac{r}{c}\right)$$

Product of antenna patterns of tx and rx antennas at freq f in the direction (θ, ϕ)



L2

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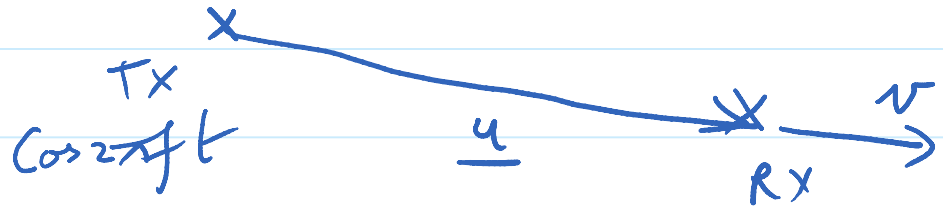
2) Free space, moving antenna



$$\underline{u}(t) = (r(t), \theta, \phi)$$

$$(r_0 + vt$$

$$E_{rx}(f, t, (r_0 + vt, \theta, \phi)) = \frac{\alpha(\theta, \phi, f) \cos\left(2\pi f \left(t - \frac{r_0 + vt}{c}\right)\right)}{r_0 + vt}$$

$$= \frac{\alpha(\theta, \phi, f)}{r_0 + vt} \cos\left[2\pi f\left(1 - \frac{v}{c}\right)t - \frac{2\pi f r_0}{c}\right]$$

L2

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This channel is not LTI.

- Input at freq $f \longrightarrow$ output at $f\left(1-\frac{v}{c}\right)$

  Freq. shift $-f\frac{v}{c}$ (Doppler shift), depends on f

- Time-varying attenuation ($r_0 + vt$ in the denominator)

  If vt is small compared to r_0, this effect is negligible

$r_0 = 1\,\text{km}$

$v = 30\,\text{km/hr}$

for $vt = 0.1 r_0$

$t = 125$

L 2

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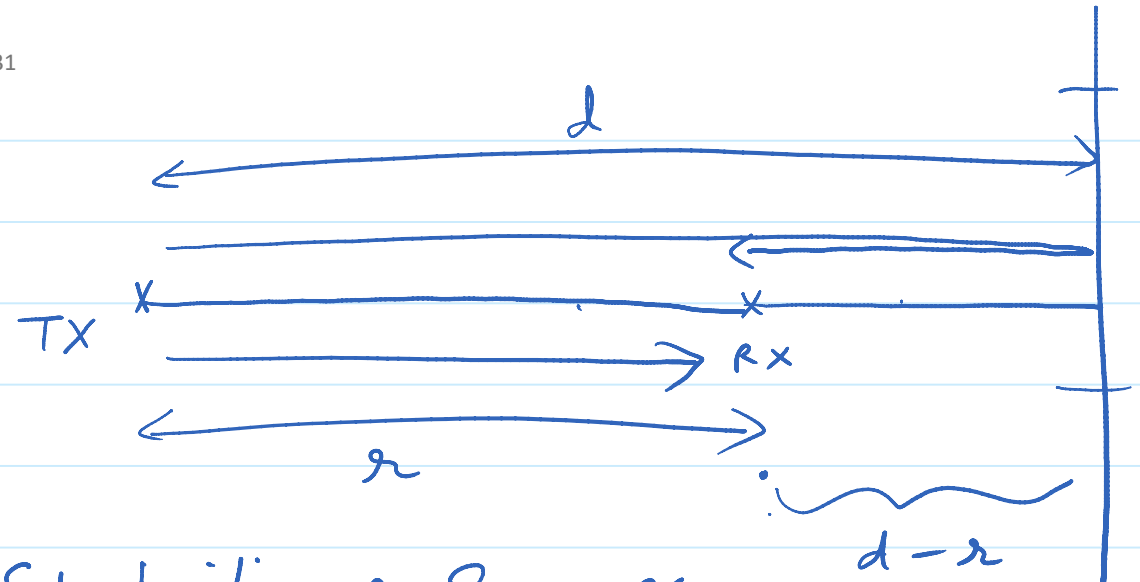
3)



Rx : Superposition of 2 waves

Reflecting wall

Assuming wall is very large

$$E_r(f,t) = \frac{\alpha \cos 2\pi f\left(t - \frac{r}{c}\right)}{r} - \frac{\alpha \cos 2\pi f\left(t - \frac{2d-r}{c}\right)}{2d-r}$$

Phase difference between the two terms $= \frac{2\pi f(2d-r)}{c} + \pi - \frac{2\pi f r}{c}$

L 2

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$$= \underbrace{\frac{4\pi f}{c}(d-r) + \pi}_{\triangleq \Delta\theta}$$

if $\Delta\theta = 2\pi n$, 2 waves add constructively

$\Delta\theta = (2n+1)\pi$, 2 waves add destructively

* As a fn. of r (spatial location of rx. w.r.t tx) there is a pattern of constructive and destructive interference.

$$\Delta\theta = \frac{4\pi f}{c}(d-r) + \pi = \frac{4\pi}{\lambda}(d-r) + \pi$$

If $(d-r)$ increases by $\frac{\lambda}{4}$, $\Delta\theta$ increases by π

L 2

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If change in distance $\ll \frac{\lambda}{4}$, there is not much change in $\Delta\theta$.

Coherence distance Δx_c : Distance over which you can assume that change in $\Delta\theta$ is small.

$$\Delta x_c \approx \frac{\lambda}{4} \quad \text{(one possible defn)}$$

* As a fn. of f $\quad \Delta\theta = \frac{4\pi f}{c}(d-r) + \pi$

For $\Delta\theta$ to change by π, f should change by $\frac{c}{4(d-r)}$

Lecture 3

21 January 2019 07:51

Difference in delay of the 2 paths

$$\frac{2d-r}{c} - \frac{r}{c} = \frac{2(d-r)}{c}$$

Delay spread (T_d)

$$(\Delta\theta = 2\pi f T_d + \pi)$$

If f changes by $\underbrace{\frac{1}{4\frac{(d-r)}{c}}}_{\frac{1}{2T_d}}$, $\Delta\theta$ changes by π

$\frac{1}{T_d}$: coherence bandwidth

L3

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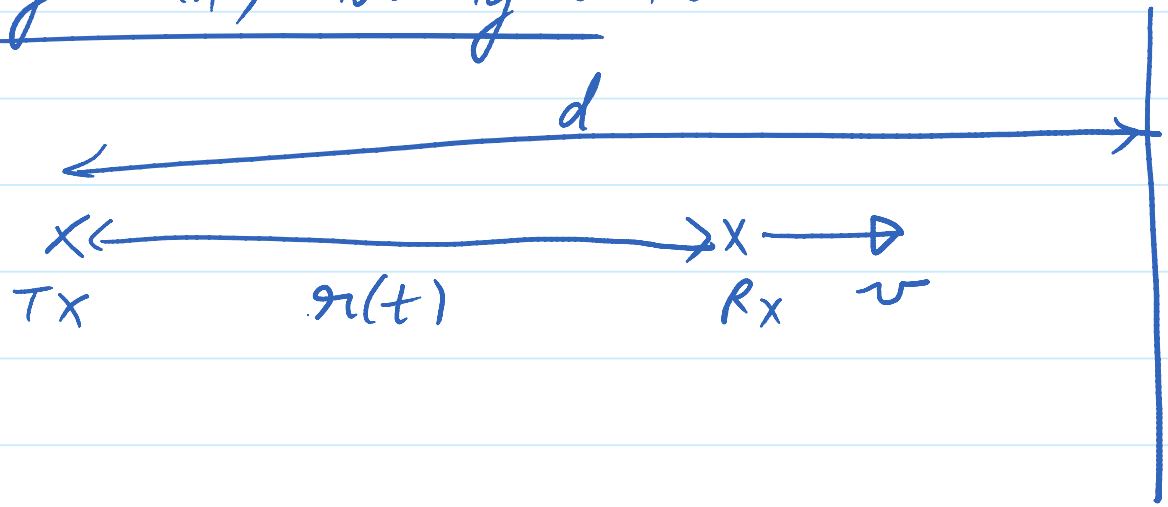
4) Reflecting wall, moving antenna.



* The strength of the received signal increases and decreases as the receiver moves. (constructive and destructive interference between the multiple paths)

↓ 2 in this case

* coh. distance $\xrightarrow[\text{by } v]{\text{divide}}$ coh. time $\frac{\lambda/4}{v} = \frac{f}{4cv}$

L3

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* We can assume channel gain is ≃ constant if our time for transmission ≪ coherence time ↓

$$\frac{\alpha \cos 2\pi f\left(\left(1-\frac{v}{c}\right)t - \frac{r_0}{c}\right)}{r_0 + vt} - \frac{\alpha' \cos 2\pi f\left(\left(1+\frac{v}{c}\right)t - \frac{2d-r_0}{c}\right)}{2d - r_0 - vt}$$

Direct path

Doppler shift $-\frac{fv}{c}$

↓ Reflected path

Doppler shift $+\frac{fv}{c}$

Doppler spread $\frac{2fv}{c}$

$f = 900\,MHz$ $\lambda/4 = \frac{1}{12}\,m$

$v = 3\,km/hr$

$= \frac{3000}{3600}\,m/s$

$= \frac{5}{6}\,m/s$

$\frac{1}{12} \times \frac{6}{5} = 0.15$

$= 100\,ms$

L3

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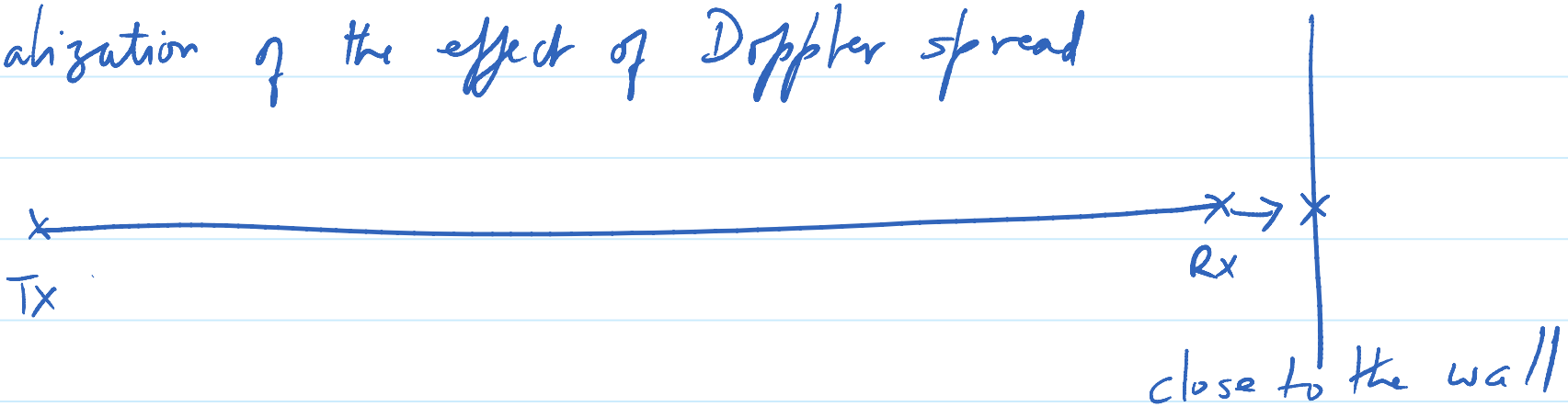
Doppler spread (D_s) $\approx \frac{1}{\text{Coherence time } (T_c)}$

eg. $v = 60$ km/hr

$f = 900$ MHz

$D_s = 100$ Hz $\quad T_c \approx 1$ ms

* Visualization of the effect of Doppler spread



Denominators $r_0 + vt \approx 2d - r_0 - vt$

L3

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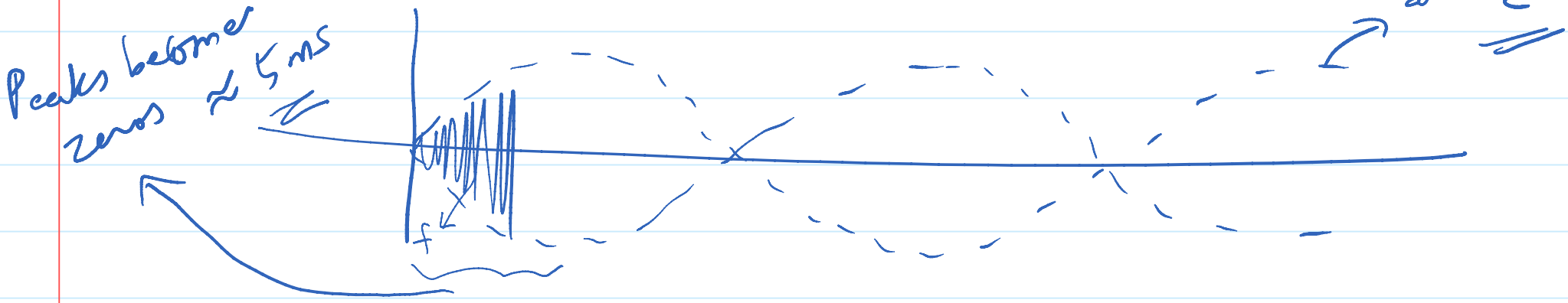
Let $\alpha = \alpha'$

$$\frac{\alpha \cos 2\pi f\left(\left(1-\frac{v}{c}\right)t - \frac{r_0}{c}\right)}{r_0 + vt} - \frac{\alpha \cos 2\pi f\left(\left(1+\frac{v}{c}\right)t - \frac{2d - r_0}{c}\right)}{r_0 + vt}$$

$$= \frac{2\alpha}{r_0 + vt} \sin 2\pi f\left(\frac{vt}{c} + \frac{r_0 - d}{c}\right) \sin 2\pi f\left(t - \frac{d}{c}\right) \cdot$$

sinusoid at $\frac{fv}{c}$ sinusoid at f



L3

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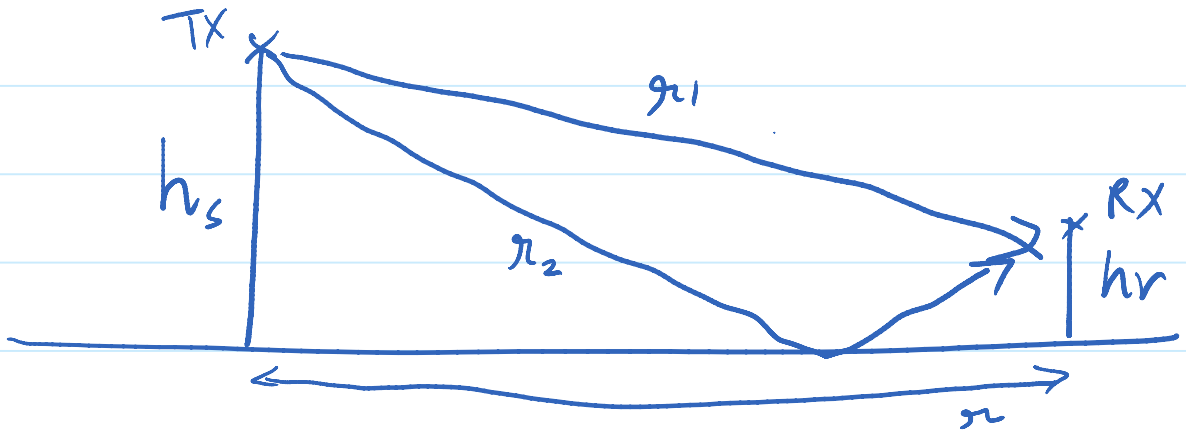
changes due time-varying attenuation $\left(\frac{1}{\Lambda_0 + rt}\right)$ happens in slower time scales (not ms) but s

5) Reflection from a ground plane



r_1 : Length of direct path

r_2 : Length of reflected path

Lecture 4

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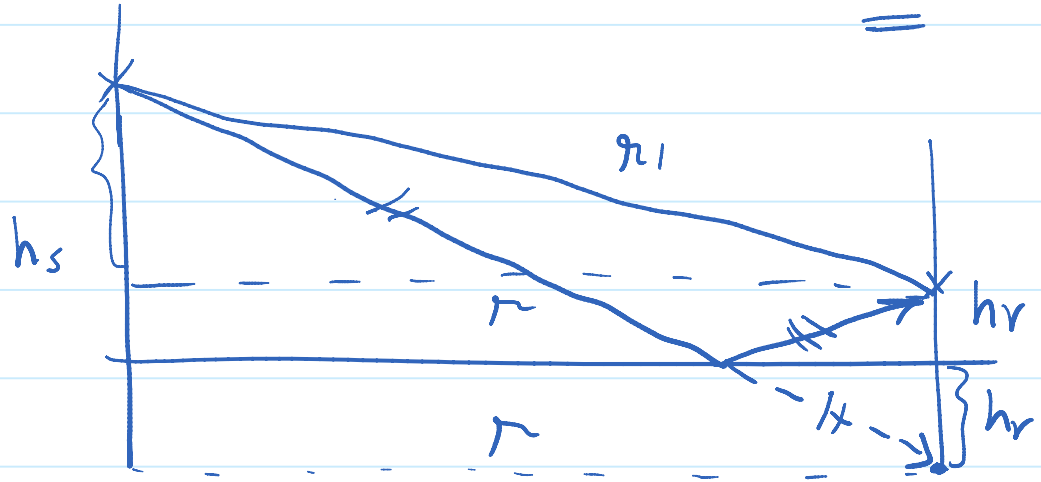
$$\frac{\alpha \cos 2\pi f\left(t - \frac{r_1}{c}\right)}{r_1} - \frac{\alpha' \cos 2\pi f\left(t - \frac{r_2}{c}\right)}{r_2}$$



$$r_1 = \sqrt{(h_s - h_r)^2 + r^2} = r\left(\sqrt{1 + \left(\frac{h_s - h_r}{r}\right)^2}\right)$$

$$r_2 = \sqrt{(h_s + h_r)^2 + r^2} = r\sqrt{1 + \left(\frac{h_s + h_r}{r}\right)^2}$$

L4

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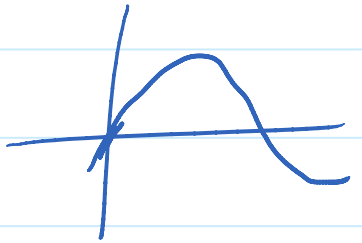
Let $\alpha = \alpha'$, $r_1 \approx r_2$ for the attenuation component



$$\frac{\alpha}{r_1}\left[\cos 2\pi f\left(t - \frac{r_1}{c}\right) - \cos 2\pi f\left(t - \frac{r_2}{c}\right)\right]$$

$$= \frac{2\alpha}{r_1} \sin 2\pi f\left(t - \frac{r_1 + r_2}{2c}\right) \sin 2\pi f\left(\frac{r_2 - r_1}{2c}\right) \quad \checkmark$$

$\rightarrow \sin\theta \approx \theta$ small θ

$$r_2 - r_1 = r\left(\sqrt{1 + \left(\frac{h_s + h_r}{r}\right)^2} - \sqrt{1 + \left(\frac{h_s - h_r}{r}\right)^2}\right)$$

$(1+x)^{1/2} \approx 1 + \frac{x}{2}$

$$(\text{large } r) \approx r\left(1 + \frac{(h_s + h_r)^2}{2r^2} - 1 - \frac{(h_s - h_r)^2}{2r^2}\right)$$

$$= \frac{2h_s h_r}{r}$$

L 4

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$$\frac{2\alpha}{r_1}\sin\left(2\pi f\left(t-\frac{r_1+r_2}{2c}\right)\right)\underbrace{2\pi f\left(\frac{r_2-r_1}{2c}\right)}$$

$$\approx \frac{2\pi f}{2c}\left(\frac{2h_s h_r}{r_2}\right)$$

As r increases, signal magnitude goes down as $\frac{1}{r^2}$.

signal power goes down as $\frac{1}{r^4}$.

(In free space, signal power goes down as $\frac{1}{r^2}$).

L4

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Power decay with distance (large scale)

Free space : Power $\propto r^{-2}$

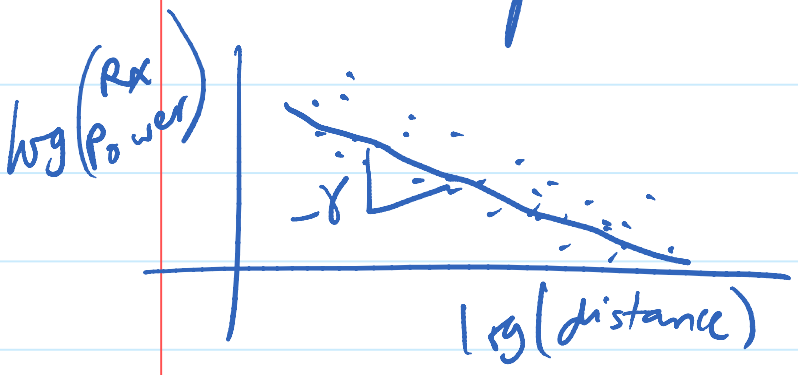
With obstacles : decays faster than r^{-2}

2-ray example : r^{-4}

Difficult to exactly model for each case

Simple models : $r^{-\gamma}$ are used.

Some options for γ based on measurements



Usually $\gamma > 2$

L 4

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Moving antenna, multiple reflectors

* Ray tracing is not simple
* Linearity still holds
  - Received signal is a sum of responses from different paths
* Another type of reflection: scattering (rough surfaces)

Using insights from these examples, & measurements, come up with a mathematical model for the channel.

L 4

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Continuous time Passband channel

$\sum_{i=0}^{?}$ signals from paths

how many

attenuation, delay

Input-output model for the wireless channel

* So far, $\phi(t) = \cos 2\pi f t$ Transmit signal

$\sum_i a_i(f,t)\,\phi(t - \tau_i(f,t))$ Received signal

$\tau_i(f,t)$ → Delay of path i – Distance

$a_i(f,t)$ → attenuation of path i

- ant. pattern of tx
- ant. pattern of rx
- Nature of reflections
- Distance
- frequency (slowly varying fn. of freq.) → usually Carrier freq ≫ Bandwidth

Baseband channel

Discrete-time channel model

statistical model

L 4

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For narrowband communication (Bandwidth << carrier freq)

assume $a_i(f,t)$, $\tau_i(f,t)$ are independent of f.

output $y(t) = \sum_i a_i(t)\, x(t-\tau_i(t))$

x: arbitrary input (superposition)

Time varying Linear filter model

Special case

General linear time varying filter

$$y(t) = \int_{-\infty}^{\infty} h(\tau,t)\, x(t-\tau)\, d\tau$$

$h(\tau,t)$: response at time t due to impulse at time $t-\tau$.

L 4

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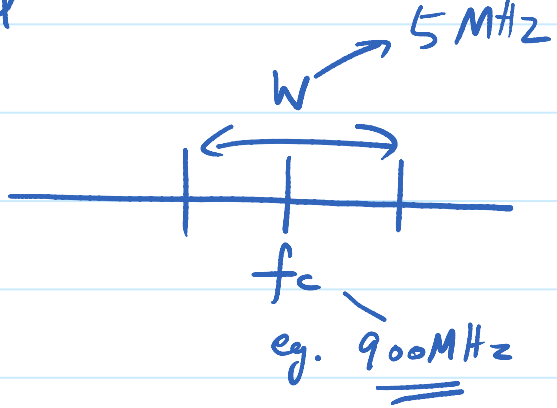
$$h(t,\tau) = \sum_i a_i(t)\,\delta(\tau - \tau_i(t))$$

Baseband equivalent model for the channel



Lecture 5

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$$\text{output } y(t) = \sum_i a_i(t)\, x(t - \tau_i(t))$$

↑ sum of many copies of the input

↑ input signal

each has an attenuation $a_i(t)$

delay $\tau_i(t)$

Remarks

1) $a_i(t)$, $\tau_i(t)$ instead of $a_i(f,t)$ and $\tau_i(f,t)$ ignoring the dependence on f

2) Overall response is still a fn of 'f' (when there is more than one term in the summation)

L 5

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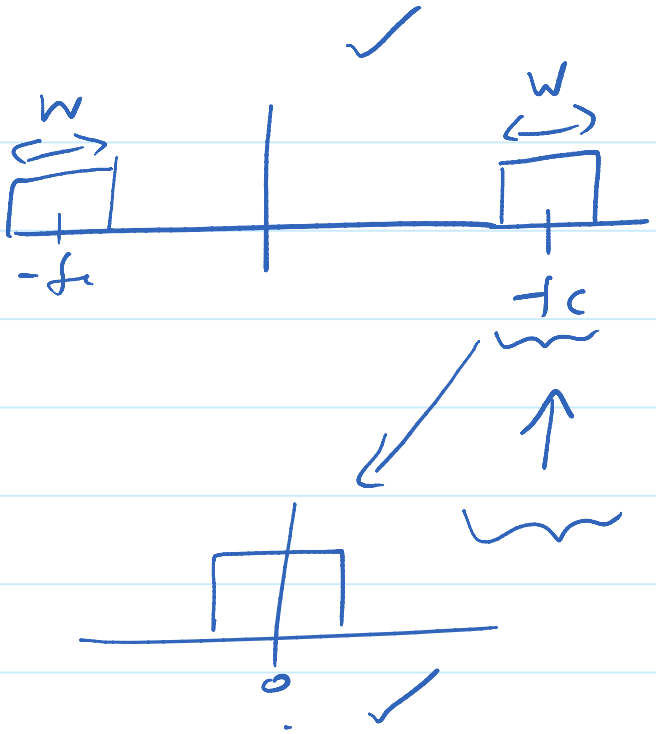
Equivalent baseband model



$$x(t) = Re\left(x_b(t)\, e^{j2\pi f_c t}\right)$$

$$y(t) = Re\left(y_b(t)\, e^{j2\pi f_c t}\right)$$

$y_b(t)$ in terms of $x_b(t)$

L 5

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$$\mathrm{Re}\left[y_b(t)\, e^{j2\pi f_c t}\right] = \sum_i \underbrace{a_i(t)}_{\text{real}} \mathrm{Re}\left[x_b(t-\tau_i(t))\, e^{j2\pi f_c (t-\tau_i(t))}\right]$$

$$= \mathrm{Re}\left[\sum_i a_i(t)\, x_b(t-\tau_i(t))\, e^{j2\pi f_c t}\, e^{-j2\pi f_c \tau_i(t)}\right]$$

$$= \mathrm{Re}\left[\left(\sum_i a_i(t) e^{-j2\pi f_c \tau_i(t)}\, x_b(t-\tau_i(t))\right) e^{j2\pi f_c t}\right]$$

$$y_b(t) = \sum_i \underbrace{a_i(t) e^{-j2\pi f_c \tau_i(t)}}_{\triangleq\, a_i^b(t)}\, x_b(t-\tau_i(t))$$

L 5

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Baseband equivalent input-output model

$$y_b(t) = \sum_i a_i^b(t)\, x_b(t - \tau_i(t))$$

Baseband equivalent linear time varying filter

$$h_b(t, \tau) = \sum_i a_i^b(t)\, \delta(t - \tau_i(t))$$

$$a_i^b(t) = a_i(t)\, e^{-j 2\pi f_c \tau_i(t)}$$

$a_i(t)$: changes slowly (changes in path lengths) order of 's'

$e^{-j 2\pi f_c \tau_i(t)}$: phase changes by $\frac{\pi}{2}$ if $\tau_i(t)$ changes by $\frac{1}{4 f_c}$

i.e. path length changes by $\frac{c}{4 f_c}$

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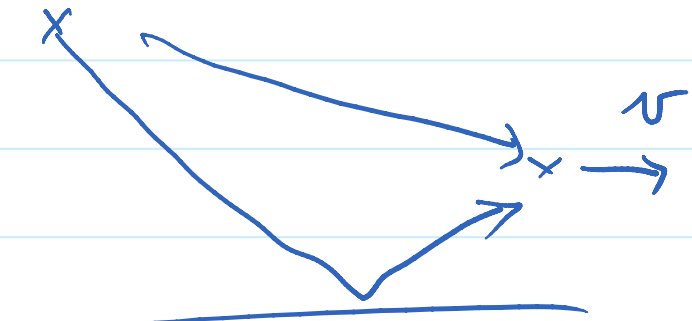
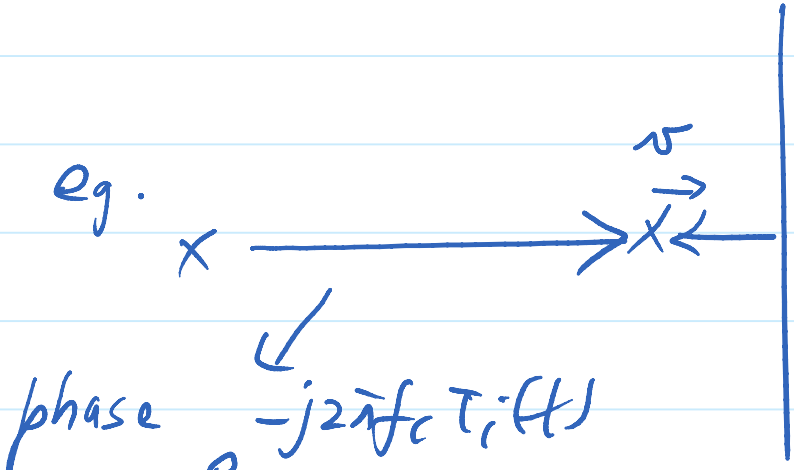
at velocity v, the time taken for a phase change of $\frac{\pi}{2}$ is $\frac{c}{4 f_c v} \approx$ 'ms'



for each path, phase changes at a different rate

eg.



phase $e^{-j2\pi f_c \tau_i(t)}$

for each path will change at the same rate, but in opposite dir.

L5

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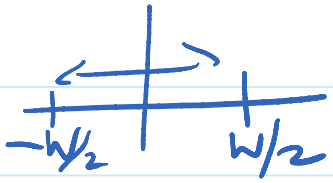
Discrete-time baseband model

$$y_b(t) = \sum_i a_i^b(t)\, x_b(t - \tau_i(t))$$

* Assume that the input is bandlimited to W.

(passband) $x(t)$

$x_b(t)$ is then bandlimited to $\frac{W}{2}$.



$x_b(t)$ can be written in terms of its samples as

$$x_b(t) = \sum_n \underbrace{x_b\left(\frac{n}{W}\right)}_{\triangleq x[n]} \operatorname{sinc}(Wt - n)$$

(sampling rate W)

$\left[\operatorname{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t}\right]$

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Due to Doppler spread (time varying system), bandwidth of the output is slightly larger than input bandwidth.

(Doppler spread $\approx$ 100s of Hz, BW $\approx$ few MHz)

We will ignore this. And we will sample $y_b(t)$ also at sampling rate W.

$$y_b(t) = \sum_i a_i^b(t) \underbrace{\sum_n x[n]\,\mathrm{sinc}(W(t-\tau_i(t)) - n)}_{x_b(t-\tau_i(t))}$$

$$= \sum_n x[n] \sum_i a_i^b(t)\,\mathrm{sinc}(W(t-\tau_i(t)) - n)$$

$$y_b\left(\frac{m}{W}\right) = ?$$

L5

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$$y_b\left(\frac{m}{W}\right) = \sum_n x[n] \sum_i a_i^b\left(\frac{m}{W}\right) \operatorname{sinc}\left(m - W\tau_i\left(\frac{m}{W}\right) - n\right)$$

Let $l = m - n$

$n = m - l$

$$= \sum_l x[m-l] \underbrace{\sum_i a_i^b\left(\frac{m}{W}\right) \operatorname{sinc}\left(l - W\tau_i\left(\frac{m}{W}\right)\right)}_{\triangleq h_l[m]}$$

$$y[m] = \sum_l x[m-l]\, h_l[m]$$

$y[m] \rightarrow y_b\left(\frac{m}{W}\right)$

$h_l[m]$: l^{th} tap of the filter at time m.

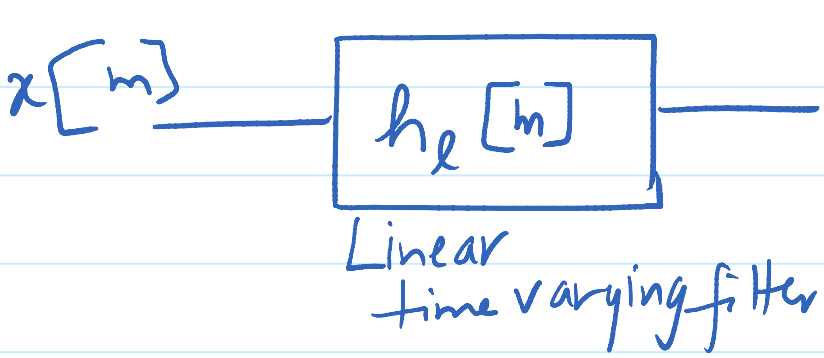
↓

linear time-varying

L5

25 January 2019 10:47

$$h_\ell[n] = \sum_i a_i{}^b\left(\frac{n}{W}\right) \operatorname{sinc}\left(\ell - W\tau_i\left(\frac{n}{W}\right)\right)$$



$$y[n] = \sum_\ell x[n-\ell]\, h_\ell[n]$$

Lecture 6

28 January 2019 08:01

$$h_l[m] = \sum_i a_i^b\left(\frac{m}{W}\right) \text{sinc}\left(l - W\tau_i\left(\frac{m}{W}\right)\right)$$

In case $a_i^b(t)$, $\tau_i(t)$ are time-invariant,

$$h_l = \sum_i a_i^b \, \text{sinc}(l - \tau_i W) \quad \left[\text{No dependence on } m\right]$$

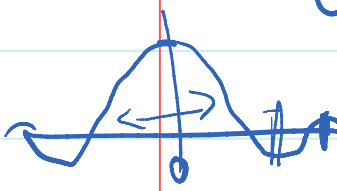
$\downarrow$

convolution of $\text{sinc}(Wt)$ with $\sum_i a_i^b \, \delta(t - \tau_i)$

sampled at $t = \frac{l}{W}$.

* Consider the ith path $(a_i(t), \tau_i(t))$

Contribution of ith path to l^{th} tap $(h_l[m])$ $a_i^b\left(\frac{m}{W}\right) \text{sinc}\left(l - W\tau_i\left(\frac{m}{W}\right)\right)$

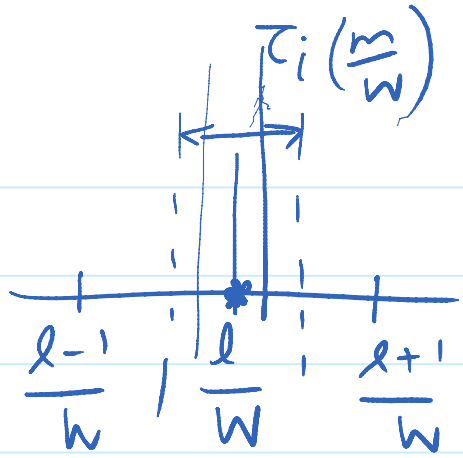


If $l - W\tau_i\left(\frac{m}{W}\right)$ is close to 0, there will be a significant contribution. Max contribution is when $\frac{l}{W}$ close to $\tau_i\left(\frac{m}{W}\right)$

L6

28 January 2019 08:09

$$\tau_i\left(\frac{m}{W}\right) \in \left[\frac{\ell}{W} - \frac{1}{2W}, \frac{\ell}{W} + \frac{1}{2W}\right]$$



* ℓ^{th} tap is mainly an aggregation of physical paths with delay in $\left[\frac{\ell}{W} - \frac{1}{2W}, \frac{\ell}{W} + \frac{1}{2W}\right]$

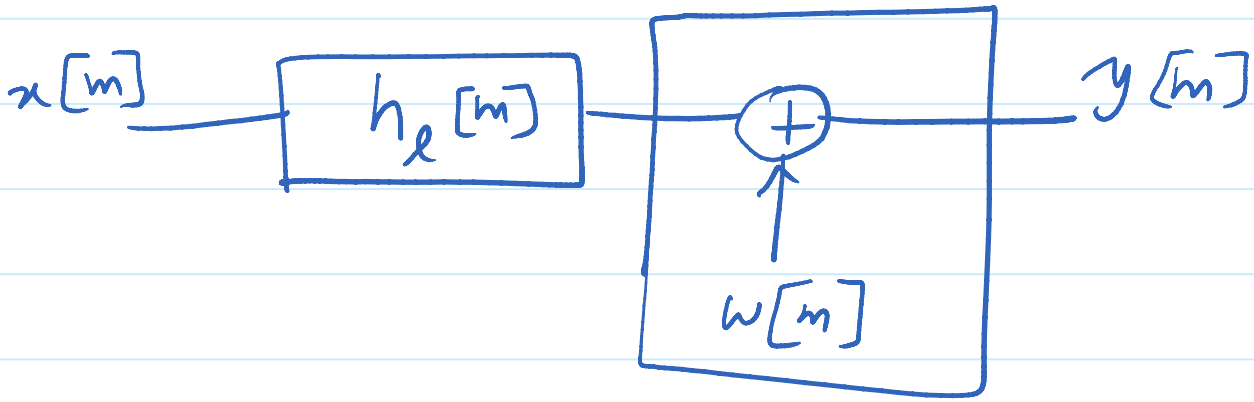
$$h_\ell[m] = \sum_{i} a_i^b\left(\frac{m}{W}\right) \operatorname{sinc}\left(\ell - W\tau_i\left(\frac{m}{W}\right)\right)$$

i: mainly paths in $\left[\frac{\ell}{W} - \frac{1}{2W}, \frac{\ell}{W} + \frac{1}{2W}\right]$

L6

28 January 2019 08:13

Additive White Gaussian noise (AWGN)



$$y[m] = \sum_{\ell} h_\ell[m]\, x[m-\ell] + w[m]$$

$$y(t) = \sum_{i} a_i(t)\, x_i\left(t - \tau_i(t)\right) + w(t)$$

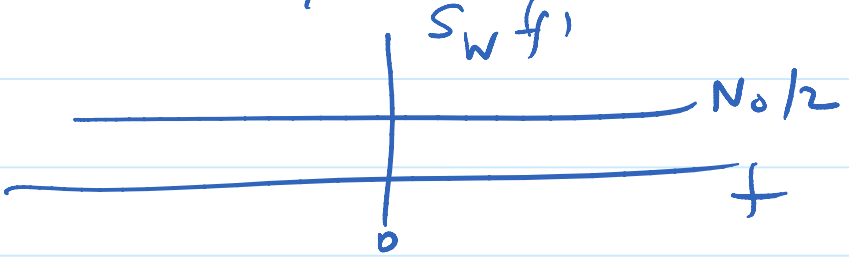
$w(t)$: AWGN

Lowpass Filtered (W) and sampled at $\frac{m}{W}$.

L6

28 January 2019 08:16

If we start with $w(t)$ that white Gaussian noise



we can show $\{w[n]\}$ is a discrete-time white noise process & circularly symmetric

$$\left(E[w[n]\,w^*[l]] = \begin{cases} 0 & l \neq n \\ \sigma^2 & l = n \end{cases} \right)$$

L6

28 January 2019 08:19

Time and frequency coherence

1) Coherence time and Doppler spread

$$h_\ell[m] = \sum_i a_i^b\left(\frac{m}{W}\right) \mathrm{sinc}\left(\ell - W\tau_i\left(\frac{m}{W}\right)\right)$$

$$= \sum_i a_i\left(\frac{m}{W}\right) e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)} \mathrm{sinc}\left(\ell - W\tau_i\left(\frac{m}{W}\right)\right)$$

How fast does $h_\ell[m]$ change with m?

- $a_i\left(\frac{m}{W}\right)$: attenuation as distance to Tx changes
  Time scale of change $\approx$ secs
- $\tau_i\left(\frac{m}{W}\right)$: —— " ——

L6

28 January 2019 08:22

$e^{-j2\pi f_c \tau_i(\frac{m}{W})}$: changes much faster

Time scale $\approx$ ms.

(Doppler shift for i^{th} path $f_c \tau_i'(t)$)

$\left(f_c \tau_i \approx \frac{\pi}{2} \rightarrow \tau_i \approx \frac{10}{\text{ms}} \right)$

$h_l[m] = \sum_i$ (phase changes $\approx$ 10 ms)

Coherence time $T_c \approx$ 10 ms

$\approx$ changes in time scale $\approx$ 10 ms.

Doppler spread $D_s = \max_{i,j} f_c |\tau_i'(t) - \tau_j'(t)| \approx \frac{1}{\text{coh. time}}$

L6

28 January 2019 08:29

2) Coherence bandwidth and delay spread.

Delay spread $T_d = \max_{i,j} |\tau_i(t) - \tau_j(t)|$

Coherence bw $W_c \approx \frac{1}{T_d}$

Fading

Time selective

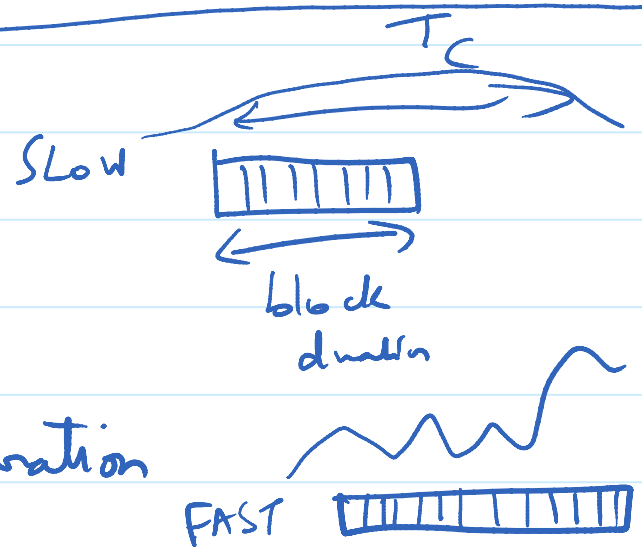
Fast

$T_c \ll$ coding block duration

Flat

slow

T_c coh. time $\gg$ coding block duration



L6

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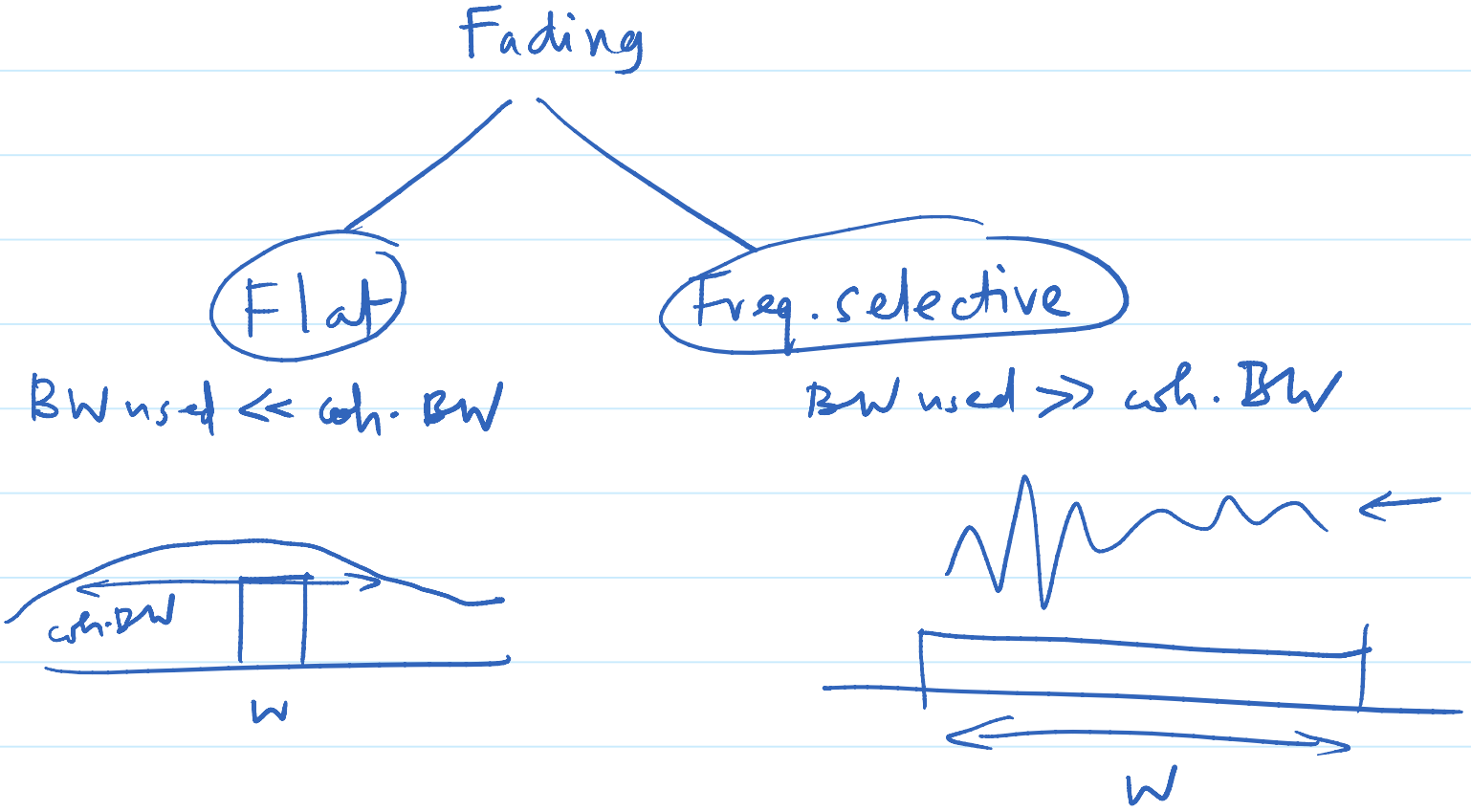


✓ Given f_c, W, Distance b/w tx & rx, velocity/mobility model

estimate T_c, D_s, T_d, W_c → use in design

L6

28 January 2019 08:39

$$y[m] = \sum_{\ell} h_\ell[m]\, x[m-\ell] + w[m]$$

* $h_\ell[m] = \sum_{i} a_i^{b}\left(\frac{m}{W}\right) \mathrm{sinc}\left(\ell - W\tau_i\left(\frac{m}{W}\right)\right)$

Deterministic model

Need to know the exact propagation environment

* Now, give a statistical characterization

  - How many taps $h_\ell[m]$ are necessary?
  - How do $h_\ell[m]$ change with m?
  - What is the dist. for each $h_\ell[m]$?

$h_\ell[m]$: random process

L6

28 January 2019 08:45

Rayleigh fading model

$$h_\ell[m] = \sum_i a_i^b\left(\frac{m}{W}\right) \operatorname{sinc}\left(\ell - W\tau_i\left(\frac{m}{W}\right)\right)$$

(Rich scattering) → Large # of statistically independent paths

i in $\left[\frac{\ell}{W} - \frac{1}{2W}, \frac{\ell}{W} + \frac{1}{2W}\right]$

$e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)}$ term: Phase $2\pi f_c \tau_i\left(\frac{m}{W}\right) = 2\pi \frac{d_i}{\lambda}$ (d_i: distance travelled by i^{th} path)

$\left(\frac{d_i}{c} \cdot f_c = \frac{d_i}{\lambda}\right)$

$d_i \gg \lambda$ (≈ 1000 m, $\frac{1}{3}$ m)

f_c 900 MHz

Lecture 7

29 January 2019 11:58

Assume $2\pi f_c \tau_i$: uniformly distributed in $[0, 2\pi]$

$(d_i \gg \lambda)$

Assume phases of different paths are independent.

$$h_l[m] = \sum_i a_i\left(\frac{m}{W}\right) e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)} \operatorname{sinc}\left[l - \tau_i\left(\frac{m}{W}\right) W\right]$$

$2\pi f_c \tau_i\left(\frac{m}{W}\right)$ → uniform $[0, 2\pi]$

large no. of terms

$\sim$ Gaussian (central limit theorem)

L7

29 January 2019 12:09

$h_l[m] \sim CN(0, \sigma_l^2)$ for each l

$h_l[m]$: small-scale fading model

σ_l^2: Variance different for each l

It can also be time-varying (a fn. of m) over large timescales (over several secs)

called Rayleigh fading model because

if $h_l[m] \sim CN(0, \sigma_l^2)$, then $X = |h_l[m]| \sim$ Rayleigh distributed

$$f_X(x) = \frac{x}{\sigma_l^2} e^{-x^2/2\sigma_l^2} \quad \text{for } x \geq 0$$

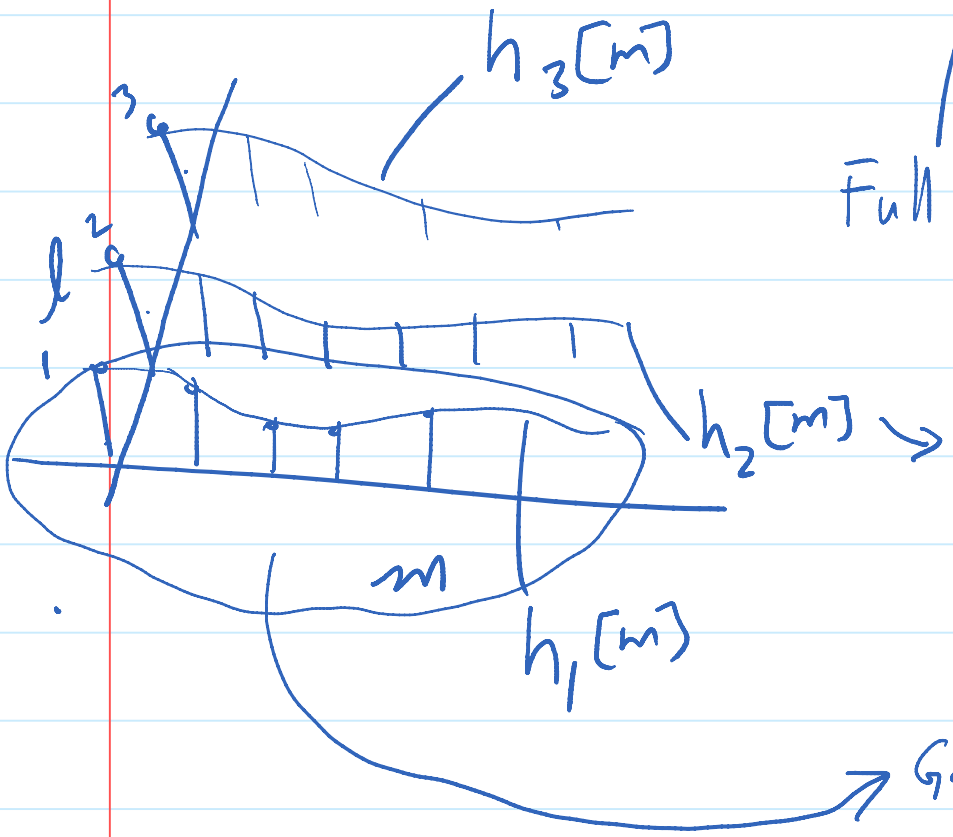
$|h_l[m]|^2 \sim$ exponential $\left(\frac{1}{\sigma_l^2} e^{-x/\sigma_l^2}, x \geq 0\right)$

L7

29 January 2019 12:16

$h_\ell[n]$: Discrete-time Random process

Complex Gaussian (for each n zero mean variance σ_ℓ^2)



Gaussian process

Full description

✓ mean fn $\rightarrow 0$

Autocorrelation fn $\rightarrow R_h[n]$

$$E\left[h_\ell^*[m]\, h_\ell[m+n]\right]$$

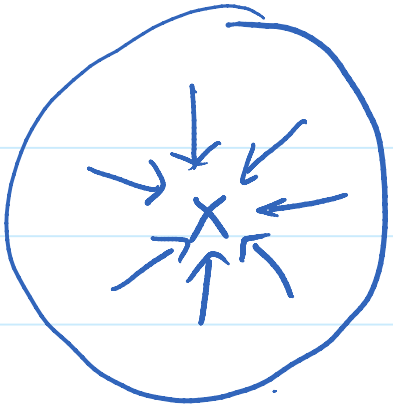
✓ $R_h[0] = \sigma_\ell^2$

$h_\ell[m]$ & $h_{\ell'}[m']$ are indep. for $\ell \neq \ell'$

L7

29 January 2019 12:23

Modeling the autocorrelation



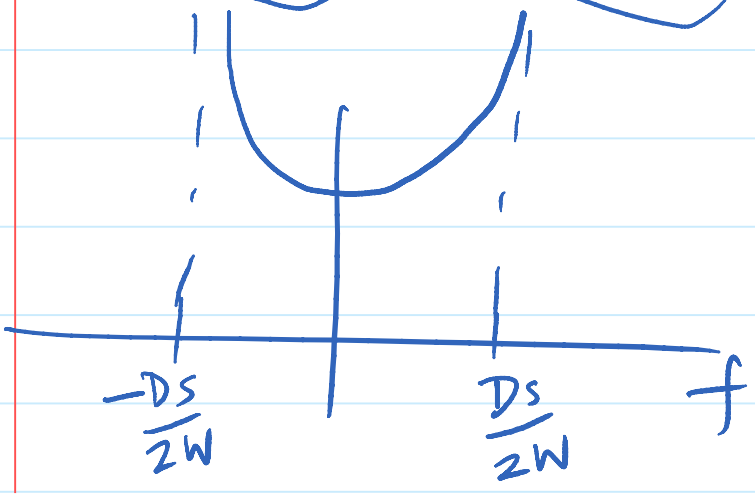
Clarke's model

$$R_h[n] = 2\pi a^2 J_0\left(n\pi \frac{D_s}{W}\right)$$

↑ increases with velocity

Problem 2.17 →

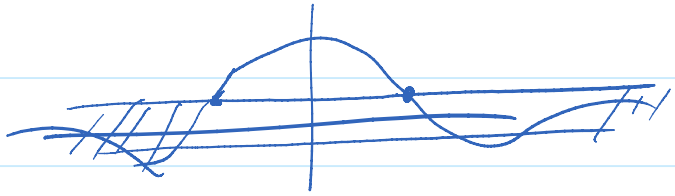




$$S_h(f) = \begin{cases} \dfrac{4a^2W}{D_s\sqrt{1-\left(\frac{2fW}{D_s}\right)^2}} & \text{for } \dfrac{-D_s}{W} \le f \le \dfrac{D_s}{W} \\ 0 & \text{else} \end{cases}$$

L7

29 January 2019 12:33



For this model

$R_l[0] : E[|h_l[n]|^2]$ energy of lth tap

$\sum_l R_l[0]$: Total energy

Multipath spread / Delay spread T_d : (Range of l that contains most of the energy) / W

↑ eg. 99% 100%

Coherence time T_C : $\begin{cases} \text{smallest } n \text{ such that} \\ \dfrac{R_h[n]}{R_h[0]} \text{ below a certain threshold} \end{cases}$

eg. 0.9 0.8

L7

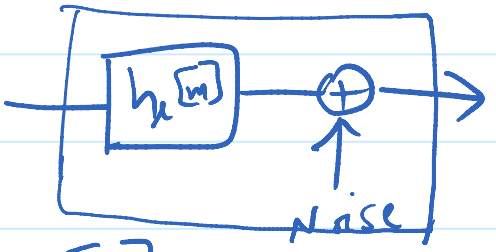
29 January 2019 12:38

Note: If $W \uparrow$, taps are separated by $\frac{1}{W} \downarrow$.

$\left[\frac{l}{W} - \frac{1}{2W}, \frac{l}{W} + \frac{1}{2W}\right]$ $\Rightarrow$ fewer actual physical paths contribute to each $h_l[n]$

$\Rightarrow$ Rayleigh model will be poorer for large W

summary of small scale fading model:



Rayleigh {
- channel with multiple time-varying taps $h_l[n]$
- $\{h_l[n]\}$ is a stationary Gaussian process for each l, mean 0, $R_h[n]$: clarke's model
- taps are independent of each other

L7

29 January 2019 12:46

Rician fading

$$h_l[n] = \left(\sqrt{\frac{K}{K+1}}\,\sigma_l\, e^{j\theta}\right) + \left(\sqrt{\frac{1}{K+1}}\,\mathcal{CN}(0, \sigma_l^2)\right)$$

Line-of-sight

$\left(\sqrt{\frac{K}{K+1}}\,\sigma_l\, e^{j\theta}\right)$: constant

$\left(\sqrt{\frac{1}{K+1}}\,\mathcal{CN}(0, \sigma_l^2)\right)$: Rayleigh

$\frac{K\sigma_l^2}{K+1}$: Energy

$\frac{\sigma_l^2}{K+1}$: Energy

K small → Rayleigh

K large → Constant

$+$

σ_l^2

Lecture 8

31 January 2019 11:05

Statistical models for large scale fading

$\frac{1}{\overline{PL}(r)}$ → σ_l^2 for each path → Parameters in the small scale model.

Freespace propagation

- Average rx power decreases as $\frac{1}{r^2}$

2-ray ground reflection model

- Average rx power decreases as $\frac{1}{r^4}$.

General model that includes the above special cases:

$\frac{\text{Avg tx power}}{\text{Avg rx power}}$ ← Average path loss $\overline{PL}(r) \propto \left(\frac{r}{r_0}\right)^{\gamma}$

γ: Path loss exponent r_0: reference distance

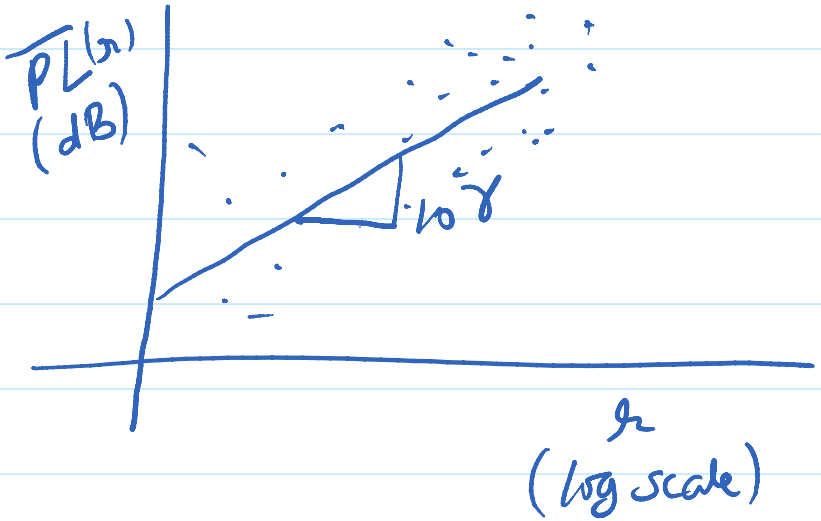
L8

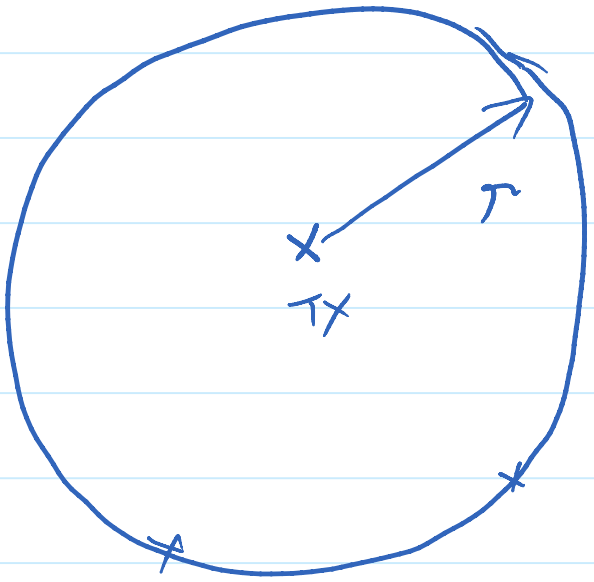
31 January 2019 11:12

in log scale (dB scale)

$$\underbrace{10\log_{10}\overline{PL}(r)}_{\overline{PL}_{(dB)}(r)} = \gamma\, 10\log_{10}\frac{r}{r_0} + \underbrace{10\log_{10}K}_{\text{constant}}$$

$$\overline{PL}_{(dB)}(r) = \underbrace{10\log_{10}K}_{\text{constant}} + 10\gamma\log_{10}\left(\frac{r}{r_0}\right)$$





1 km → 2 km

extra PL $\approx 10\gamma(0.3) = 3\gamma$

6 dB ($\gamma=2$)

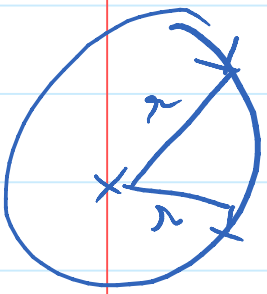
12 dB ($\gamma=4$)

L 8

31 January 2019 11:17

* The above model is too simple.

At the same distance from tx, we will generally have different path loss at diff. locations



⇒ We have to model this

$$PL(r)\ (dB) = \overline{PL}(r)\ (dB) + X_\sigma\ (dB)$$

$\overline{PL}(r)$ (dB) $\rightarrow$ r^γ

X_σ (dB) $\sim N(0, \sigma^2)$ — 'Shadowing' — e.g. 8 dB

'Log normal shadowing model.'

$\log(Z) \sim N(\ )$ — log normal

L8

31 January 2019 11:27

There are more sophisticated large scale fading models for specific scenarios.

Eg: Longley-Rice model
Hata model
Extensions of Hata model
Okamura model

X

L8

31 January 2019 11:30

Point-to-point communication

1) Orthogonal transmission over a Rayleigh fading channel

Single-tap channel (or) flat fading (in freq.)

$$y[m] = \underbrace{h[m]}_{\sim CN(0,1)} \underbrace{x[m]}_{} + \underbrace{w[m]}_{\sim CN(0,N_0)}$$

[Assumption on autocorrelation of $\{h[n]\}$ not reqd. now. Will talk about it when required]

Channel constant over one transmission time (one symbol period)

L8

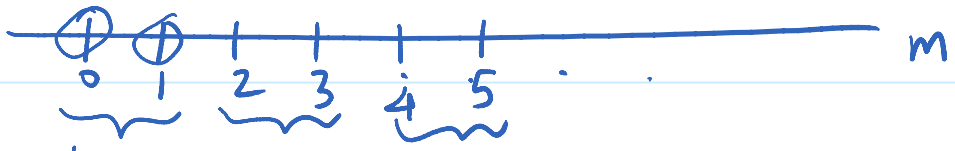
31 January 2019 11:38

* $h[m]$ is not known at the receiver
  (But it is known that $h[m] \sim \mathcal{CN}(0,1)$)

* Since phase of $h[m]$ is uniform in $[0, 2\pi]$, and unknown at rx, we cannot use the phase to send information

* Consider orthogonal signalling as follows

'over 2 samples'



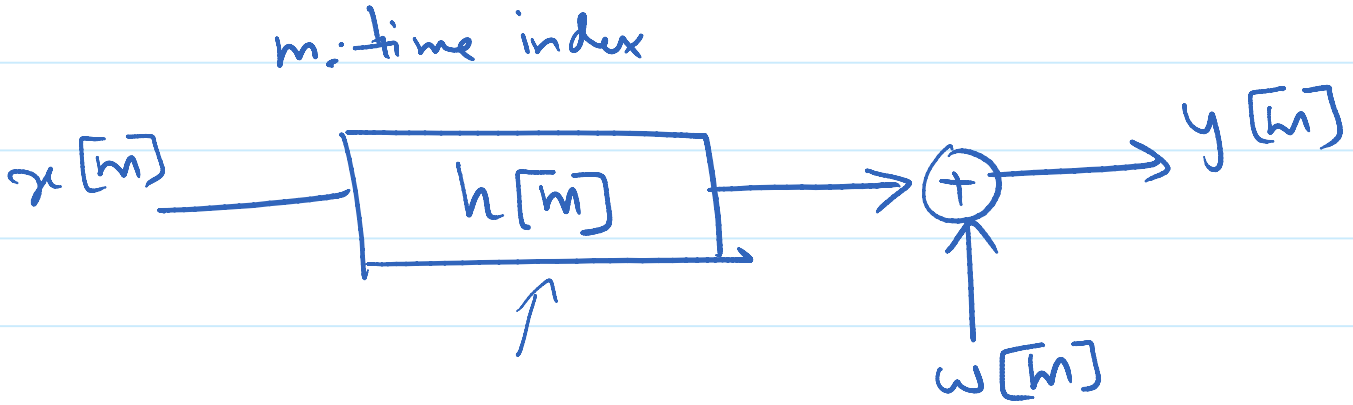
2 possible symbols

Symbol A: $\underline{x}_A = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Symbol B: $\underline{x}_B = \begin{pmatrix} 0 \\ a \end{pmatrix}$

1 bit of information

L8

31 January 2019 11:45



Want to send A B A A B B A . . . .

$x[0]$ $x[1]$ $x[2]$ $x[3]$ $x[4]$ $x[5]$ $x[6]$ $x[7]$

$$\begin{cases} y[0] = h[0]\,a + w[0] \\ y[1] = 0 + w[1] \end{cases}$$

$$\begin{cases} y[2] = 0 + w[2] \\ y[3] = h[3]\,a + w[3] \end{cases}$$

$\vdots$

a 0 | 0 a | a 0 | a 0

A B A A

$x[8]$ $x[9]$ $x[10]$ $x[11]$ . . . .

0 a | 0 a

B B

How should the rx decide what was transmitted? — optimal scheme

— Prob. of error

Lecture 9

04 February 2019 07:58

$\underline{x}, \underline{y}$ orthogonal if $\underline{x}^H \underline{y} = 0$

Optimal receiver for this scheme.

Minimize Prob. error.

Let $\hat{\underline{x}}$ be the decision.

$$Pr(\hat{\underline{x}} \neq \underline{x}) = Pr(\hat{\underline{x}} \neq \underline{x} / \underline{x} = \underline{x}_A) P(\underline{x} = \underline{x}_A) + Pr(\hat{\underline{x}} \neq \underline{x} / \underline{x} = \underline{x}_B) P(\underline{x} = \underline{x}_B)$$

($\underline{x}$: Transmitted)

$$= Pr(\hat{\underline{x}} = \underline{x}_B / \underline{x} = \underline{x}_A) P(\underline{x} = \underline{x}_A) + Pr(\hat{\underline{x}} = \underline{x}_A / \underline{x} = \underline{x}_B) P(\underline{x} = \underline{x}_B)$$ ✓

$$\hat{\underline{x}} \text{ (MAP rule)} = \arg \max_{\underline{x}'} P[\underline{x} = \underline{x}' / \underline{y}]$$

L 9

04 February 2019 08:11

$$P[\hat{x} \neq x] = \int P[\hat{x} \neq x / y] \, f_y(y) \, dy$$

$P[\hat{x} \neq x]$ ↓

$f_y(y) > 0$

$P[\hat{x} \neq x / y]$ ↓ minimize this for each y

choose $\hat{x}$ such that $P[\hat{x} \neq x / y]$ is min

$P[\hat{x} = x / y]$ is max (MAP rule)

If $P[x = x_A / y] > P[x = x_B / y]$, $\hat{x} = x_A$

$P[x = x_A / y] \leq P[x = x_B / y]$, $\hat{x} = x_B$

} optimal decision rule

L9

04 February 2019 08:16

Bayes rule

$$P[\underline{x} = \hat{\underline{x}} / \underline{y}] = \frac{f_{\underline{y}/\underline{x}=\hat{\underline{x}}}(\underline{y})\, P[\underline{x} = \hat{\underline{x}}]}{f_{\underline{y}}(\underline{y})}$$

If $f_{\underline{y}/\underline{x}=\underline{x}_A}(\underline{y})\, P[\underline{x} = \underline{x}_A] > f_{\underline{y}/\underline{x}=\underline{x}_B}(\underline{y})\, P[\underline{x} = \underline{x}_B]$,

$\hat{\underline{x}} = \underline{x}_A$

else $\hat{\underline{x}} = \underline{x}_B$

To decide on the first symbol, we need use only $y[0]$ & $y[1]$

($y[2], \ldots$ are independent of first symbol)

To decide on $(m+1)^{th}$ symbol, use $y[2m], y[2m+1]$.

L9

04 February 2019 08:23

$$\underline{y} = \begin{bmatrix} y[2m] \\ y[2m+1] \end{bmatrix} \begin{matrix} \rightarrow h[2m]a + w[2m] \\ \rightarrow 0 + w[2m+1] \end{matrix} \sim CN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a^2+N_0 & 0 \\ 0 & N_0 \end{bmatrix} \right)$$

($h[2m] \sim CN(0,1)$, $w[2m] \sim CN(0,N_0)$, $w[2m+1] \sim CN(0,N_0)$)

$$\underline{y} \text{ given } \underline{x} = \underline{x}_A \sim CN\left(\underline{0}, \underbrace{\begin{bmatrix} a^2+N_0 & 0 \\ 0 & N_0 \end{bmatrix}}_{C_A} \right) \checkmark$$

$$\underline{y} \text{ given } \underline{x} = \underline{x}_B \sim CN\left(\underline{0}, \underbrace{\begin{bmatrix} N_0 & 0 \\ 0 & a^2+N_0 \end{bmatrix}}_{C_B} \right) \checkmark$$

If $f_{\underline{y}/\underline{x}=\underline{x}_A}(\underline{y})\, P[\underline{x}=\underline{x}_A] > f_{\underline{y}/\underline{x}=\underline{x}_B}(\underline{y})\, P[\underline{x}=\underline{x}_B]$, $\hat{\underline{x}} = \underline{x}_A$.

(assume equal)

else, $\hat{\underline{x}} = \underline{x}_B$

L9

04 February 2019 08:28

If $f_{y|x=x_A}(y) > f_{y|x=x_B}(y)$, $\hat{x} = x_A$.

else, $\hat{x} = x_B$

If $\frac{1}{(2\pi)\,|C_A|^{1/2}} e^{-\frac{1}{2}(y^H C_A^{-1} y} > \frac{1}{2\pi\,|C_B|^{1/2}} e^{-\frac{1}{2} y^H C_B^{-1} y}$, $\hat{x} = x_A$

$|C_A| = |C_B| = (a^2 + N_0) N_0$

If $y^H C_A^{-1} y < y^H C_B^{-1} y$, $\hat{x} = x_A$

$$y^T C_A^{-1} y = \begin{bmatrix} y^*[2m] & y^*[2m+1] \end{bmatrix} \begin{bmatrix} \frac{1}{a^2+N_0} & 0 \\ 0 & \frac{1}{N_0} \end{bmatrix} \begin{bmatrix} y[2m] \\ y[2m+1] \end{bmatrix} = \frac{|y[2m]|^2}{a^2+N_0} + \frac{|y[2m+1]|^2}{N_0}$$

L9

04 February 2019 08:32

If $\frac{|y[2m]|^2}{a^2+N_0} + \frac{|y[2m+1]|^2}{N_0} < \frac{|y[2m]|^2}{N_0} + \frac{|y[2m+1]|^2}{a^2+N_0}$,

then $\hat{x} = x_A$

$$|y[2m]|^2\left(\frac{1}{a^2+N_0} - \frac{1}{N_0}\right) + |y[2m+1]|^2\left(\frac{1}{N_0} - \frac{1}{a^2+N_0}\right) < 0$$

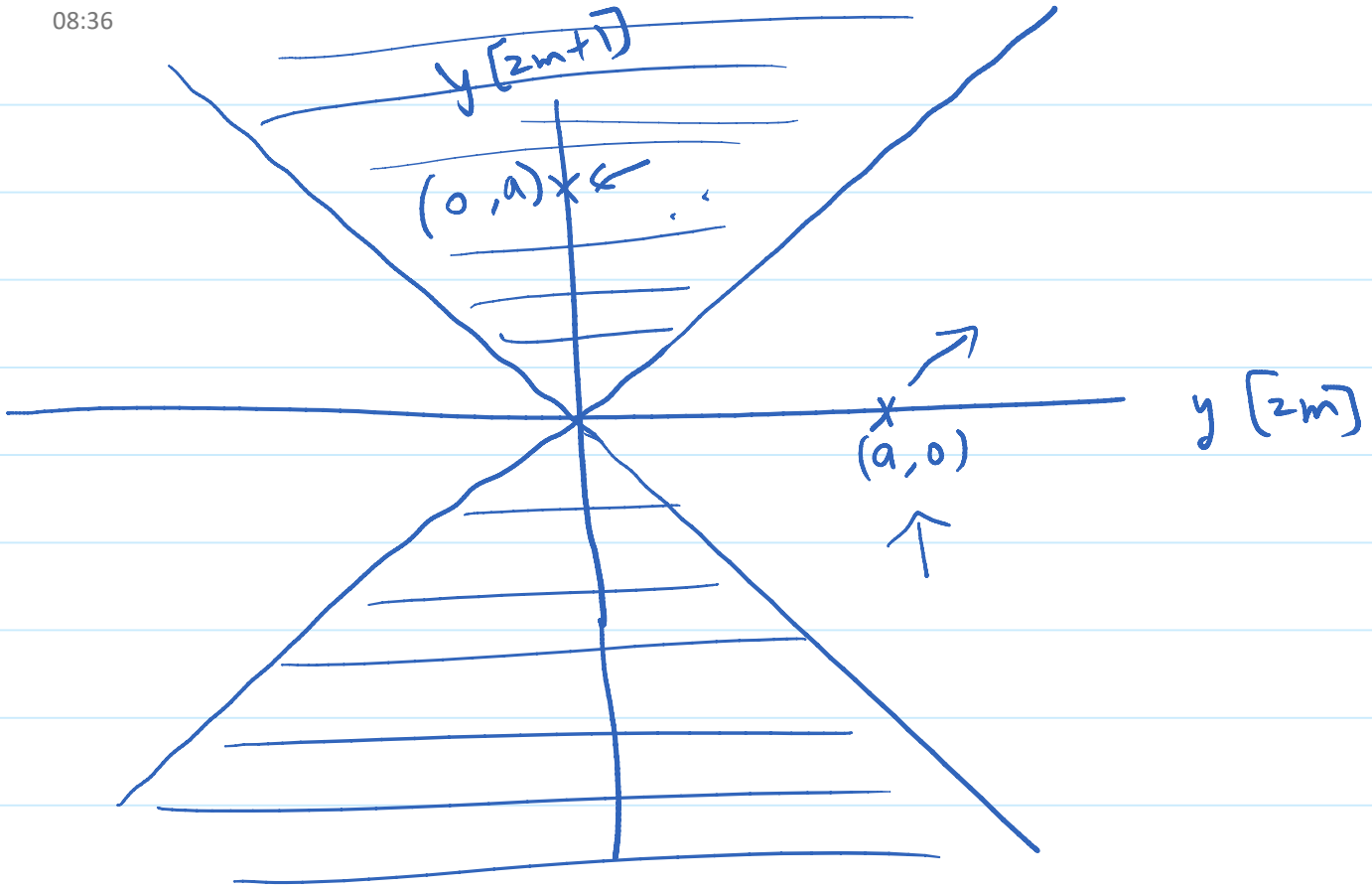
$$|y[2m]|^2\left(\frac{-a^2}{N_0(a^2+N_0)}\right) + |y[2m+1]|^2\left(\frac{a^2}{(a^2+N_0)N_0}\right) < 0$$

If $|y[2m+1]|^2 - |y[2m]|^2 < 0$, $\hat{x} = x_A$

else, $\hat{x} = x_B$

L9

04 February 2019 08:36



shaded region

Decide (0,a)

else

(a,0)

Note: Phase of y[n] not used

$$Pr(error) = P[\hat{x} = x_A / x = x_B] P[x = x_B] \Big/ \tfrac{1}{2} + P[\hat{x} = x_B / x = x_A] P[x = x_A] \Big/ \tfrac{1}{2}$$

L9

04 February 2019 08:41

By symmetry, $P[\hat{\underline{x}} = \underline{x}_A \mid \underline{x} = \underline{x}_B] = P[\hat{\underline{x}} = \underline{x}_B \mid \underline{x} = \underline{x}_A]$

$$P[\hat{\underline{x}} = \underline{x}_B \mid \underline{x} = \underline{x}_A] = P[|y[2m+1]|^2 > |y[2m]|^2 \mid \underline{x} = \underline{x}_A]$$

$\underline{x} = \underline{x}_A$,

$$\begin{bmatrix} y[2m] \\ y[2m+1] \end{bmatrix} \sim \mathcal{CN}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a^2+N_0 & 0 \\ 0 & N_0 \end{bmatrix} \right)$$

$|y[2m+1]|^2$, $|y[2m]|^2$ are independent

($y[2m]$, $y[2m+1]$ are indep)

$|y[2m+1]|^2 \triangleq Z_1 \sim$ exponential (N_0)

$|y[2m]|^2 \triangleq Z_0 \sim$ exponential (a^2+N_0)

L13

11 February 2019 08:19

L9

04 February 2019 08:45

given $\underline{x} = \underline{x}_A$

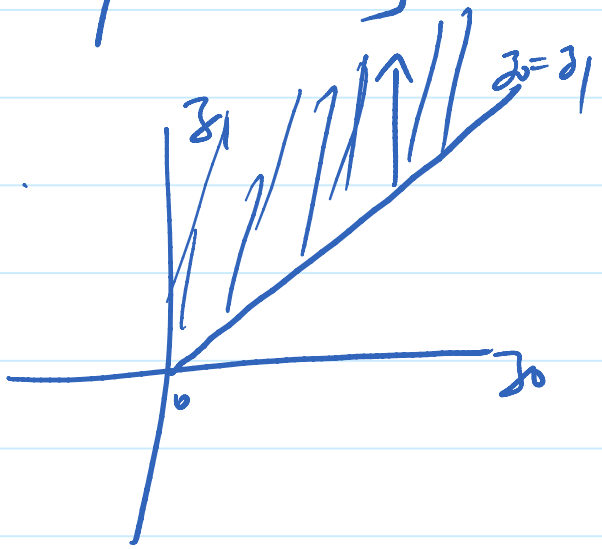
$$f_{Z_0}(z_0) = \frac{1}{a^2+N_0} e^{-\frac{z_0}{a^2+N_0}} \qquad f_{Z_1}(z_1) = \frac{1}{N_0} e^{-\frac{z_1}{N_0}}.$$

$$P(\text{error}) = \frac{1}{2} P(\text{error} \mid \underline{x} = \underline{x}_A) + \frac{1}{2} P(\text{error} \mid \underline{x} = \underline{x}_B)$$

$$= P(\text{error} \mid \underline{x} = \underline{x}_A) = P\left[Z_1 > Z_0 \mid \underline{x} = \underline{x}_A\right]$$

$$= \int_0^\infty \int_{z_0}^\infty f_{Z_0}(z_0) f_{Z_1}(z_1) \, dz_1 \, dz_0$$

$$= \frac{1}{2 + \frac{a^2}{N_0}}.$$



Lecture 10

05 February 2019 11:57

$$Pr(error) = \int_0^{\infty} \left[\int_{\gamma_0}^{\infty} f_{Z_1}(\gamma_1)\, d\gamma_1 \right] f_{Z_0}(\gamma_0)\, d\gamma_0$$

$$\int_{\gamma_0}^{\infty} \frac{1}{N_0} e^{-\gamma_1/N_0}\, d\gamma_1 = \frac{1}{N_0} \frac{e^{-\gamma_1/N_0}}{(-1/N_0)} \Bigg|_{\gamma_0}^{\infty} = -e^{-\gamma_1/N_0} \Big|_{\gamma_0}^{\infty}$$

$$= e^{-\gamma_0/N_0}$$

$$\int_0^{\infty} e^{-\gamma_0/N_0} \frac{1}{a^2+N_0} e^{-\gamma_0/(a^2+N_0)}\, d\gamma_0 = \frac{1}{a^2+N_0} \frac{e^{-\gamma_0\left(\frac{1}{N_0}+\frac{1}{a^2+N_0}\right)}}{-\left(\frac{1}{N_0}+\frac{1}{a^2+N_0}\right)} \Bigg|_0^{\infty}$$

L10

05 February 2019 12:09

$$= \frac{1}{a^2+N_0} \frac{1}{\left(\frac{1}{N_0}+\frac{1}{a^2+N_0}\right)} = \frac{1}{\frac{a^2+N_0}{N_0}+1}$$

$$= \frac{1}{2+a^2/N_0}$$

a^2 : Average transmitted/received signal energy over 2 sym intervals

N_0 : Noise variance for each symbol interval

$\frac{a^2}{2N_0}$: Average SNR

$$P_e = \frac{1}{2+\frac{a^2}{N_0}} = \frac{1}{2(1+SNR)} \propto (SNR)^{-1}$$

for high SNR

L 10

05 February 2019 12:14

channel model $y[m] = h[m]\,x[m] + w[m]$

$h[m] \sim CN(0,1)$ Unknown at receiver

$w[m] \sim CN(0, N_0)$

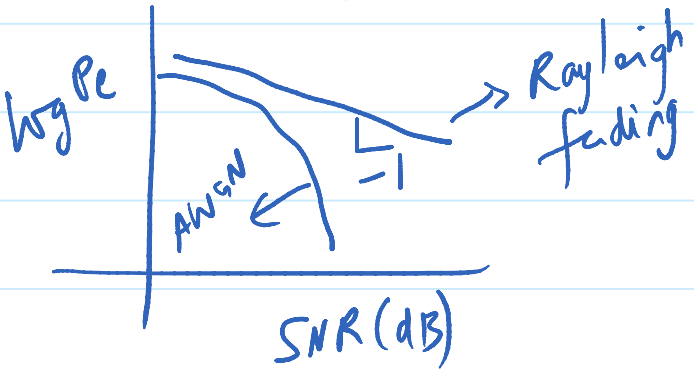
$$P_e = \frac{1}{2(1+SNR)}$$

AWGN $y[m] = x[m] + w[m]$

$w[m] \sim CN(0, N_0)$

P_e goes down exponentially with SNR

$$P_e < ae^{-SNR\,b}$$



L10

05 February 2019 12:21

2) $h[m]$ is known at the receiver

$$y[m] = h[m]\,x[m] + w[m]$$

($h[m]$ has been estimated at the receiver perfectly.
Possible if coherence time $\gg$ symbol time)

$x[m]$ is $+a$ or $-a$ with equal probability

What is the optimal decision rule?

to decide on $x[m]$, it is enough to use $y[m]$

(Drop index m) $\quad y = h\,x + w$

h: known

$w \sim CN(0, N_0)$

L 10

05 February 2019 12:27

If $P[x=a/y] > P[x=-a/y]$, $\hat{x}=a$

else $\hat{x}=-a$

$$\text{If } \frac{f_{y/x=a}(y)\,P[x=a]}{f_y(y)} > \frac{f_{y/x=-a}(y)\,P[x=-a]}{f_y(y)}, \quad \hat{x}=a.$$

$y = ha + w$

$CN(ha, N_0)$ $\qquad$ $CN(-ha, N_0)$

If $|y-ha|^2 < |y+ha|^2$, then $\hat{x}=a$

else, $\hat{x}=-a$.

$$|y|^2 + |h|^2a^2 - 2Re(h^*y)a < |y|^2 + |h|^2a^2 + 2Re(h^*y)a$$

If $Re(h^*y) > 0$, $\hat{x}=a$, else, $\hat{x}=-a$

L10

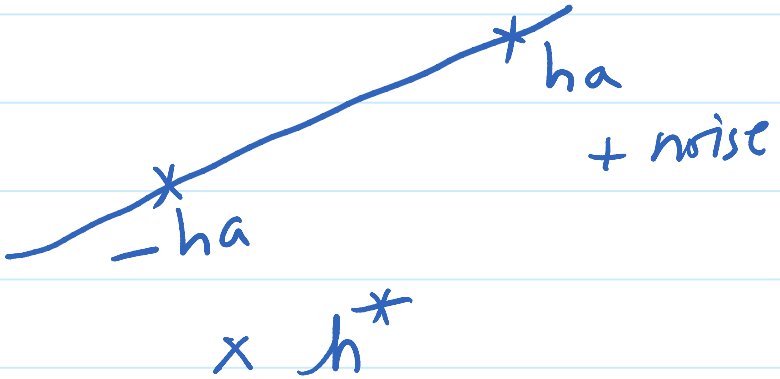
05 February 2019 12:31

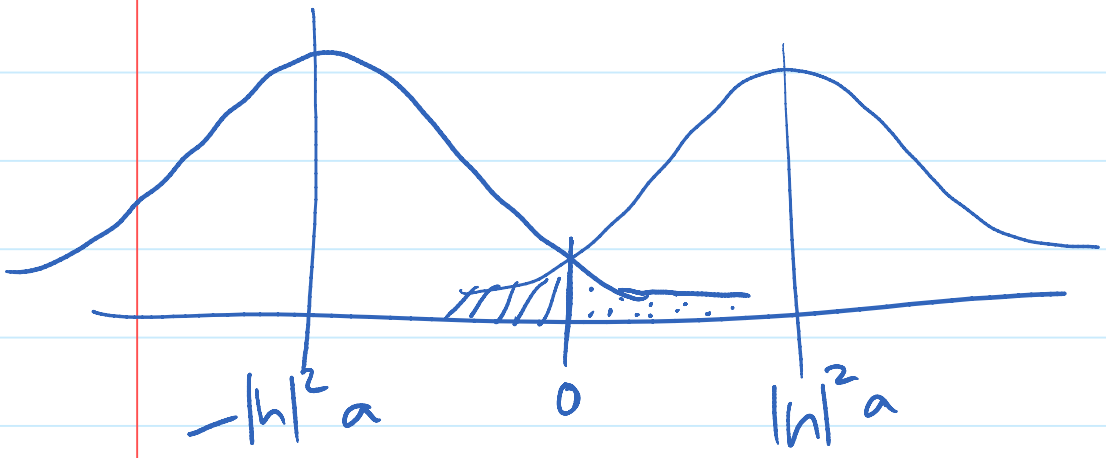
$$Pr(error) = \int \underbrace{P(error/h)}\, f_h(h)\, dh$$

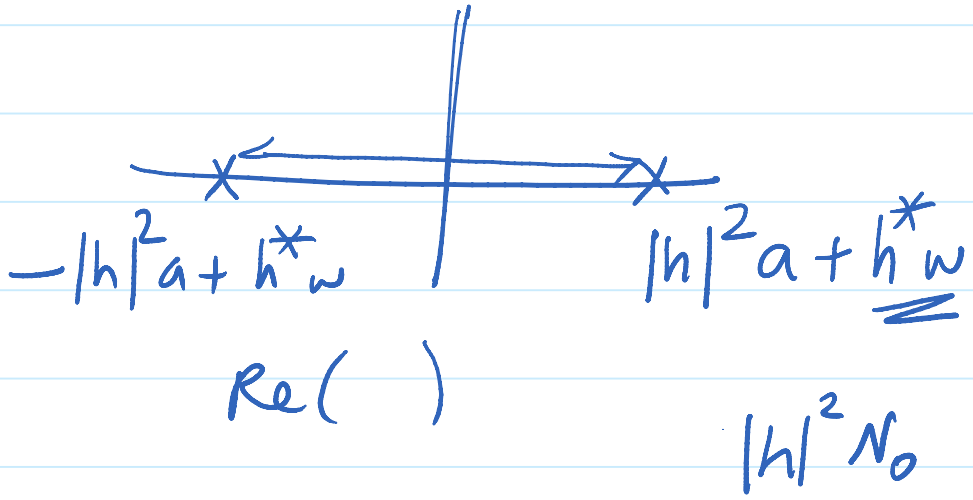


$$Re(h^* y) \overset{x=a}{\sim} N\left(|h|^2 a, \frac{|h|^2 N_0}{2}\right)$$

$$\overset{x=-a}{\sim} N\left(-|h|^2 a, \frac{|h|^2 N_0}{2}\right)$$







L 10

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$$P(\text{error}/h) = Q\left(\frac{|h|^2 a}{\sqrt{\frac{|h|^2 N_0}{2}}}\right)$$

$$= Q\left(\sqrt{2|h|^2 \frac{a^2}{N_0}}\right)$$

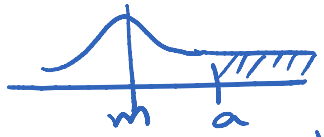
$$= Q\left(\sqrt{2|h|^2 \, SNR}\right)$$

$$\left(\text{Average SNR} \triangleq \frac{a^2}{N_0}\right)$$

$$P(\text{error}) = \int_0^\infty Q\left(\sqrt{2|h|^2 SNR}\right) f_{|h|^2}(|h|^2)\, d(|h|^2)$$

$h \sim CN(0,1)$

$|h|^2 \sim \exp(1)$



$$\left(X \sim N(m, \sigma^2)\right.$$

$$P[X > a]$$

$$= Q\left(\frac{a-m}{\sigma}\right)$$

$$\left.(a > m)\right)$$

L10

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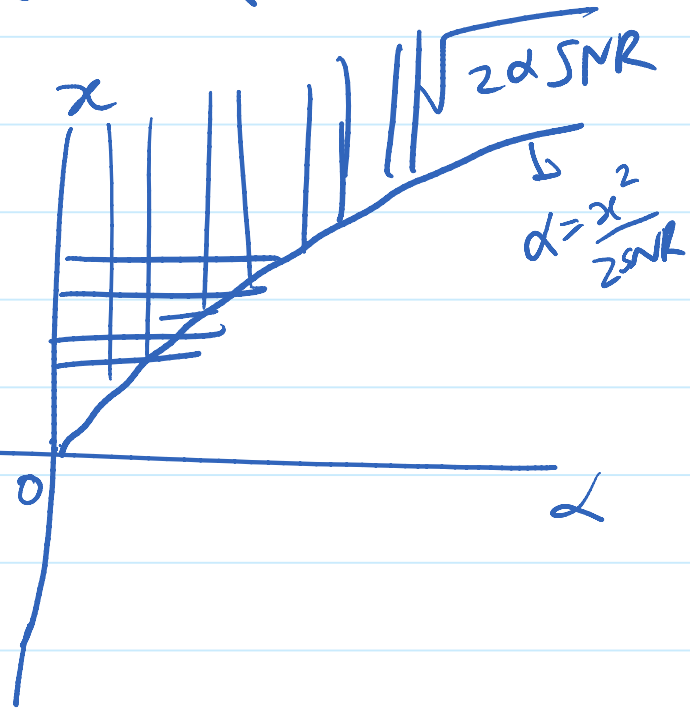
$$P(\text{error}) = \int_0^\infty \underbrace{Q\left(\sqrt{2\alpha\, SNR}\right)}_{|h|^2 \triangleq \alpha} e^{-\alpha}\, d\alpha$$

$$= \int_0^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\alpha SNR}}^\infty e^{-x^2/2}\, dx\right] e^{-\alpha}\, d\alpha$$

$$= \int_0^\infty \int_0^{\frac{x^2}{2SNR}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, e^{-\alpha}\, d\alpha\, dx$$



L10

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$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} \left(1 - e^{-x^2/2SNR}\right) dx$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2\left(\frac{1}{2} + \frac{1}{2SNR}\right)} dx$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{1+\frac{1}{SNR}}}} \int_0^\infty e^{-\frac{x^2}{2}\left(1+\frac{1}{SNR}\right)} dx$$

$\hookrightarrow = \frac{1}{2}$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{SNR}{1+SNR}} \leftarrow \text{ANSWER } Pr(\text{error})$$

Lecture 11

07 February 2019 10:54

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{SNR}{1+SNR}} = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right)$$

$$\sqrt{\frac{SNR}{1+SNR}} = \left(\frac{SNR}{1+SNR}\right)^{1/2} = \left(1 + \frac{1}{SNR}\right)^{-1/2}$$

$$\approx 1 - \frac{1}{2}\cdot\frac{1}{SNR}$$

$(1+x)^{\alpha} \approx 1 + \alpha x$ for small x

At high SNR

$$P_e \approx \frac{1}{2}\left(1 - \left(1 - \frac{1}{2}\frac{1}{SNR}\right)\right) = \frac{1}{4\,SNR} \propto SNR^{-1}$$

In both cases, (1) orthogonal signaling + $h[m]$ not known at the rx

(2) antipodal signaling + $h[m]$ known at the rx

$P_e \propto SNR^{-1}$

L 11

07 February 2019 11:07

Closer look at why performance is poor:

DOMINANT ERROR EVENT at high SNR

For a given h, $P_{e/h} = Q\left(\sqrt{2|h|^2 SNR}\right)$

$|h|^2$ can be small

$|h|^2 SNR$ can be small

$$Pr\left(|h|^2 SNR < 1\right) = Pr\left(|h|^2 < \frac{1}{SNR}\right)$$

$$= \int_0^{1/SNR} e^{-x}\, dx = \frac{e^{-x}}{-1}\Bigg|_0^{1/SNR}$$

$$= 1 - e^{-\frac{1}{SNR}}$$

L11

07 February 2019 11:10

At high SNR

$$P(|h|^2 SNR < 1) \approx 1 - \left(1 - \frac{1}{SNR}\right) = \frac{1}{SNR}$$

e^{-x} for small x

$1 - x + \frac{x^2}{2!} - \cdots$

"Deep fade"

$y = x + w$

✓ For an AWGN channel, dominant error event is just noise being large

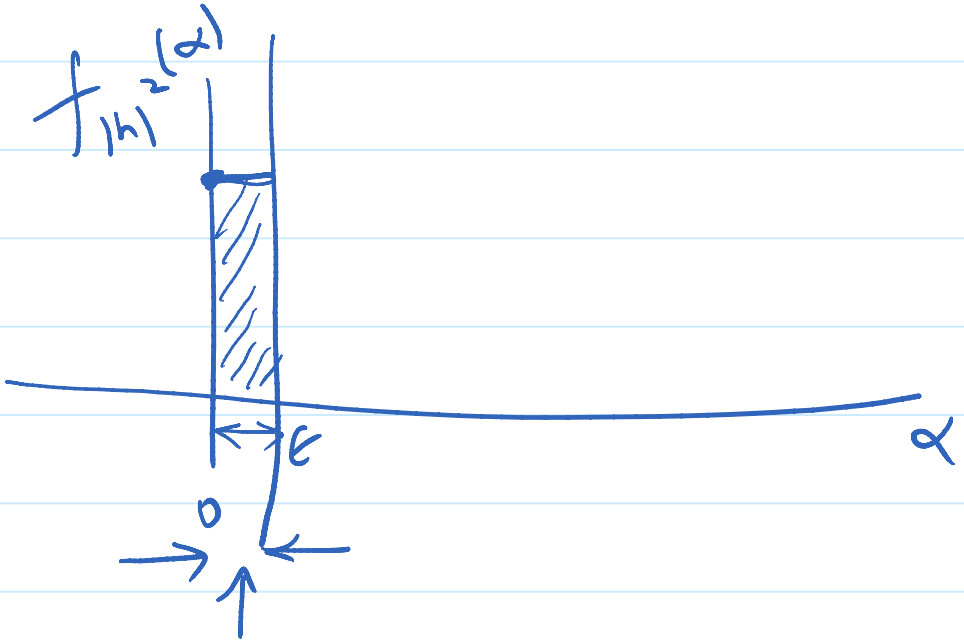
✓ Here, dominant error event is "deep fade".

How much does the above conclusion depend on the 'Rayleigh fading' assumption?

$$P_r(|h|^2 < \epsilon) \approx \epsilon \quad \epsilon = \frac{1}{SNR}$$

L 1h

07 February 2019 11:23

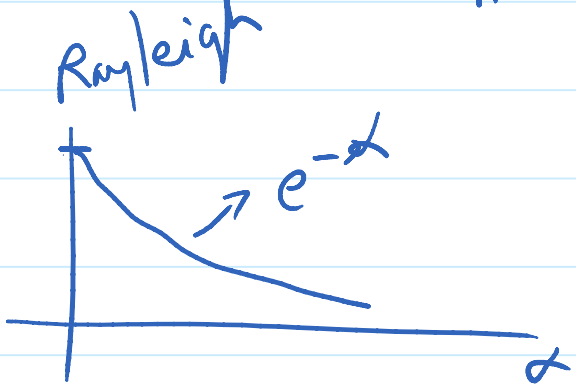


$Pr(|h|^2 < \epsilon)$ = Area under PDF in $[0, \epsilon]$

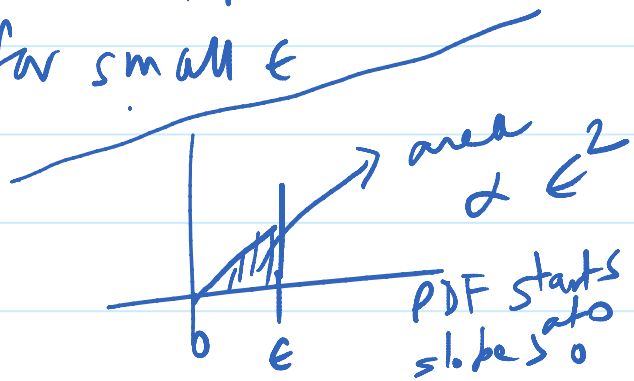
If PDF of $|h|^2$ is nonzero at 0, then

Area $\approx f_{|h|^2}(0) \cdot \epsilon$

$Pr(|h|^2 < \epsilon) \propto \epsilon$ if $f_{|h|^2}(0) > 0$



for small ϵ



L11

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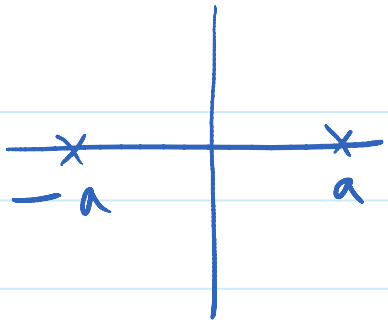
QPSK instead of BPSK ($\pm a$)

(for $h[m]$ known at the rx)

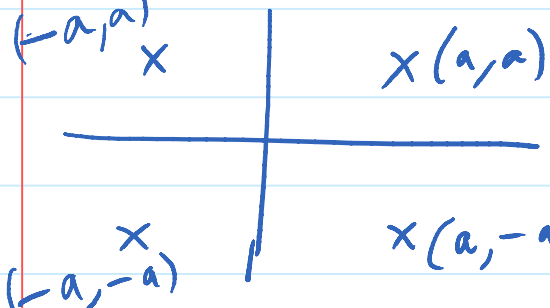


— We used real transmit symbols and discarded $Im(h^* y)$

— We can use both real & imag parts

complex channel has 2 "degrees of freedom"

$SNR_q = \frac{2a^2}{N_0}$



For AWGN channel P(bit error) in BPSK = P(bit error) in QPSK for the same energy per bit

Rayleigh fading case $P(\text{bit error}) = \frac{1}{2}\left(1 - \sqrt{\frac{SNR_q}{2 + SNR_q}}\right)$

L11

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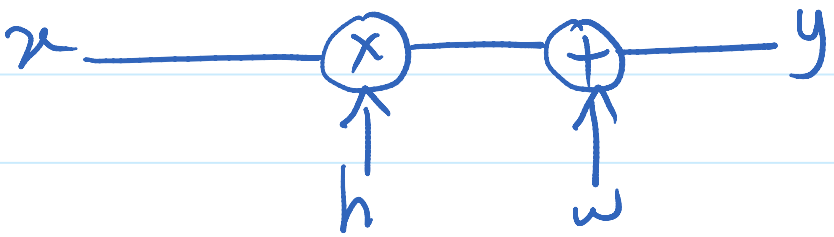
So far, $P_e \propto SNR^{-1}$.

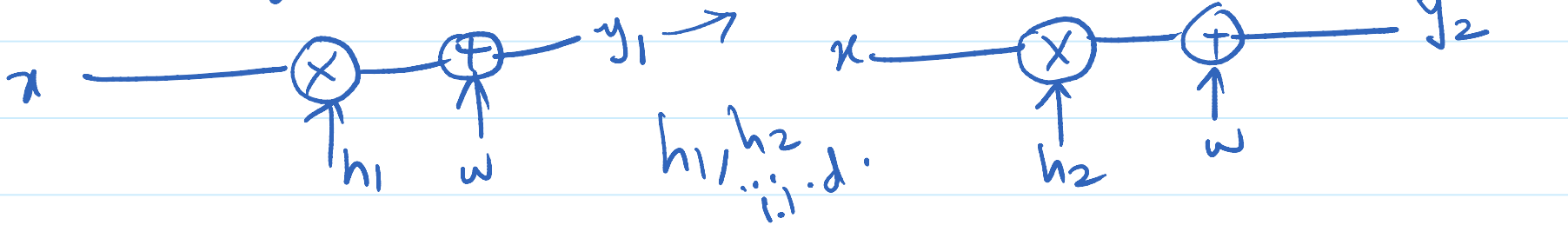
How do we improve upon this?

For eg $Pr(\text{deep fade}) \propto SNR^{-2}$



Diversity

↳ Ensure that information symbols are received through multiple independent channels.



L 11

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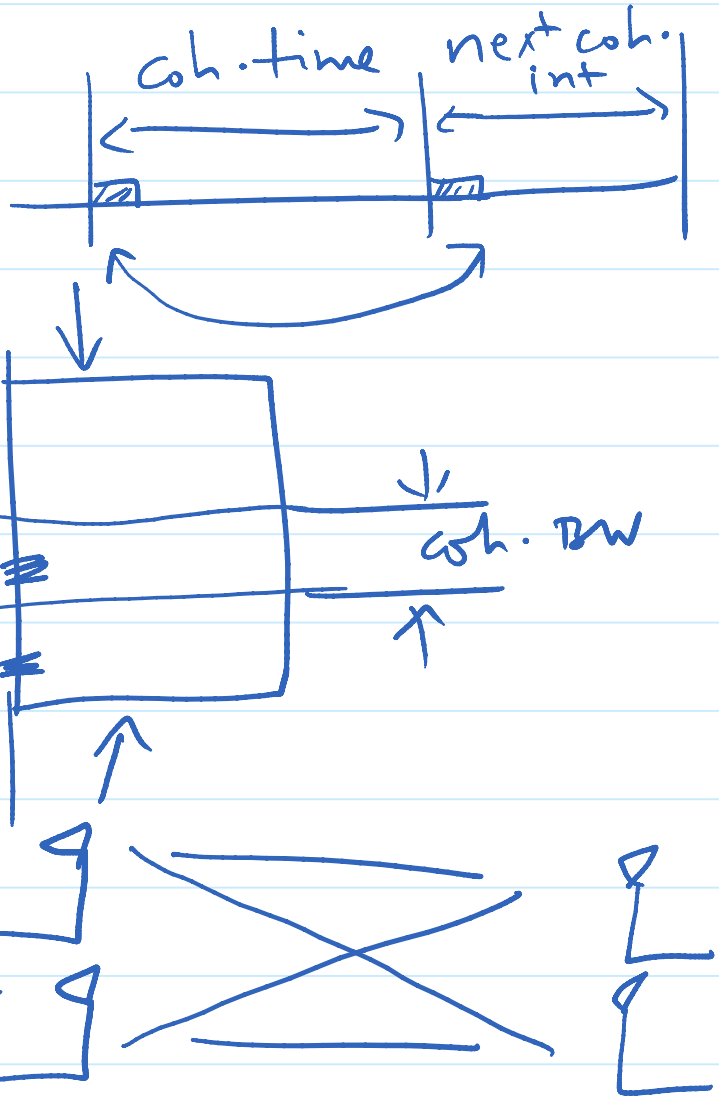
How do we achieve diversity?

- Diversity in time -

- Diversity in freq -

or any combination

- Diversity in space



Lecture 12

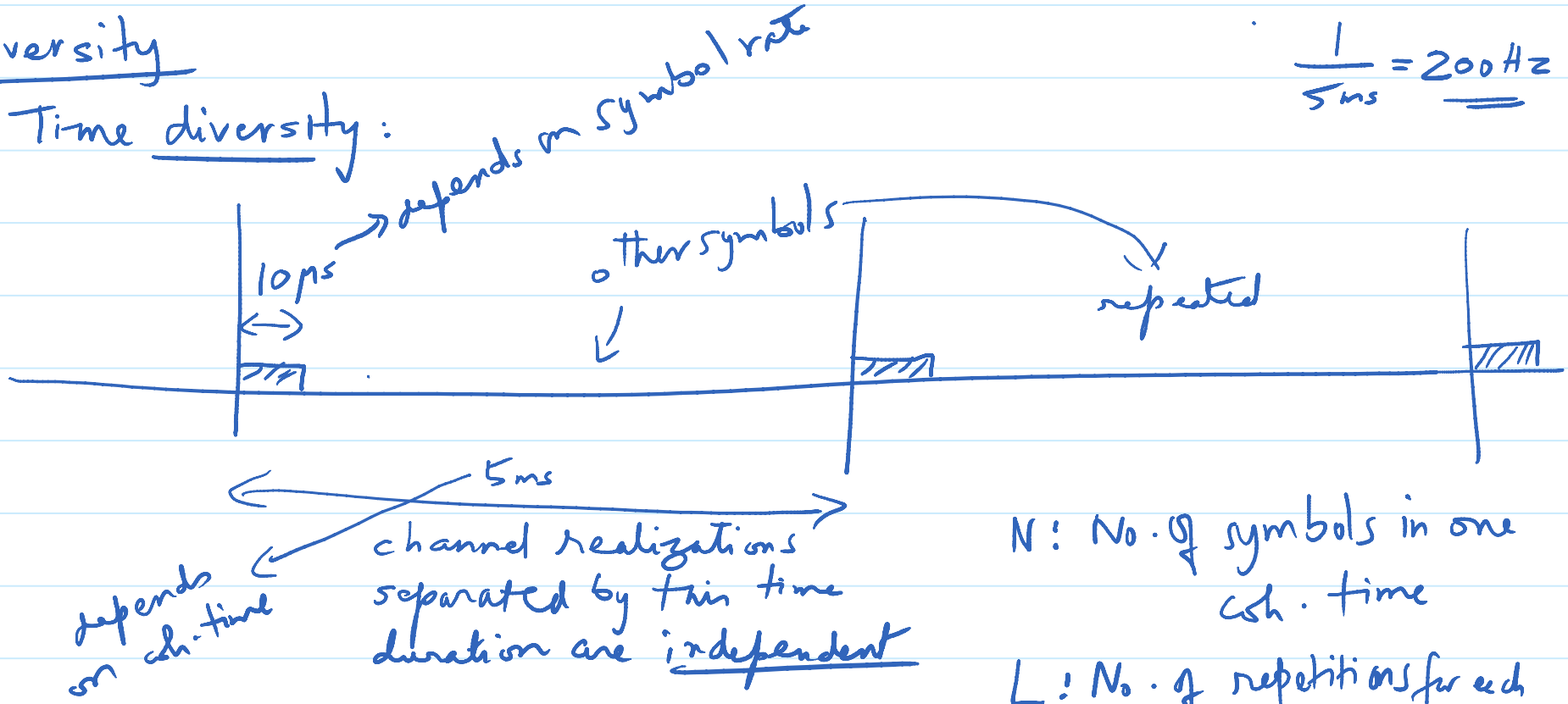
08 February 2019 10:01

Coh time = 5 ms

$\frac{1}{5\,ms} = 200\,Hz$

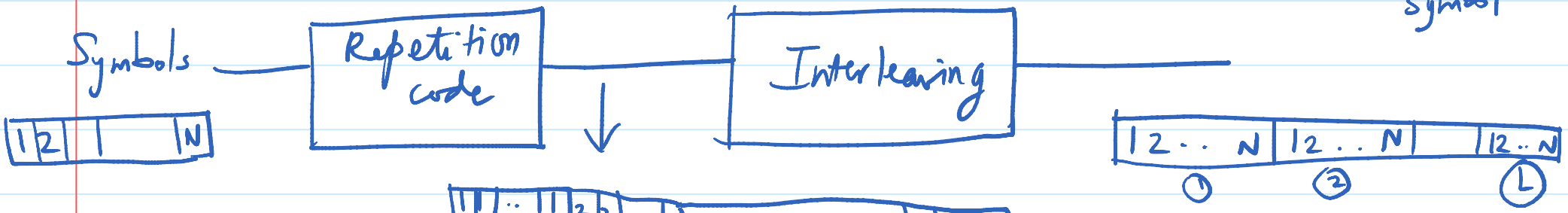
Diversity

Time diversity:



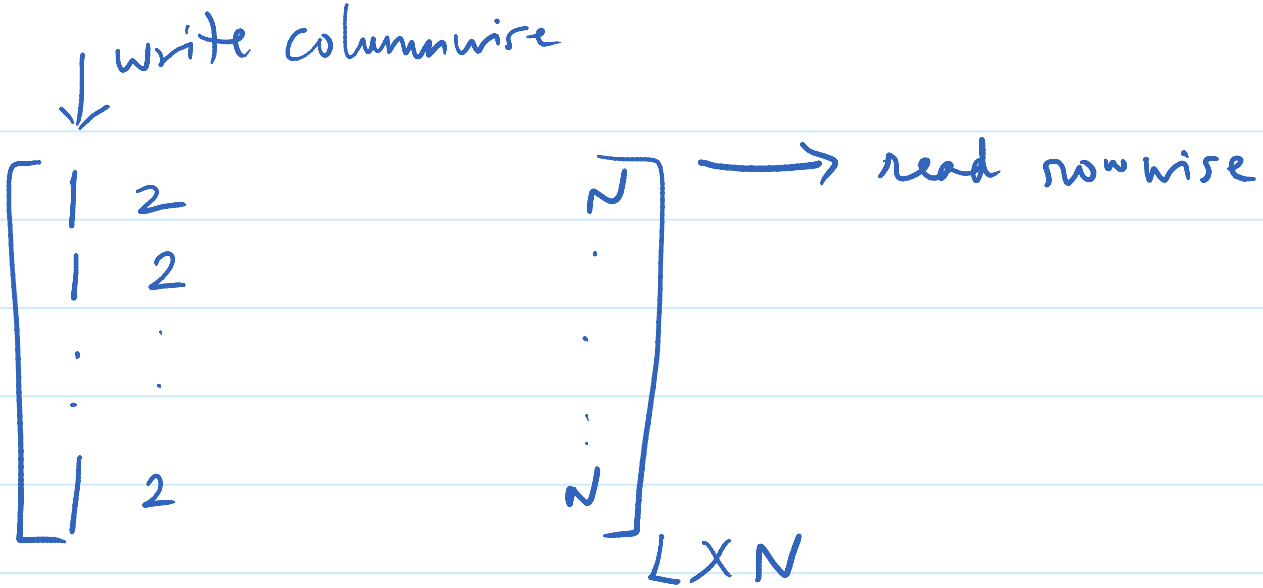
N: No. of symbols in one coh. time

L: No. of repetitions for each symbol



L 12

08 February 2019 10:14



* What is the optimal detection rule to minimize Pr(sym error)?

— For each symbol, we have L received outputs

x : symbol

$$y_1 = h_1 x + w_1$$
$$y_2 = h_2 x + w_2$$
$$\vdots$$
$$y_L = h_L x + w_L$$

h_i's i.i.d $CN(0,1)$

w_i's i.i.d $CN(0, N_0)$.

L12

08 February 2019 10:20

$$\underline{y} = \underline{h}\, x + \underline{w} \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} \quad \underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{pmatrix}$$

- Assume $\underline{h}$ is known at the receiver

x as $\pm a$ (equally likely)

$$\max_{x \to \{a,-a\}} f_{\underline{y}/x}(\underline{y})$$

$$f_{\underline{y}/a}(\underline{y}) : CN(\underline{h}a, N_0 I)$$

$$f_{\underline{y}/-a}(\underline{y}) : CN(-\underline{h}a, N_0 I)$$

$$P[x/\underline{y}] \qquad 1/2$$

$$= \frac{f_{\underline{y}/x}(\underline{y})\, P(x)}{f_{\underline{y}}(\underline{y})}$$

does not depend on x

L12

08 February 2019 10:25

$$f_{\underline{y}|x}(\underline{y}) = \frac{1}{(2\pi)^{L/2}\,|N_0 \underline{I}|^{1/2}}\, e^{-\frac{1}{2}(\underline{y}-\underline{h}x)^H (N_0\underline{I})^{-1}(\underline{y}-\underline{h}x)}$$

$\max$ over x; a or $-a$; $\min$

$$\min_x \; (\underline{y}-\underline{h}x)^H(\underline{y}-\underline{h}x)$$

$$\underline{y}^H\underline{y} - \underline{y}^H\underline{h}x - x^*\underline{h}^H\underline{y} + x^*\underline{h}^H\underline{h}x$$

same for a or $-a$

$$-2\,\mathrm{Re}\left(\underline{h}^H\underline{y}\right)x$$

$\pm a$ (assuming $a>0$)

Decision rule: If $\mathrm{Re}(\underline{h}^H\underline{y}) > 0$, $\hat{x} = a$, else $\hat{x} = -a$

L12

08 February 2019 10:33

* Enough for the optimal rule to look at $\underline{h}^H \underline{y}$ alone

$$\checkmark \quad \underline{y} = \underline{h} x + \underline{w}$$

↓ Sufficient statistic for detection of x.

$$\checkmark \quad r = \underline{h}^H \underline{y} = \boxed{\|\underline{h}\|^2} x + \underline{h}^H \underline{w}$$

scalar

$\underline{h}^H \underline{w} \sim CN(\underline{0}, \underline{h}^H (N_0 I) \underline{h})$

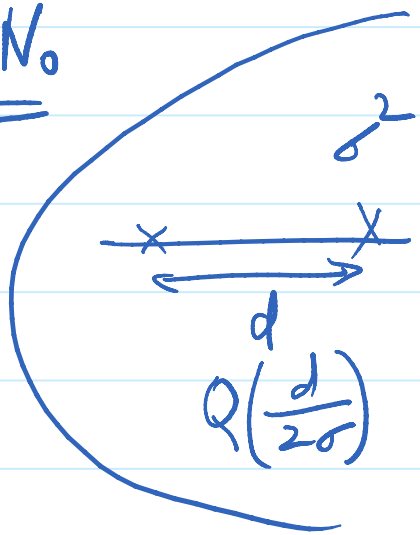
$= \|\underline{h}\|^2 N_0$

If $Re(r) > 0$, $\hat{x} = a$, else, $\hat{x} = -a$.



$$Pr(\text{error}/\underline{h}) = Q\left(\frac{2\|\underline{h}\|^2 a}{2\sqrt{\|\underline{h}\|^2 (N_0/2)}}\right)$$

$$= Q\left(\sqrt{2\|\underline{h}\|^2 \frac{a^2}{N_0}}_{SNR}\right)$$



LIR

08 February 2019 10:40

$$P_r(\text{error}) = \int_0^\infty Q\left(\sqrt{2 \times SNR}\right) \underbrace{f_X(x)}_{\downarrow} \, dx$$

e^{-x} for $L=1$

$$X = \|\underline{h}\|^2 = \sum_{i=1}^{L} |h_i|^2$$

FACT $f_X(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x}$ for $x \geq 0$ chi-square distribution with $2L$ degrees of freedom

$$P_r(\text{error}) = \left(\frac{1-\mu}{2}\right)^L \sum_{\ell=0}^{L-1} \binom{L-1+\ell}{\ell} \left(\frac{1+\mu}{2}\right)^\ell$$

where $\mu = \sqrt{\frac{SNR}{1+SNR}}$.

$$= \frac{1}{2}\left[1 - \mu \sum_{\ell=0}^{L-1} \binom{2\ell}{\ell} \left(\frac{1-\mu^2}{4}\right)^\ell\right]$$

$\left(L=1,\ P_r(\text{error}) = \frac{1-\mu}{2}\right)$

L 12

08 February 2019 10:47

At high SNR

$$\frac{1+\mu}{2} \approx 1 \rightarrow$$

$$\frac{1-\mu}{2} \approx \frac{1}{4SNR}$$

$$\sum_{l=0}^{L-1} \binom{L-1+l}{l} = \binom{2L-1}{L}$$

$$Pr(\text{error}) \approx \binom{2L-1}{L} \frac{1}{(4SNR)^L} \propto (SNR)^{-L}$$

Dominant error event : $Pr\left(\|\underline{h}\|^2 < \frac{1}{SNR}\right) \approx \frac{1}{L!} \frac{1}{(SNR)^L}$

Deep fade

Lecture 13

11 February 2019 07:56

$$P(\text{"Deep fade"}) = P\left(\|\underline{h}\|^2 < \frac{1}{SNR}\right) = \int_0^{\frac{1}{SNR}} f_{\|\underline{h}\|^2}(x)\,dx$$

For small x

$$f_{\|\underline{h}\|^2}(x) = \frac{1}{(L-1)!}\, x^{L-1} \underbrace{e^{-x}}_{\approx (1-x)} \approx \frac{1}{(L-1)!}\, x^{L-1}$$

$$P_r(\text{"Deep fade"}) = \frac{x^L}{L!}\Bigg|_0^{\frac{1}{SNR}} = \frac{1}{SNR^L} \cdot \frac{1}{L!}$$

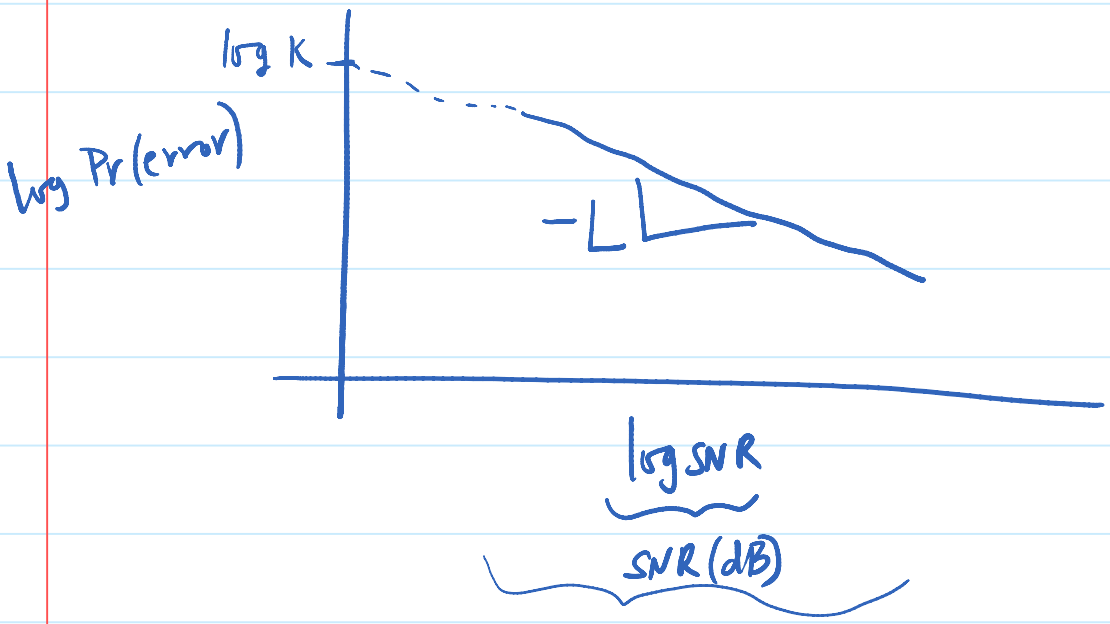
$\int(\quad)$

$$\|\underline{h}\|^2 = \sum_{l=1}^{L} |h_l|^2$$

L : Diversity gain of the system

L13

11 February 2019 08:04



Div. gain = L

$$Pr(error) \approx K \frac{1}{SNR^L}$$

$$\log Pr(error) \approx \log K - L \log SNR$$

Repeat L times over L independent fading channels to get diversity L. (Receiver: $\underline{h}^H \underline{y}$ a sufficient statistic)

Rate of transmission:

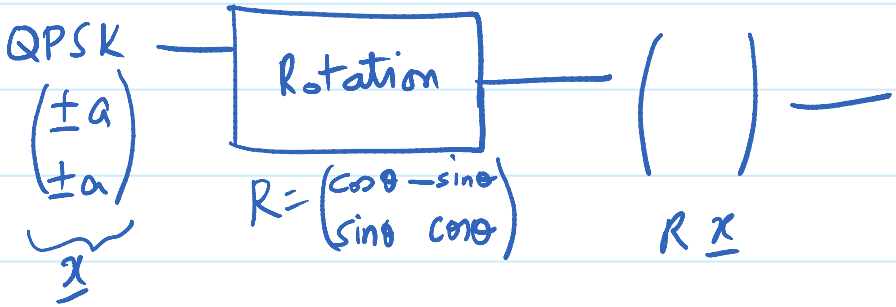
- ± a repeated L times — 1 bit in L channel uses
- ↓ increase in constellation size
- QPSK repeated L times — 2 bits in L channel uses

L13

11 February 2019 08:12

Another way to increase rate : change repetition code (without affecting diversity gain).

Rotation code: 2 bits / channel use, **diversity 2**.



[With repetition 1 symbol in 2 channel uses (diversity 2)

↓

(16 symbol constellation for 2 bits/channel use]

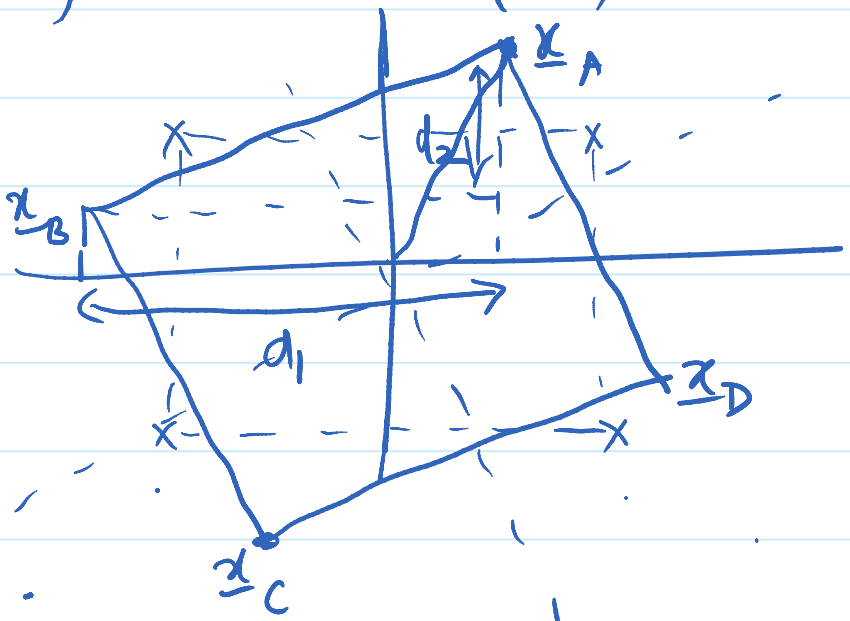
L13

11 February 2019 08:18

Back to rotation code.

4 bits over 2 channel uses

$R\begin{pmatrix}\pm a\\ \pm a\end{pmatrix}$ $\underline{x}_A = R\begin{pmatrix}a\\ a\end{pmatrix}$ $\underline{x}_B = R\begin{pmatrix}-a\\ a\end{pmatrix}$ $\underline{x}_C = R\begin{pmatrix}-a\\ -a\end{pmatrix}$ $\underline{x}_D = R\begin{pmatrix}a\\ -a\end{pmatrix}$



- Each coordinate of $\underline{x}$ will be transmitted thro' an independent channel.
- Do the same in both real & imag. parts.

L13

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h_1 (first coordinate) , h_2 (second coordinate)

Receiver has to decide between

$$\begin{pmatrix} h_1 a \\ h_2 a \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \text{(or)} \quad \begin{pmatrix} h_1(-a) \\ h_2 a \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \text{(or)} \quad \begin{pmatrix} h_1(-a) \\ h_2(-a) \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\text{(or)} \quad \begin{pmatrix} h_1(a) \\ h_2(-a) \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Receiver knows h_1, h_2, use ML decoding $\left[\left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} \quad \\ \uparrow \end{pmatrix} \right\|^2 \right]$

$$Pr(\text{error}) = Pr(\text{error} / \underline{x}_A \text{ is transmitted})$$

(symmetry in constellation, all symbols equally likely)

L13

11 February 2019 08:32

For a given h_1, h_2

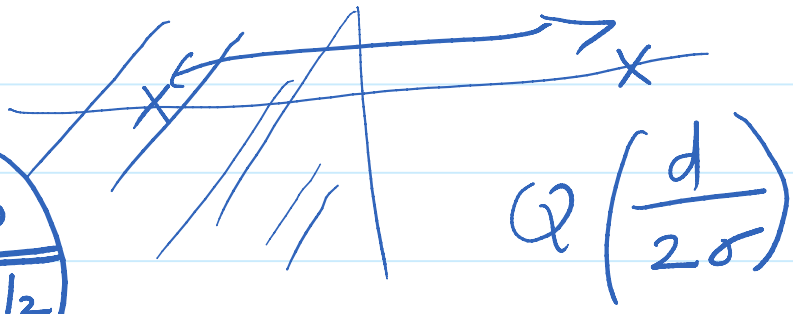
$$Pr(\text{error} \mid h_1, h_2) = Pr\left(\underbrace{\underline{x}_A \to \underline{x}_B}_{\text{Transmit } \underline{x}_A, \text{ decode } \underline{x}_B} \middle| h_1, h_2\right) + Pr\left(\underline{x}_A \to \underline{x}_C \middle| h_1, h_2\right) + Pr\left(\underline{x}_A \to \underline{x}_D \middle| h_1, h_2\right)$$

$$\leq \mathbb{P}\left(\underline{x}_A \to \underline{x}_B \middle| h_1, h_2\right) + \mathbb{P}\left(\underline{x}_A \to \underline{x}_C \middle| h_1, h_2\right) + \mathbb{P}\left(\underline{x}_A \to \underline{x}_D \middle| h_1, h_2\right)$$

Pairwise error prob

(Prob of decoding $\underline{x}_B$ when $\underline{x}_A$ is transmitted, $\underline{x}_A$ and $\underline{x}_B$ are the only 2 constellation pts ~~evidently~~)

$$= Q\left(\frac{d_{A,B}}{2\sqrt{N_0/2}}\right) + Q\left(\frac{d_{A,C}}{2\sqrt{N_0/2}}\right) + Q\left(\frac{d_{A,D}}{2\sqrt{N_0/2}}\right)$$



$$Q\left(\frac{d}{2\sigma}\right)$$

L13

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$$d_{A,B}^2 = \left\| \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \underbrace{\left[R\begin{pmatrix} a \\ a \end{pmatrix} - R\begin{pmatrix} -a \\ a \end{pmatrix} \right]}_{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}} \right\|^2$$

(If there was no R, $d_{A,B}$ will depend only on h_1 ($d_2 = 0$))

$$= |h_1|^2 |d_1|^2 + |h_2|^2 |d_2|^2$$

$$P(\underline{x}_A \to \underline{x}_B / h_1, h_2) = Q\left(\frac{1}{2} \sqrt{\frac{|h_1|^2 |d_1|^2 + |h_2|^2 |d_2|^2}{N_0/2}} \right)$$

$$E_{|h_1|^2, |h_2|^2}[\ \downarrow\ ] = ? \propto \frac{1}{SNR^2}$$

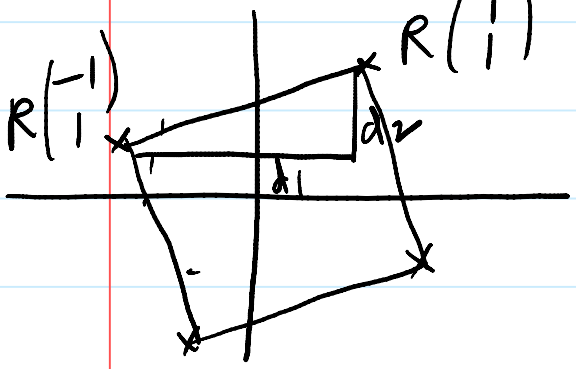
Lecture 14

12 February 2019 12:02

Use $Q(x) \le e^{-x^2/2}$

$$\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\alpha^2/2}\, d\alpha \le e^{-x^2/2}$$

$$P(\underline{x}_A \to \underline{x}_B / h_1, h_2) \le \exp\left(-\frac{a^2}{4}\frac{(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{N_0/2}\frac{1}{2}\right)$$



$$= \exp\left(-a^2\frac{(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{4N_0}\right)$$

$$SNR = \frac{a^2}{N_0}$$

$$= \exp\left(-\frac{SNR}{4}(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)\right)$$

L 14

12 February 2019 12:13

$$E_{|h_1|^2,|h_2|^2}\left(P\left(\underline{x}_A \to \underline{x}_B \mid h_1, h_2\right)\right) = \int_0^\infty \int_0^\infty e^{-\frac{SNR}{4}\left(t_1|d_1|^2 + t_2|d_2|^2\right)} \cdot e^{-t_1} e^{-t_2} \, dt_1 dt_2$$

($|h_1|^2$, $|h_2|^2$: $\exp(1)$ $\exp(1)$)

$$= \left(\int_0^\infty e^{-\left(\frac{SNR}{4}|d_1|^2+1\right)t_1} dt_1\right)\left(\int_0^\infty e^{-\left(\frac{SNR}{4}|d_2|^2+1\right)t_2} dt_2\right)$$

$$= \frac{e^{-\left(\frac{SNR}{4}|d_1|^2+1\right)t_1}}{-\left(\frac{SNR}{4}|d_1|^2+1\right)}\Bigg|_0^\infty \cdot \frac{e^{-\left(\frac{SNR}{4}|d_2|^2+1\right)t_2}}{-\left(\frac{SNR}{4}|d_2|^2+1\right)}\Bigg|_0^\infty$$

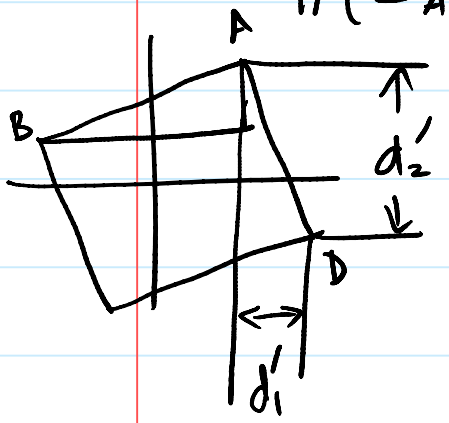
$$= \left(\frac{1}{\frac{SNR}{4}|d_1|^2+1}\right)\left(\frac{1}{\frac{SNR}{4}|d_2|^2+1}\right)$$

L14

12 February 2019 12:17

$$P(\underline{x}_A \to \underline{x}_B) \le \frac{1}{\left(1+\mathrm{SNR}\frac{d_1^2}{4}\right)} \frac{1}{\left(1+\mathrm{SNR}\frac{d_2^2}{4}\right)} \overset{\text{high SNR}}{\approx} \frac{16}{\mathrm{SNR}^2 d_1^2 d_2^2} \quad \text{if } d_1, d_2 > 0$$

$$P(\underline{x}_A \to \underline{x}_D) \le \frac{1}{\left(1+\mathrm{SNR}\frac{d_1'^2}{4}\right)} \frac{1}{\left(1+\mathrm{SNR}\frac{d_2'^2}{4}\right)} \approx \frac{16}{\mathrm{SNR}^2 d_1'^2 d_2'^2}.$$



$\underline{d_1, d_2 > 0}$ as long as rotation angle is not $\underline{0, 90^\circ, 180^\circ, 270^\circ}$

$$\Pr(\text{error}) \le P(\underline{x}_A \to \underline{x}_B) + P(\underline{x}_A \to \underline{x}_D) + \underbrace{P(\underline{x}_A \to \underline{x}_C)}_{\text{redundant}}$$

$$\approx \frac{16}{\mathrm{SNR}^2}\left(\frac{1}{d_1^2 d_2^2} + \frac{1}{d_1'^2 d_2'^2}\right) \le \frac{16}{\mathrm{SNR}}\left(\frac{2}{\min d_1^2 d_2^2}\right)$$

L14

12 February 2019 12:32

* choose the rotation angle θ such that

$\boxed{\min \delta_{ij}}$ as large as possible

δ_{ij} = Product distance between $\underline{x}_i$ and $\underline{x}_j$

$= \underbrace{d_{ij1}^2} \cdot \underbrace{d_{ij2}^2}$

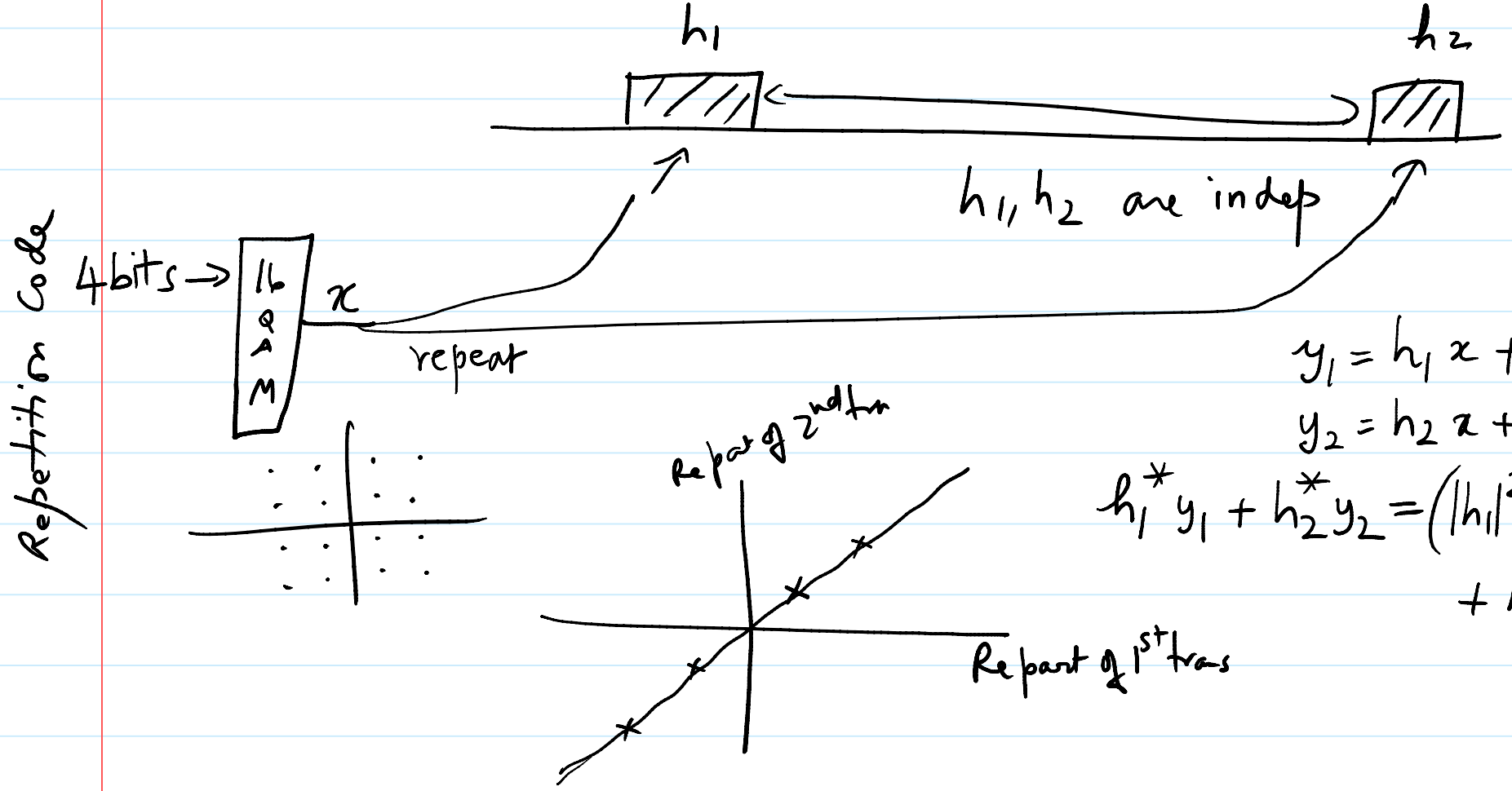
Can be shown that $\theta^* = \frac{1}{2}\tan^{-1}(2)$ is the optimal choice

L14

12 February 2019 12:36

Summary (Rotation code vs Repetition code)

Rate 2 bits/channel use, code over 2 channel uses



$$y_1 = h_1 x + w_1$$

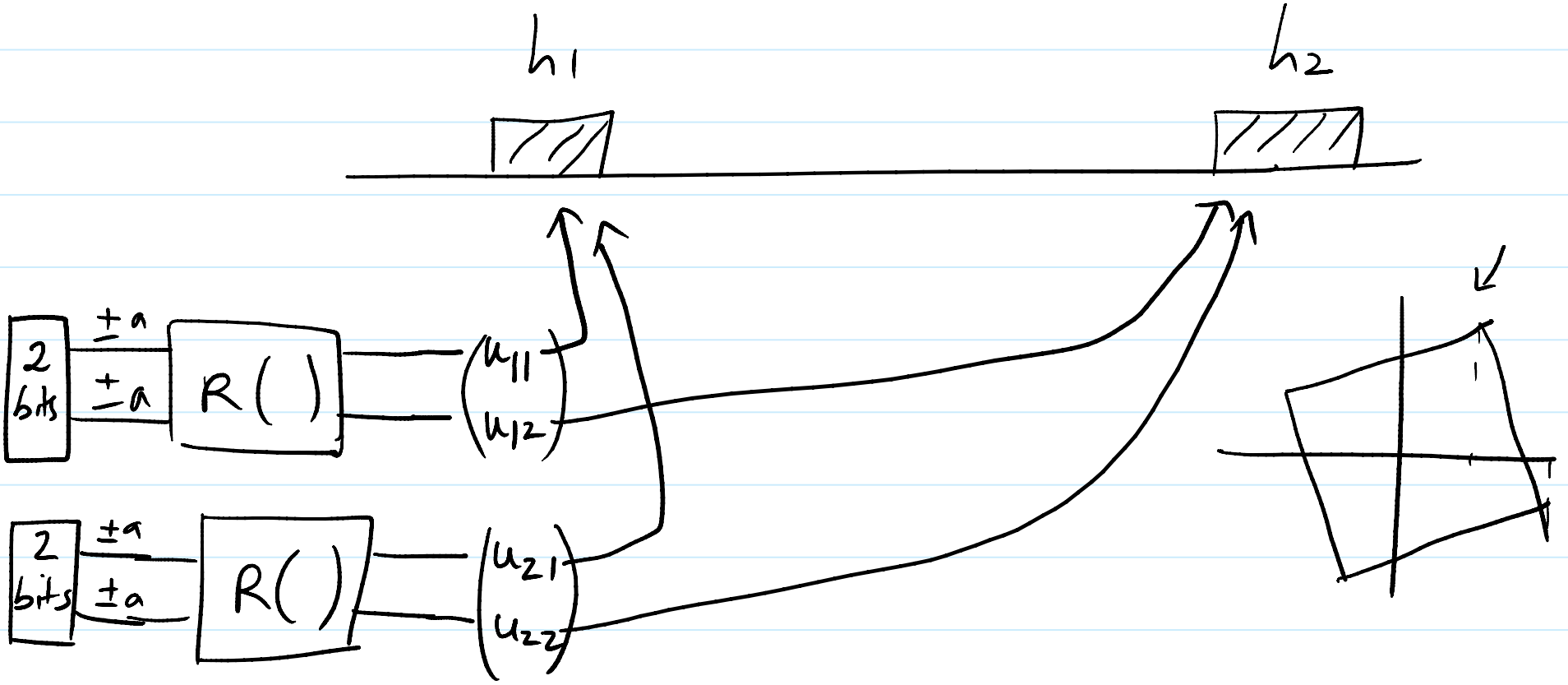
$$y_2 = h_2 x + w_2$$

$$h_1^* y_1 + h_2^* y_2 = \left(|h_1|^2 + |h_2|^2\right) x + h_1^* w_1 + h_2^* w_2$$

L14

12 February 2019 12:40

Rotation code



$$y_1 = (u_{11} + j u_{21}) h_1 + w_1 \qquad y_2 = (u_{12} + j u_{22}) h_2 + w_2$$

$$e^{-j\angle h_1} y_1 = |h_1| (u_{11} + j u_{21}) + w_1 e^{-j\angle h_1}$$

$$e^{-j\angle h_2} y_2 = |h_2| (u_{12} + j u_{22}) + w_2 e^{-j\angle h_2}$$

$\rightarrow$ Re( ) $\rightarrow$ decide $(\pm a)$

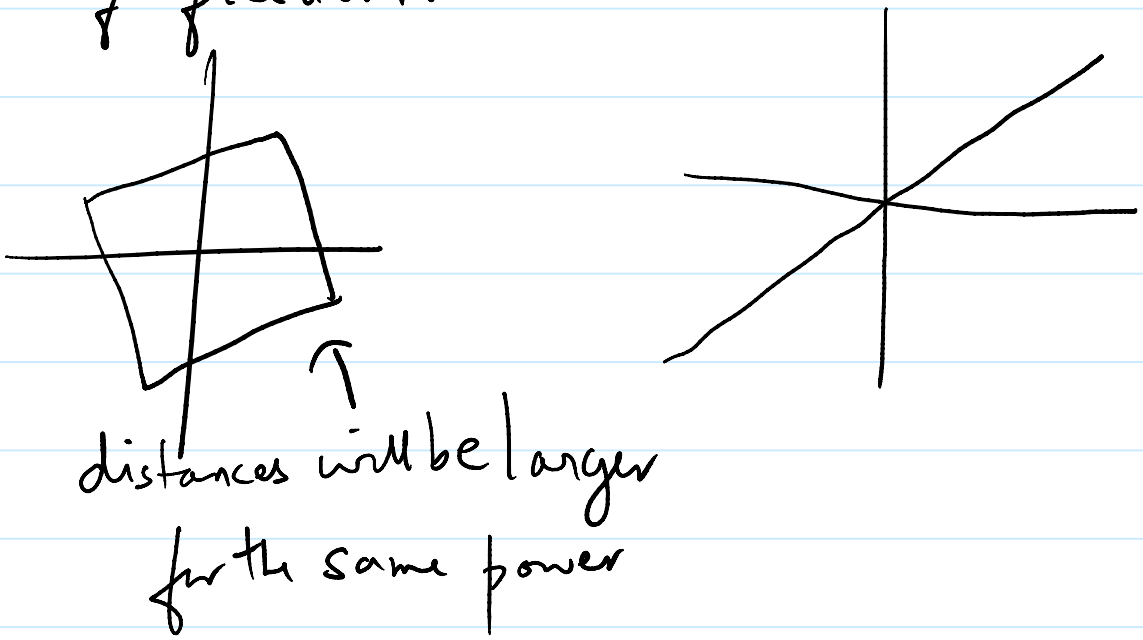
$\rightarrow$ Im( ) $\rightarrow$ decide $(\pm a)$

L 14

12 February 2019 12:48

* Both rotation code and repetition code have diversity 2.

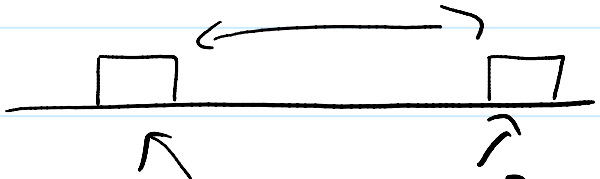
* But, rotation code will perform better. It uses all the available degrees of freedom.



Lecture 15

19 February 2019 12:01

Last class: Rotation code



4 bits over 2 symbol intervals

Better than repetition code (even though both have same diversity)

Generalization to any time diversity code:

- Uses L symbol intervals
- M possible codewords ($\frac{\log_2 M}{L}$ bits/sym.int) ← rate
  - $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_M$
    - $L \times 1$ vector

L15

19 February 2019 12:07

Assume channel independent in each of the L transmissions

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} = \begin{pmatrix} h_1 x_1 \\ h_2 x_2 \\ \vdots \\ h_L x_L \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{pmatrix}$$

$$h_1, h_2, \ldots, h_L \text{ indep} \sim CN(0,1)$$

$$\frac{1}{ML} \sum_{i=1}^{M} \|\underline{x}_i\|^2 = 1 \quad \text{(Power constraint)}$$

$$\Pr(\text{error}) = \sum_{i=1}^{M} \Pr(\text{error} / \underline{x}_i)\, P(\underline{x}_i \text{ is trans})$$

$\Pr(\text{error})$ ↓ codeword error

L15

19 February 2019 12:11

$$\Pr(\text{error} \mid \underline{x}_i) = \sum_{j \neq i} P(\underline{x}_j \text{ decoded} \mid \underline{x}_i)$$

$$\leq \sum_{j \neq i} P(\underline{x}_i \to \underline{x}_j)$$

$$= \sum_{j \neq i} E_{\underline{h}}\left[P(\underline{x}_i \to \underline{x}_j \mid \underline{h})\right]$$

$$P(\underline{x}_i \to \underline{x}_j) \leq \prod_{l=1}^{L} \frac{1}{1 + \frac{SNR}{4}\left|x_{il} - x_{jl}\right|^2}$$

$Q(y) < e^{-y^2/2}$ → avg. $\left(SNR = \frac{1}{N_0}\right)$

Diversity ← L_{ij} : # components in which $\underline{x}_i$ and $\underline{x}_j$ differ

L15

19 February 2019 12:18

Diversity of the overall code = $\boxed{\min_{\substack{i,j \\ i \neq j}} L_{ij}}$

If $L_{ij} = L$ for all $i, j, i \neq j$, then the code achieves full diversity L.

At high SNR

$$\mathbb{P}(\underline{x}_i \to \underline{x}_j) \approx \frac{1}{\left(\frac{SNR}{4}\right)^{L_{ij}} \underbrace{\left(\prod_{L_{ij} \text{ terms}} |x_{i\ell} - x_{j\ell}|^2\right)}_{\delta_{ij}}}$$

L15

19 February 2019 12:25

Suppose $\Pr(\text{error}/\underline{x}_i)$ same for all i (Symmetric codebook)
— Diversity L

$$\Pr(\text{error}) = \Pr(\text{error}/\underline{x}_i) \leq \sum_{i \neq j} P(\underline{x}_i \to \underline{x}_j)$$

$$\leq \prod_{l=1}^{L} \frac{1}{1 + \frac{SNR}{4}|x_{il} - x_{jl}|^2}$$

$$\approx \sum_j \left(\frac{4}{SNR}\right)^L \frac{1}{\delta_{ij}}$$

$$\leq \left(\frac{4}{SNR}\right)^L \frac{1}{\min_j \delta_{ij}} (M-1)$$

$$\delta_{ij} = \prod_{l=1}^{L} |x_{il} - x_{jl}|^2$$

L15

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Design guidelines for a time diversity code:

Choice of codewords $\underline{x}_1, \ldots, \underline{x}_M$ → fixes rate

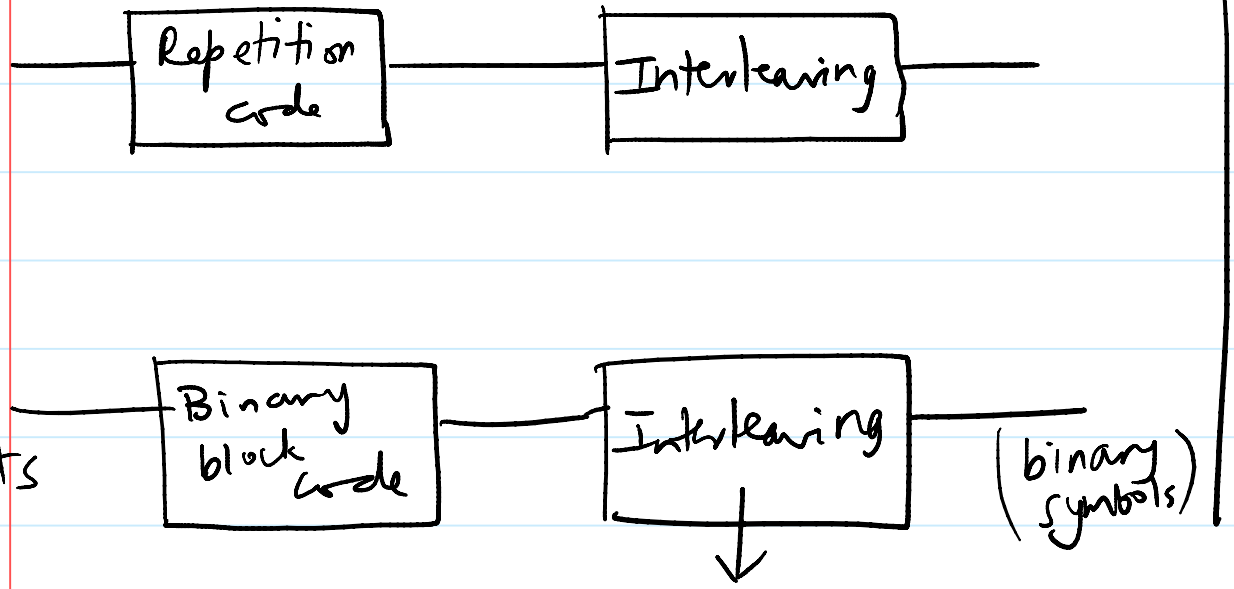
✓ Diversity = $\min_{\substack{i,j \\ i \neq j}} L_{ij}$ → Try to maximize

✓ $\min_{\substack{i,j \\ i \neq j}} \delta_{ij}$ → Try to maximize

Analysis tells us how to optimize the code.

L15

19 February 2019 12:33

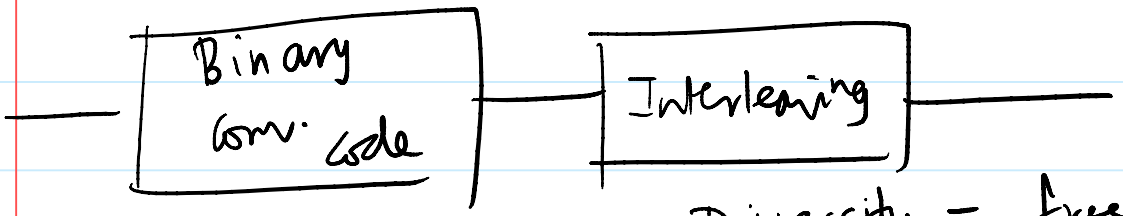


each codeword
bits

$$\frac{1}{\log M} \leq \frac{\#\text{bit errors}}{\log M} \leq \frac{\log M}{\log M}$$

(when codeword is in error)

Diversity = minimum Hamming distance of the code



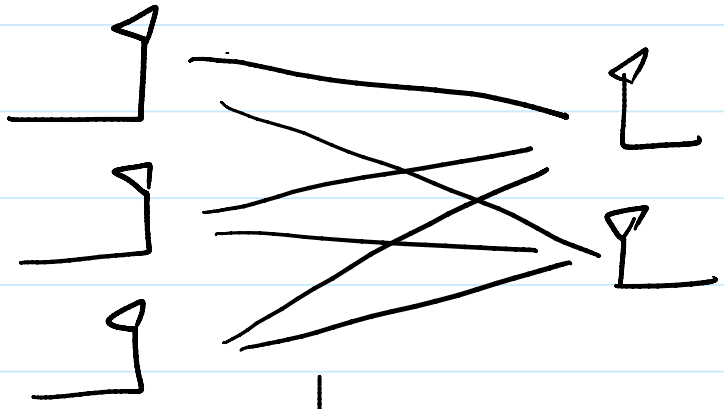
Diversity = free distance of conv. code.

L15

19 February 2019 12:41

Antenna diversity

– obtained by using multiple antennas at the transmitter (or) receiver (or) both



↓

channels between different pairs of antennas are independent if

* antennas are spaced sufficiently apart

depends on ✓carrier freq
✓scattering env.

L15

19 February 2019 12:47

- More on MIMO channel modeling in chap 7 of book
- For now, we will assume that independent channels can be achieved.

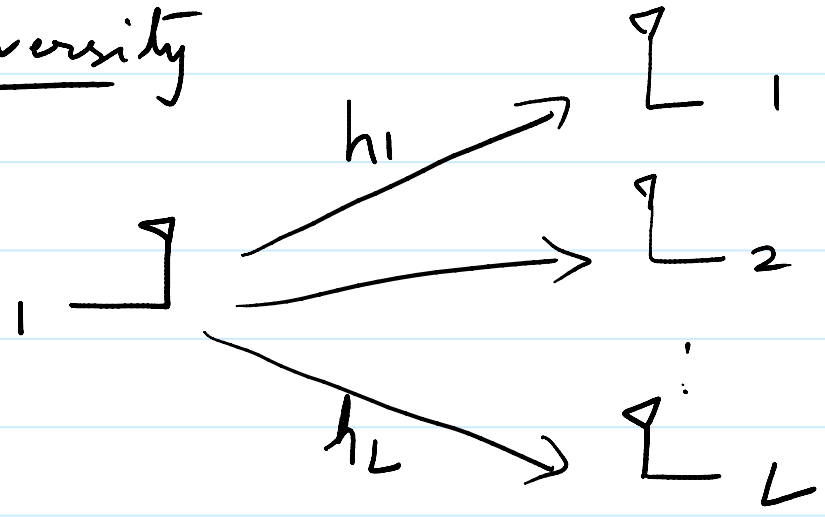
3 cases

1) Receive diversity (single input multiple output) (SIMO)
2) Transmit diversity (Multiple input single output) (MISO)
3) 2 x 2 MIMO

Lecture 16

25 February 2019 07:55

1) Receive diversity



$$y_l[m] = h_l[m]\,x[m] + w_l[m]$$

$l = 1, 2, \ldots, L$

$\underline{h}$ known at the rx.

$w_l[m] \sim CN(0, N_0)$

$h_l[m] \sim CN(0,1)$ independent Rayleigh fading for each h_l.

Detect $x[m]$ based on $y_1[m], y_2[m], \ldots, y_L[m]$

$\hat{x}$ detected based on $\underline{h}^H \underline{y} = \sum_{l=1}^{L} h_l^*[m]\, y_l[m]$

L 16

25 February 2019 08:08

$$\sum_{l=1}^{L} h_l^*[m] y_l[m] = \underbrace{\left(\sum_{l=1}^{L} |h_l[m]|^2\right)}_{\|\underline{h}\|^2} x[m] + \underbrace{\left(\sum_{l=1}^{L} h_l^*[m] w_l[m]\right)}_{\sim \mathcal{CN}(0, \|\underline{h}\|^2 N_0)}$$

BPSK: $\hat{x} = \begin{cases} a & \text{if } \mathrm{Re}(\underline{h}^H \underline{y}) > 0 \\ -a & \text{else} \end{cases}$

$$\Pr(\text{error}/\underline{h}) = Q\left(\frac{2\|\underline{h}\|^2 a}{2\sqrt{\frac{\|\underline{h}\|^2 N_0}{2}}}\right)$$

$$= Q\left(\sqrt{2\|\underline{h}\|^2 \underbrace{\left(\frac{a^2}{N_0}\right)}_{\triangleq SNR}}\right)$$

$$= Q\left(\sqrt{2\|\underline{h}\|^2 SNR}\right)$$

L 16

25 February 2019 08:15

$\|\underline{h}\|^2 SNR$ can be thought of as $\left(\frac{1}{L}\|\underline{h}\|^2\right)(L\,SNR)$

$E\left[\frac{1}{L}\|\underline{h}\|^2\right] = 1$

$E(\;)$ increases linearly with L

* By adding multiple receive antennas, received power (total) increases linearly with L.

(There are some limitations for very large L. Total rx power cannot be larger than tx power)

L 16

25 February 2019 08:21

* In time-diversity repetition coding, $Pr(\text{error}/\underline{h}) = Q\left(\sqrt{2\|\underline{h}\|^2 \underbrace{\frac{a^2}{N_0}}_{SNR}}\right)$.

Transmit power per symbol = $a^2 \cdot L$

If we keep transmit power per symbol $= a^2$ for any L,

(i.e. $\frac{a^2}{L}$ per transmission time)

$$Q\left(\sqrt{2\|\underline{h}\|^2 \frac{a^2}{LN_0}}\right) = Q\left(\sqrt{2\left(\frac{1}{L}\|\underline{h}\|^2\right)\underbrace{\frac{a^2}{N_0}}}\right)$$

$\frac{a^2}{N_0}$ → SNR instead of L SNR in RX diversity

L16

25 February 2019 08:24

* Rx diversity : Doubling L with give 2 times tx power (3dB improvement)

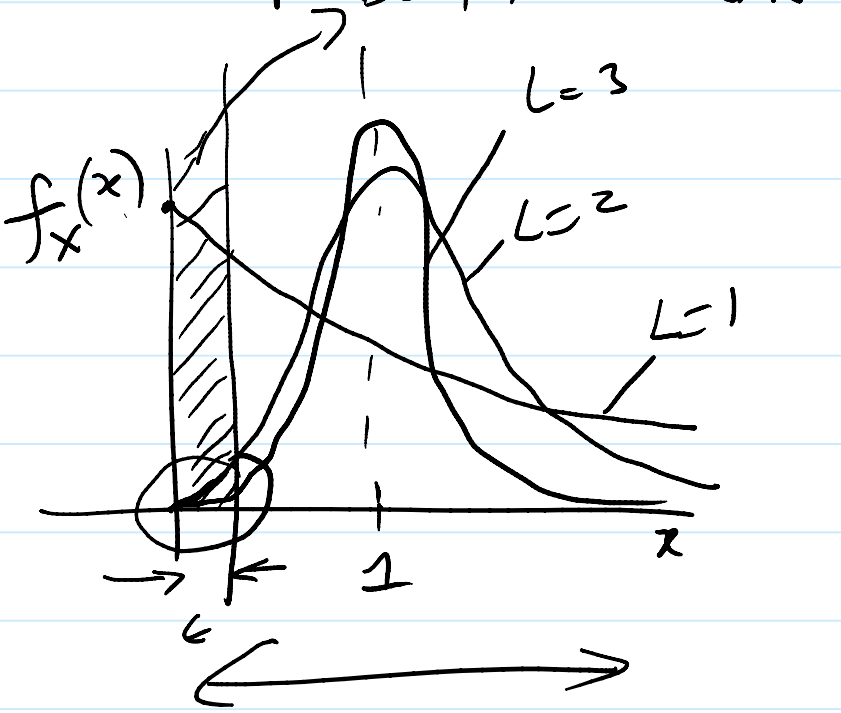
This gain is called array gain.

$$P_r(\text{error}) = \int P(\text{error}/\underline{h})\, f_{\|\underline{h}\|^2}(\ )\, d\|\underline{h}\|^2.$$

$$Q\left(\sqrt{2\frac{1}{L}\|\underline{h}\|^2 L\,SNR}\right)$$

$$\left\{ X = \frac{1}{L}\|\underline{h}\|^2 \xrightarrow[a.s]{n.f} 1 \quad \text{as } L \to \infty \right\}$$

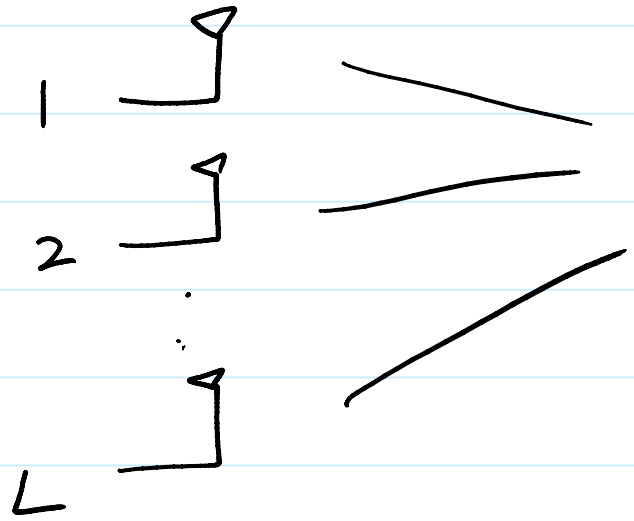
(Law of large numbers)



L16

25 February 2019 08:32

2) Transmit diversity



$$y[m] = \sum_{l=1}^{L} h_l[m]\, x_l[m] + w[m]$$

$h_l[m]$: i.i.d. $\mathcal{CN}(0,1)$

$x_l[m]$: Tx symbol from l^{th} ant.

$w[m]$: $\mathcal{CN}(0, N_0)$

Channel known at rx, Not known at tx.

* Total power constraint (over L ant) → $\sum_{l=1}^{L} E\left[|x_l[m]|^2\right] = \text{constant}$
* Suppose $x_l[m] = \frac{1}{\sqrt{L}} x[m]$ for all l

L16

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$$y[m] = \sum_{\ell=1}^{L} h_\ell[m] \frac{1}{\sqrt{L}} x[m] + w[m]$$

$$= \left(\frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} h_\ell[m] \right) x[m] + w[m]$$

$$\rightarrow \sim \mathcal{CN}(0, 1)$$

Same distribution for all L.

All copies are mixed up & we cannot separate them.

(interference)

* Transmit from one antenna in each interval

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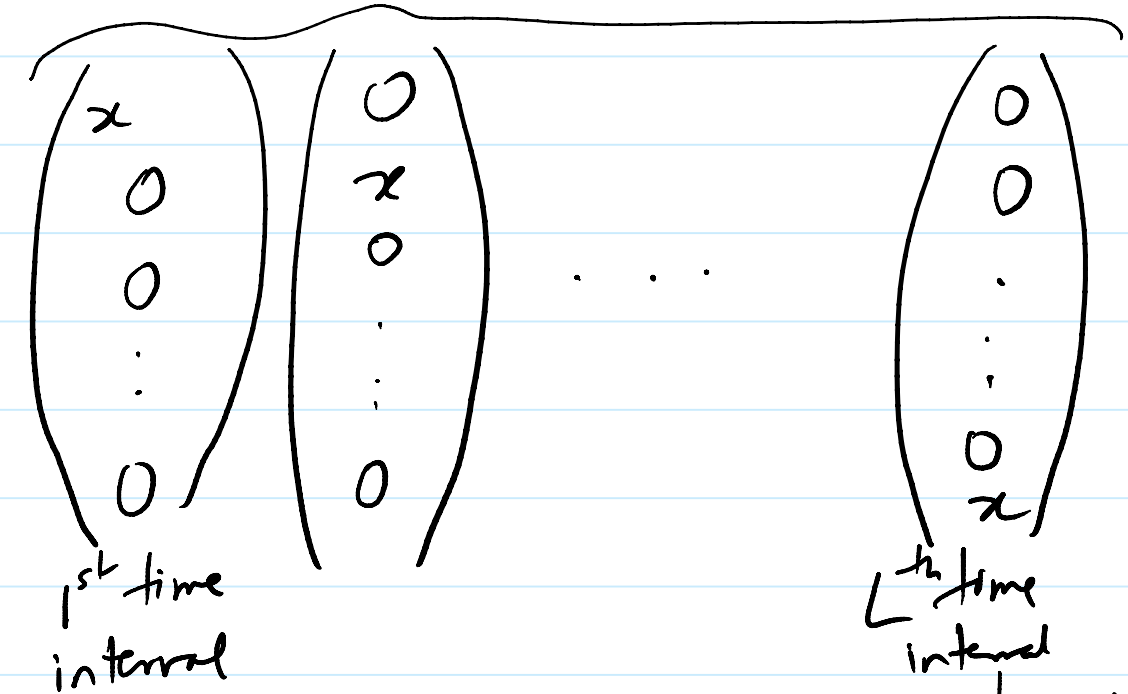

$y[1] = h_1[1]\,x + w_1[1]$

$y[L] = h_L[1]\,x + w_L[1]$

Diversity L

L16

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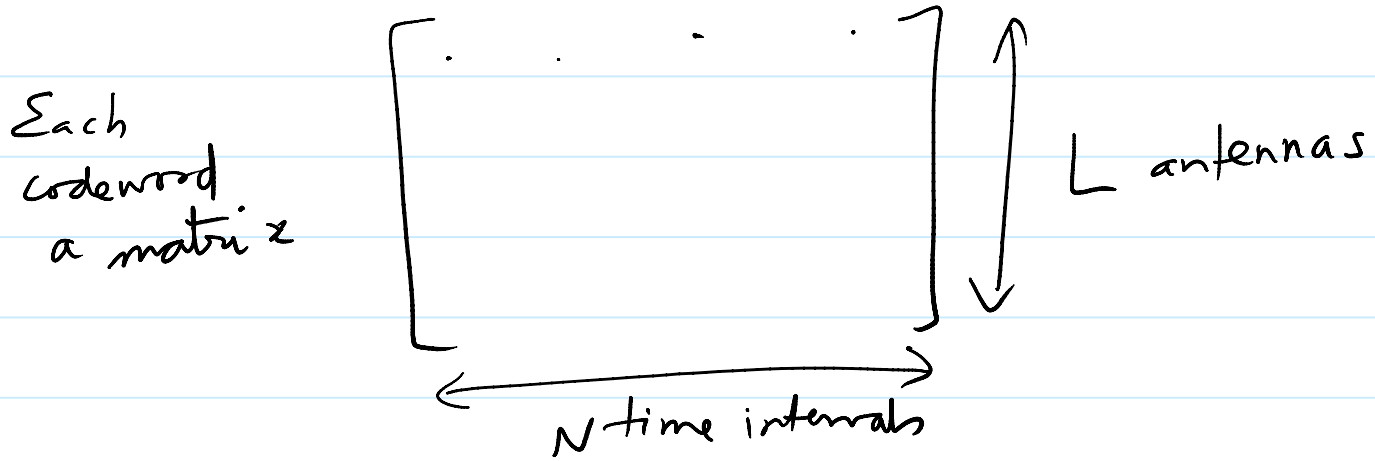
* Any time diversity code of block length L can be used.
  (Provide coding gain over repetition code)



* We can think of designing codewords across space & time



Given a set of matrices, analyze Pr(error)
Then, use this analysis to arrive at design rules for the matrices

Lecture 17

26 February 2019 11:56

Design of space-time code matrix

$$\begin{pmatrix} & \\ & \end{pmatrix}_{L \times N}$$

$L=2$ Repetition code

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{matrix} \times h_1 \\ \times h_2 \end{matrix} \qquad x \in \text{Some constellation}$$

$$(h_1 \;\; h_2)\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = (h_1 x \;\; h_2 x)$$

$$\begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} x + \begin{pmatrix} & \\ & \end{pmatrix}$$

h_1

h_2

$\mathbb{C}$

L17

26 February 2019 12:08

Another code

u_1, u_2

$$\begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} \begin{matrix} \leftarrow \text{ant 1} \\ \leftarrow \text{ant 2} \end{matrix}$$

$\uparrow$ time 1 $\quad$ $\uparrow$ time 2

$u_1, u_2 \in$ some constellation

$$y[1] = \begin{pmatrix} h_1[1] & h_2[1] \end{pmatrix} \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} + w[1]$$

Assume that $\downarrow$ are equal

$$y[2] = \begin{pmatrix} h_1[2] & h_2[2] \end{pmatrix} \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} + w[2]$$

L17

26 February 2019 12:12

$$\rightarrow \begin{pmatrix} y[1] & y[2] \end{pmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} + \begin{pmatrix} w[1] & w[2] \end{pmatrix} .$$

At the receiver: h_1, h_2 known. Need to decode u_1, u_2.

Rewrite as follows

$$\rightarrow \begin{pmatrix} y[1] \\ y^*[2] \end{pmatrix} = \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} w[1] \\ w^*[2] \end{pmatrix}$$

circularly symmetric noise $\sim CN(0, N_0)$

$$y[1] = h_1 u_1 + h_2 u_2 + w[1]$$

$$y[2] = h_1(-u_2^*) + h_2(u_1^*) + w[2] \rightarrow y^*[2] = h_1^*(-u_2) + h_2^*(u_1) + w^*[2]$$

L17

26 February 2019 12:16

$$\underline{y} = H \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underline{w}$$

$$\sim CN(H\underline{u}, N_0 I)$$

$$H = \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}$$

$\begin{pmatrix} h_1 \\ h_2^* \end{pmatrix}$ ↑ u_1 $\quad$ $\begin{pmatrix} h_2 \\ -h_1^* \end{pmatrix}$ ↑ u_2

To minimize prob of error choose $\underline{u}$ that minimizes $\|\underline{y} - H\underline{u}\|^2$

(Assuming all symbols (codewords) equally likely)

$$\|\underline{y} - H\underline{u}\|^2 = (\underline{y} - H\underline{u})^H (\underline{y} - H\underline{u}) = \underline{y}^H \underline{y} - \underline{y}^H H\underline{u} - \underline{u}^H H^H \underline{y} + \underline{u}^H H^H H \underline{u}$$

$H^H \underline{y}$ is sufficient for detection

L17

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$$\underline{r} \triangleq H^H \underline{y} = H^H H \underline{u} + H^H \underline{w}$$

$$H = \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} \qquad H^H H = \begin{pmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix}$$

↓

Orthogonal matrix

$$= \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{pmatrix}$$

$$= (|h_1|^2 + |h_2|^2)\, I$$

$$\underline{r} = (|h_1|^2 + |h_2|^2)\,\underline{u} + H^H \underline{w}$$

↓ $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

$$r_1 = (|h_1|^2 + |h_2|^2) u_1 + n_1$$

$$r_2 = (|h_1|^2 + |h_2|^2) u_2 + n_2$$

L17

26 February 2019 12:25

* In the repetition code, one symbol (x) was transmitted over 2 symbol intervals. Here, 2 symbols (u_1, u_2) are transmitted over 2 symbol intervals.

* In both cases, diversity = 2.

* To maintain the same total transmit power, each symbol is transmitted with half the power in each interval in the second scheme.

* Second scheme is the <u>Alamouti</u> scheme.

L17

26 February 2019 12:29

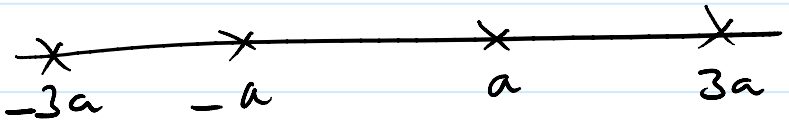
* Compare repetition scheme with Alamouti scheme.

Rate: 2 bits in 2 symbol intervals

Repetition

$x \rightarrow$ 4-PAM full power

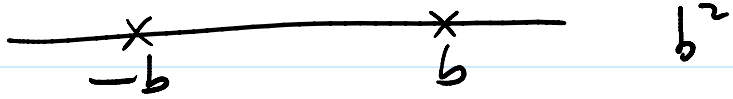
$\frac{20a^2}{4} = 5a^2$



Alamouti

$u_1 \rightarrow$ BPSK $\frac{1}{2}$ of total power

$u_2 \rightarrow$ BPSK $\frac{1}{2}$ of total power



For the same d_{min}, 4-PAM needs 5 times the energy as BPSK

For the same total transmit power, factor of 2 more power is available per symbol in repetition

Repetition requires 2.5 times more power than Alamouti scheme for the same performance ($\approx$ 4dB)

L17

26 February 2019 12:36

So far $N = 2, L = 2$

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$ Repetition code

$$\begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix}$$ Alamouti code.

$$\left(\quad X \quad \right)$$ Space-time code

$L \times N$

How should this be designed?

L17

26 February 2019 12:38

* In order to design a good code to exploit time diversity, we arrived at 2 criteria for the codewords.
  (1) # components that differ (diversity)
  (2) min product distance between 2 codewords (coding gain)

* Now, we have code matrices ($L \times N$ matrices)
  (L: # antennas; N: # time intervals)

We can derive design criteria now.
Assume that channel remains constant over N intervals

$$\underline{y}[1] \quad y[2] \quad \cdots \quad y[N] = \underbrace{(h_1 \; h_2 \; \cdots \; h_L)}_{\underline{h}^H} \left(\quad X \quad \right) + (w[1] \cdots w[N])$$

L17

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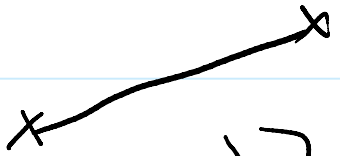
Pairwise error probability $P(X_A \to X_B)$

$$P(X_A \to X_B / \underline{h}) = Q\left(\frac{d_{AB}}{2\sigma}\right)$$

$$d_{AB} = \left\| \underline{h}^H X_A - \underline{h}^H X_B \right\|$$

$$\sigma = \sqrt{\frac{N_0}{2}}$$



$$E_{\underline{h}}\left[P(X_A \to X_B / \underline{h})\right] = E_{\underline{h}}\left[Q\left(\sqrt{\frac{SNR\ \underline{h}^H (X_A - X_B)(X_A - X_B)^H \underline{h}}{2}}\right)\right]$$

Where $SNR = \frac{1}{N_0}$

Lecture 18

28 February 2019 11:01

$$\underline{h}^H \underbrace{(X_A - X_B)(X_A - X_B)^H}\, \underline{h}$$

$\underline{h}$ $L \times 1$

$X_A - X_B$ $L \times N$

A is p.s.d if $\underline{x}^H A \underline{x} \geq 0 \; \forall \underline{x}$

$\boxed{(X_A - X_B)(X_A - X_B)^H}$ is symmetric positive semi definite

↳ can be written as $U \Lambda U^H$ where

U : unitary

Λ : diagonal

diagonals values ≥ 0.

↳ $\lambda_1^2, \lambda_2^2, \ldots, \lambda_L^2$

$$\underline{h}^H (X_A - X_B)(X_A - X_B)^H \underline{h}$$

$$= \underline{h}^H U \Lambda U^H \underline{h} = \underline{\tilde{h}}^H \Lambda \underline{\tilde{h}} \quad (\text{where } \underline{\tilde{h}} = U^H \underline{h})$$

$$= \sum_{i=1}^{L} \lambda_i^2 |\tilde{h}_i|^2$$

L18

28 February 2019 11:13

$$P_r(X_A \to X_B) \le E\left[Q\left(\sqrt{\frac{SNR}{2} \sum_{i=1}^{L} \lambda_i^2 |\tilde{h}_i|^2} \right) \right] \qquad Q(x) \le e^{-x^2/2}$$

$$\le e^{-\frac{SNR}{4} \sum_{i=1}^{L} \lambda_i^2 |\tilde{h}_i|^2}$$

$$\underline{\tilde{h}} = U \underline{h} \qquad \underline{h} \sim CN(\underline{0}, I)$$

$$U\underline{h} \sim CN(\underline{0}, I) \quad (U \text{ is unitary})$$

$$|\tilde{h}_i|^2 \sim \text{exponential}(1)$$

$$P_r(X_A \to X_B) \le \int_0^\infty \cdots \int_0^\infty e^{-\frac{SNR}{4} \sum_{i=1}^{L} \lambda_i^2 x_i} \; e^{-\sum_{i=1}^{L} x_i} \, dx_1 dx_2 \cdots dx_L$$

L18

28 February 2019 11:19

$$= \prod_{l=1}^{L} \left(\frac{1}{1 + \frac{SNR}{4} \lambda_l^2} \right)$$

$X_A - X_B : L \times N$

$\text{rank}\left[(X_A - X_B)(X_A - X_B)^H \right] \leq N.$

What can we say from the analysis?

— Diversity gain will be L if $\lambda_l^2 > 0$ for all l

and for all X_A, X_B

RANK CRITERION

For all codeword differences $X_A - X_B$, all the eig. values of $(X_A - X_B)(X_A - X_B)^H$ should be > 0. (# eig values > 0) $\leq N$.

Need $N \geq L$ for full diversity gain.

L18

28 February 2019 11:27

— At high SNR $\prod_{l=1}^{L} \left(\frac{1}{1+\frac{SNR}{4}\lambda_l^2} \right) \approx \frac{1}{(SNR)^{L'} \underbrace{\prod_{l=1}^{L'} \lambda_l^2}_{l=1}}$

($L' = L$ for a full div code)

Let $L' = L$.

Try to maximize $\prod_{l=1}^{L} \lambda_l^2 = \det\left((X_A - X_B)(X_A - X_B)^H \right)$

$$\max \underbrace{\min_{X_A, X_B} \det\left((X_A - X_B)(X_A - X_B)^H \right)}$$

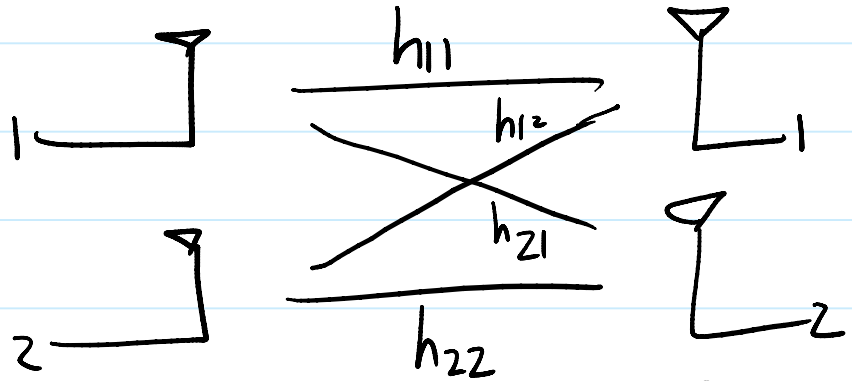
DETERMINANT CRITERION

L18

28 February 2019 11:33

3) Multiple Input Multiple Output (MIMO)

2 × 2



$$\underline{y} = H\underline{x} + \underline{w}$$

(# rx × # tx)

h_{ij} $(i,j)^{th}$ element of H

h_{ij} : channel coeff from tx antenna j to rx antenna i

h_{ij}'s are i.i.d. $CN(0,1)$

What is the max. possible div. gain? 4 ($M \times N$ channel, MN)

L 18

28 February 2019 11:40

Simple scheme to achieve diversity gain = 4

(Repetition)

Transmit the same symbol over the two antennas in two consecutive symbol intervals (use one tx antenna at a time)

Time 1 :

$$y_1[1] = h_{11}x + w_1[1]$$
$$y_2[1] = h_{21}x + w_2[1]$$

Time 2 :

$$y_1[2] = h_{12}x + w_1[2]$$
$$y_2[2] = h_{22}x + w_2[2]$$

} x has been sent over 4 independent channels

L 18

28 February 2019 11:45

Receiver will use $h_{11}^* y_1[1] + h_{21}^* y_2[1] + h_{12}^* y_1[2] + h_{22}^* y_2[2]$

to make a decision.

$$= \left(|h_{11}|^2 + |h_{21}|^2 + |h_{12}|^2 + |h_{22}|^2\right) x$$
$$+ h_{11}^* w_1[1] + h_{21}^* w_2[1]$$
$$+ h_{12}^* w_1[2] + h_{22}^* w_2[2]$$

Repetition achieves diversity of 4.

But, it does not use all degrees of freedom that are available.

L18

28 February 2019 11:48

At each time

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$\underline{y}$: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\underline{h}_1$: $\begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix}$ $\underline{h}_2$: $\begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix}$

$\begin{bmatrix} x \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ x \end{bmatrix}$ for the repetition code.

Transmit along $\underline{h}_1$

Transmit along $\underline{h}_2$

* Dimension of the received signal space is important. If $\underline{h}_1$ & $\underline{h}_2$ are linearly independent, then received signal space dimension can be 2.

Lecture 19

01 March 2019 10:04

Degrees of freedom

- Repetition code: One symbol transmitted over 2 symbol times

$$\begin{array}{c} \\ \text{Ant 1} \\ \text{Ant 2} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{Time1} & & \text{Time2} & \cdots & & \end{array} \\ \begin{pmatrix} x_1 & 0 & x_2 & 0 & x_3 & 0 & \cdots \\ 0 & x_1 & 0 & x_2 & 0 & x_2 & \end{pmatrix} \end{array}$$

(1/2 degree of freedom per symbol time)

- Alamouti scheme:

$$\begin{pmatrix} x_1 & -x_2^* & x_3 & -x_4^* & x_5 & -x_6^* & \cdots \\ x_2 & x_1^* & x_4 & x_3^* & x_6 & x_5^* & \end{pmatrix}$$

2 symbols over 2 symbol times
1 degree of freedom per symbol time.

L19

01 March 2019 10:11

Diversity of Alamouti scheme

$$\begin{pmatrix} y_1[1] \\ y_2[1] \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_1[1] \\ w_2[1] \end{pmatrix}$$

$$\begin{pmatrix} y_1[2] \\ y_2[2] \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} -x_2^* \\ x_1^* \end{pmatrix} + \begin{pmatrix} w_1[2] \\ w_2[2] \end{pmatrix}$$

$$\begin{pmatrix} y_1[1] \\ y_1^*[2] \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & -h_{11}^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_1[1] \\ w_1^*[2] \end{pmatrix}$$

L19

01 March 2019 10:16

$$\begin{pmatrix} y_2[1] \\ y_2^*[2] \end{pmatrix} = \begin{pmatrix} h_{21} & h_{22} \\ -h_{22}^* & h_{21}^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_2[1] \\ w_2^*[2] \end{pmatrix}$$

$$\begin{pmatrix} y_1[1] \\ y_1^*[2] \\ y_2[1] \\ y_2^*[2] \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ -h_{12}^* & h_{11}^* \\ h_{21} & h_{22} \\ -h_{22}^* & h_{21}^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_1[1] \\ w_1^*[2] \\ w_2[1] \\ w_2^*[2] \end{pmatrix}$$

These 2 columns are orthogonal

Diversity = 4

L19

01 March 2019 10:20

$$\begin{pmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{pmatrix}_{L \times N}$$

L | N

2 | 2

$x_1, x_2 \in$ constellation.

$$P(X_A \to X_B) \le \prod_{\ell=1}^{L} \frac{1}{1 + \frac{SNR}{4} \lambda_\ell^2}$$

$$\begin{pmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{pmatrix} \quad \begin{pmatrix} x_1' & -x_2'^* \\ x_2' & x_1'^* \end{pmatrix}$$

Single rx antenna

either $x_1 \neq x_1'$ (or) $x_2 \neq x_2'$ (or) both

$$X_A - X_B = \begin{pmatrix} x_1 - x_1' & -(x_2 - x_2')^* \\ x_2 - x_2' & (x_1 - x_1')^* \end{pmatrix} \longrightarrow \text{rank 2.}$$

L19

01 March 2019 10:25

When there are n_r receive antennas

$$P(X_A \to X_B) \leq \left[\prod_{l=1}^{L} \frac{1}{1 + \frac{SNR}{4}\lambda_l^2} \right]^{n_r}$$

Diversity = 4 for 2×2 Alamouti code

Spatial multiplexing

- Send 2 symbol in each symbol interval

$$\begin{pmatrix} x_1 & x_3 & x_5 & \cdots \\ x_2 & x_4 & x_6 & \end{pmatrix}$$

L19

01 March 2019 10:34

Degrees of freedom per symbol interval = 2.

Diversity?

$$P(X_A \to X_B) \leq \left(\prod_{l=1}^{L} \frac{1}{1 + \frac{SNR}{4} \lambda_l^2} \right)^{n_R}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 - x_1' \\ x_2 - x_2' \end{pmatrix}$$

rank = 1

$(X_A - X_B)(X_A - X_B)^H$

eig values

$\|X_A - X_B\|^2$ 0

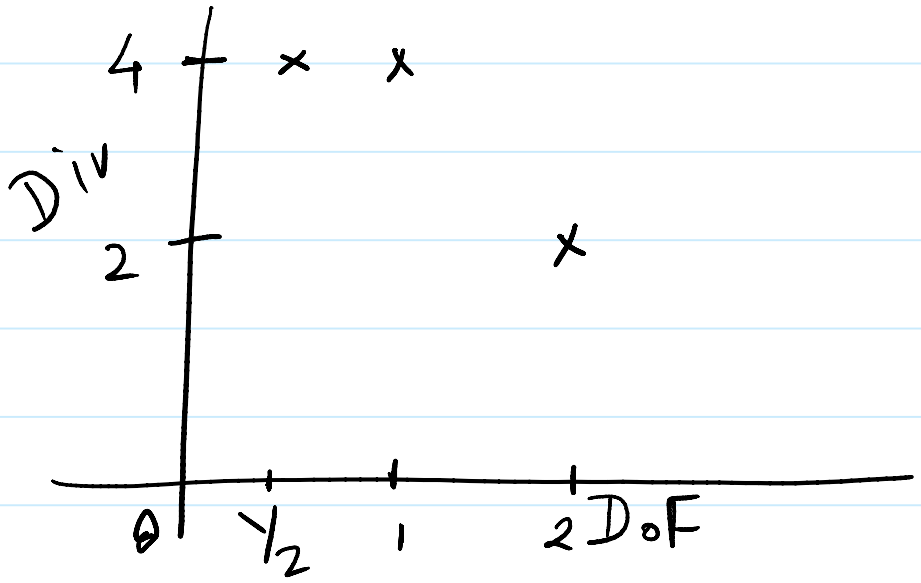
Diversity of spatial multiplexing scheme = 2 (n_R)

L19

01 March 2019 10:39

In summary (2×2)

	Diversity	DoF per sym. int.
Repetition code	4	½
Alamouti scheme	4	1
Spatial multiplexing	2	2





L19

01 March 2019 10:47

So far, we used the optimal detector that minimizes P_e (overall)
(maximum likelihood
assuming equally likely codewords)

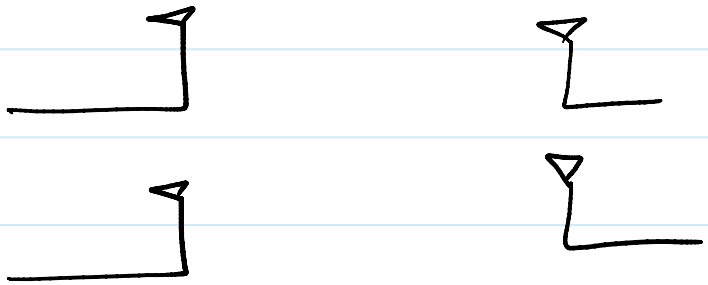
Spatial multiplexing

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ # codewords = $\left(\#\ \text{constellation pts}\right)^2$

Lecture 20

04 March 2019 08:01

Performance of spatial multiplexing with a suboptimal detector



$$\min_{\underline{x}} \|\underline{y} - H\underline{x}\|^2 \quad \text{(opt)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\underline{y} = H\underline{x} + \underline{w}$$

$$\sum_i |\tilde{y}_i - x_i|^2$$

(1) $\tilde{\underline{y}} = H^{-1}\underline{y} = \underline{x} + \underbrace{H^{-1}\underline{w}}_{\tilde{\underline{w}}}$ ← suboptimal step

(2) $\min_{\underline{x}} \|\tilde{\underline{y}} - \underline{x}\|^2$ ← suboptimal; ↓ separable

L 20

04 March 2019 08:08

This detector is called a decorrelator (or) zero-forcing detector. (ZF)

(Implemented in BellLabs V-BLAST)

$$\min_{\underline{x}} \| \underline{y} - H\underline{x} \|^2$$

$$\checkmark \min_{\underline{x}} \| H^{-1}(\underline{y} - H\underline{x}) \|^2$$

[If H is unitary, $\|\underline{y} - H\underline{x}\|^2 = \|H^{-1}(\underline{y} - H\underline{x})\|^2$

In this case, ML $\Leftrightarrow$ ZF]

L20

04 March 2019 08:15

$$\tilde{y} = \underline{x} + \tilde{\omega} \qquad \underline{\omega} \sim CN(\underline{0}, N_0 I)$$

$$\begin{bmatrix} N_0 & 0 \\ 0 & N_0 \end{bmatrix}$$

$$\tilde{\omega} \sim CN\left(\underline{0}, \underbrace{(H^{-1})(N_0 I)(H^{-1})^H}\right)$$

Not diagonal ($\Rightarrow$ step (2) is suboptimal)

$$\tilde{y} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1' \\ w_2' \end{bmatrix}$$

L20

04 March 2019 08:23

$$H^{-1} = \frac{1}{|h_{11}h_{22} - h_{12}h_{21}|}\begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{bmatrix}$$

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \quad (x_1, x_2)$$

$$= \begin{bmatrix} - & c_1 \phi_1^H & - \\ - & c_2 \phi_2^H & - \end{bmatrix}$$

direction of this vector is important

$$|c|^2 \left(|h_{22}|^2 N_0 + |h_{12}|^2 N_0 \right) \left(\frac{1}{\sqrt{|h_{12}|^2 + |h_{22}|^2}} \right)$$

scaling $\left(\begin{bmatrix} h_{22} & -h_{12} \end{bmatrix} \right)$

↓ choose scaling so that noise var remains the same

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} N_0$

L20

04 March 2019 08:34

$$y_1' = \underbrace{c_1 \phi_1^H H \underline{x}}_{} + \underbrace{c_1 \phi_1^H \underline{w}}_{\sim CN(0, N_0)}$$

$$c_1 \phi_1^H \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} x_1 + c_1 \phi_1^H \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} x_2$$

$$\begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} \sim CN(\underline{0}, I)$$

$$0$$

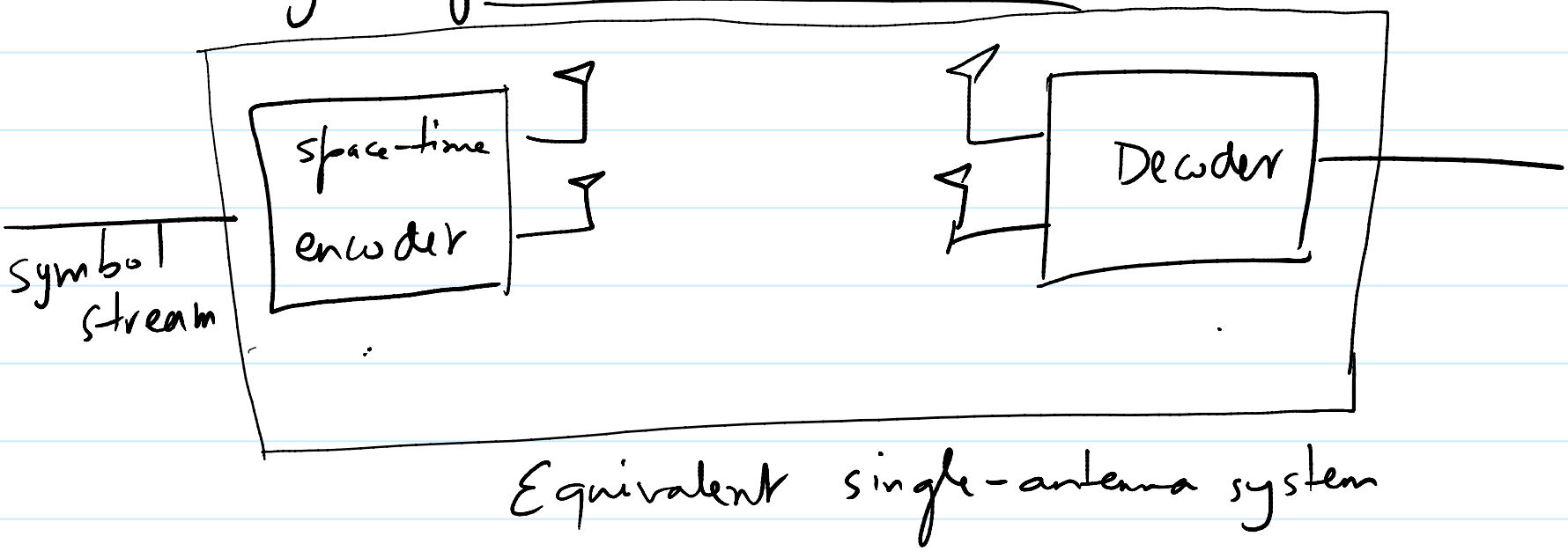
$$\sim CN(0, \sigma_n^2)$$

Diversity = 1 (Antenna dimension is used to remove interference)

L20

04 March 2019 08:41

Summary of various multi-antenna schemes

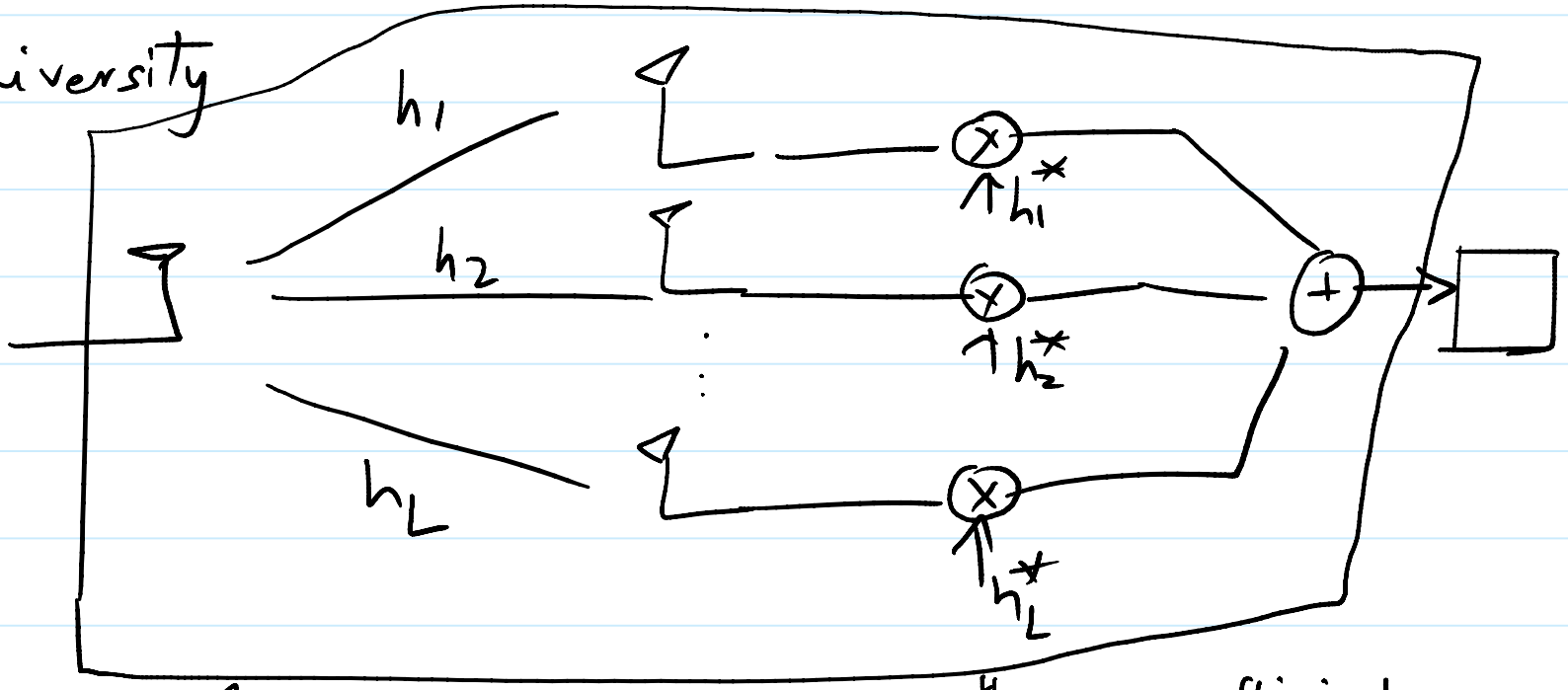


Equivalent single-antenna system

L20

04 March 2019 08:44

1) Receive diversity



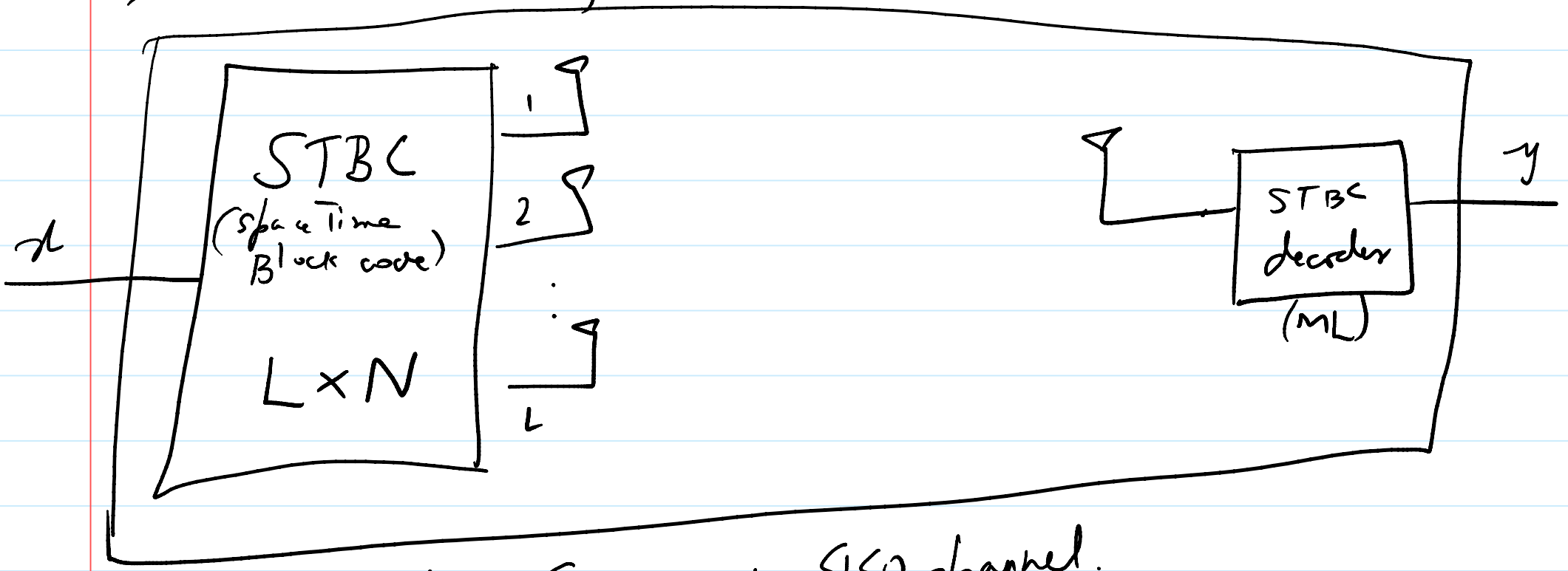
$\underline{h}^H \underline{y}$ is sufficient.

Equivalent SISO channel (No suboptimality)

L20

04 March 2019 08:46

2) Transmit diversity



2×1 Alamouti code

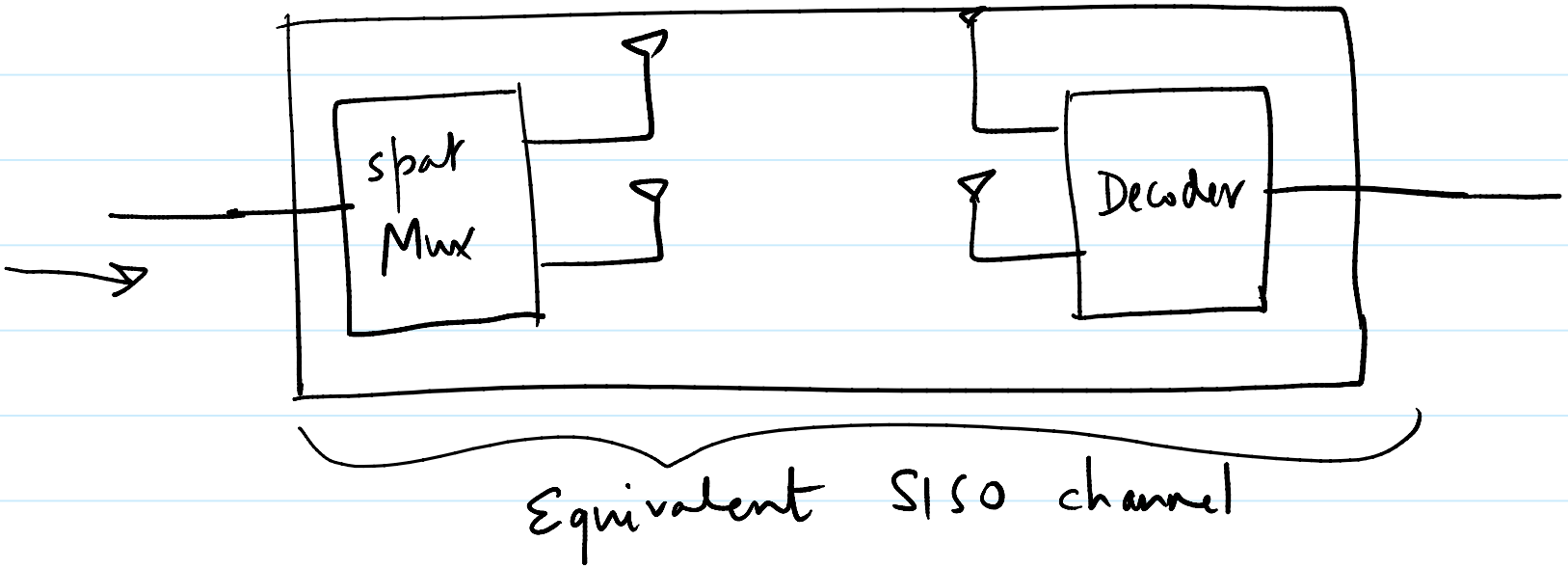
Equivalent SISO channel.

$$y = \left(|h_1|^2 + |h_2|^2\right) x + n \rightarrow CN\left(0, \left(|h_1|^2 + |h_2|^2\right) N_0\right)$$

Lecture 21

05 March 2019 12:02

3) 2 x 2 MIMO



The equivalent SISO channel view will be important to understand whether these schemes are capacity lossless or not.

L 21

05 March 2019 12:10

Frequency diversity



not freq sel

SISO Rayleigh fading channel

$P_e \propto \frac{1}{SNR}$

Diversity ↑
- Time div
- Ant div
- Freq div

Single tap filter model so far

- not frequency selective $y[n] = h[n]\,x[n] + w[n]$
- Bandwidth used < coherence BW

What if bandwidth used > coherence BW?

- Frequency selective channel.
- Linear time-varying filter with **multiple** taps

L 21

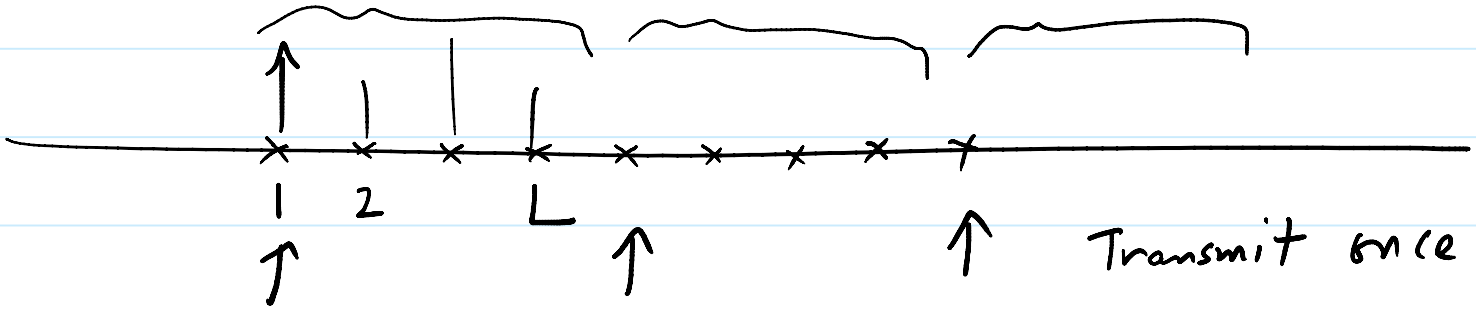
05 March 2019 12:15

$$y[m] = \sum_{l=0}^{L-1} h_l[m]\, x[m-l] + w[m]$$

↑ sum over multiple paths

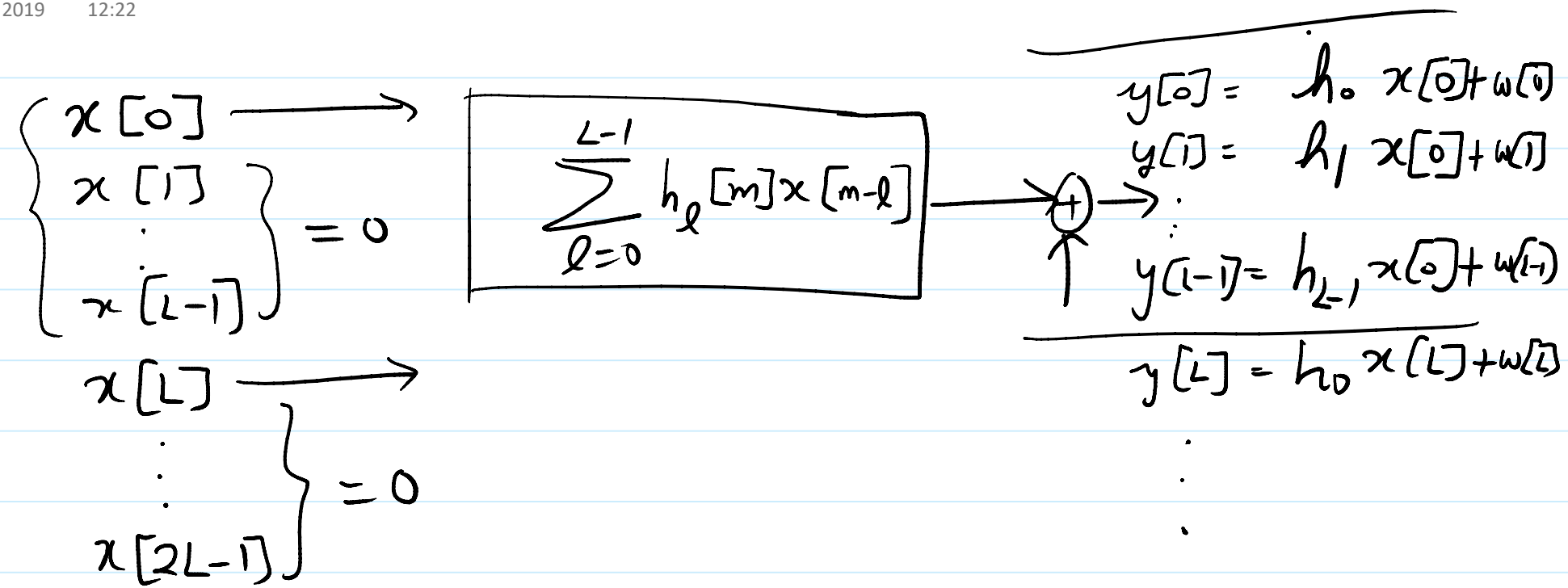
If we have L taps, then we have L independent copies of the signal. We can expect diversity of L at most. There is interference between the paths.

How can we exploit this diversity?



L21

05 March 2019 12:22



Diversity L.

h_l i.i.d

Does not use all available degrees of freedom.

L21

05 March 2019 12:27

Can we increase the symbol rate (use more degrees of freedom)?

Scheme 1: (Maximum likelihood sequence detection at the receiver)

- Send a sequence of N symbols

$$\underline{x} = (x[0] \; x[1] \; \ldots \; x[N-1])$$

- $$\underline{y} = (y[0] \; y[1] \; \ldots \; y[N-1] \; y[N] \ldots y[N+L-2])$$

- Assume that channel is not varying in this duration (coh-time > N symbol intervals)

L 21

05 March 2019 12:31

— Assume h_l are i.i.d. complex Gaussian for each l

$$(y[0] \; y[1] \; \ldots \; y[N+L-1])$$

$$= (h_0 \; h_1 \; \ldots \; h_{L-1}) \times \begin{bmatrix} x[0] & x[1] & \ldots & x[N-1] & 0 & 0 & \ldots & 0 \\ 0 & x[0] & & & x[N-1] & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \ddots & & 0 \\ 0 & \ldots & 0 & x[0] & \ldots & & & x[N-1] \end{bmatrix}$$

$L \times (N+L-1)$

$$+ (w[0] \; w[1] \; \ldots \; w[N+L-2])$$

L 21

05 March 2019 12:37

N symbols in $N+L-1$ symbol intervals

$$\frac{N}{N+L-1} \approx 1 \qquad N \gg L$$

related to Coh. time (5 ms) $\gg$ related to delay spread eg (10 μs)

— Analysis

ML decoding of $\underline{x}$ from $\underline{y}$

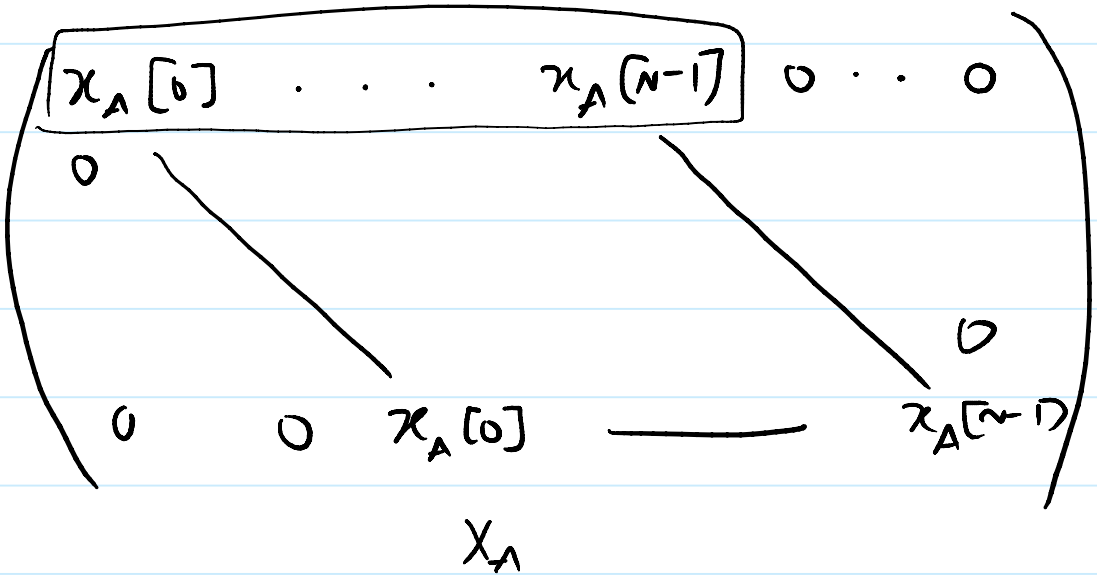
Bound pairwise error probability

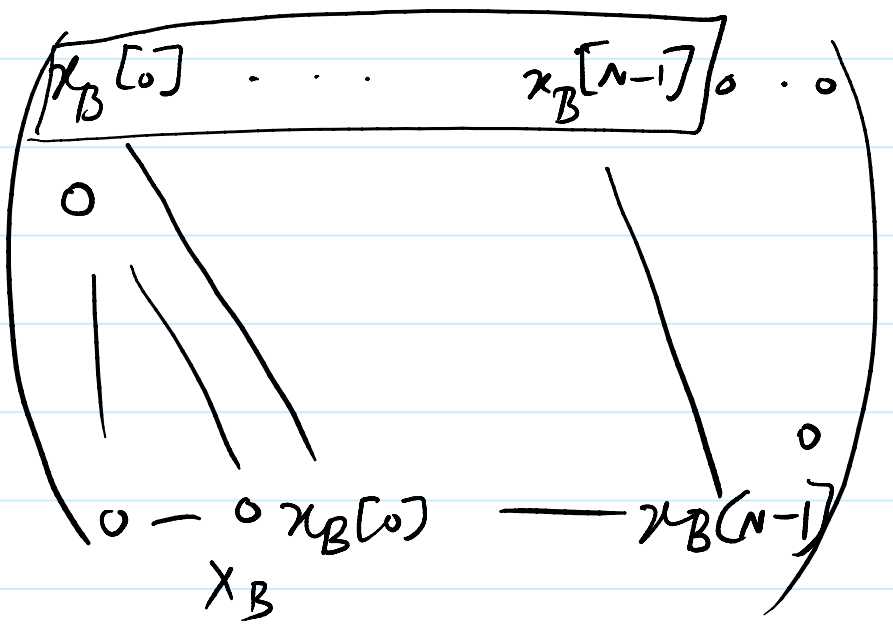
$$P_R\left(\underline{x}_A \to \underline{x}_B\right) \leq \prod_{l=1}^{L} \frac{1}{1+\frac{SNR}{4}\lambda_l^2}$$

($\underline{x}_A \downarrow X_A$, $\underline{x}_B \downarrow X_B$)

L 21

05 March 2019 12:44





Say $x_A[0] \neq x_B[0]$

$$X_A - X_B = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a \end{pmatrix}$$

$$\begin{pmatrix} a^2 & & & 0 \\ & a^2 & & \\ & & \ddots & \\ 0 & & & a^2 \end{pmatrix}$$

$$\text{rank}\left[(X_A - X_B)(X_A - X_B)^H\right] = L$$

$$(X_A - X_B)(X_A - X_B)^H$$

L21

05 March 2019 12:49

Even if $\underline{x}_A, \underline{x}_B$ differ in multiple locations, $(X_A - X_B)(X_A - X_B)^H$ is full rank.

$$P_r(\underline{x}_A \to \underline{x}_B) \leq \prod_{l=1}^{L} \frac{1}{1 + \frac{SNR}{4}\lambda_l^2} \approx \frac{1}{(SNR)^L} \cdot (\text{constant})$$

Diversity L

$$P_r(\text{error} \mid \text{given } \underline{x}_A \text{ tx}) \leq \sum_{\underline{x}_B : \underline{x}_B \neq \underline{x}_A} P_r(\underline{x}_A \to \underline{x}_B) \approx \frac{\text{Constant}}{(SNR)^L}$$

$$\text{Overall } Pr(\text{error}) \approx \frac{\text{Constant}}{(SNR)^L}$$

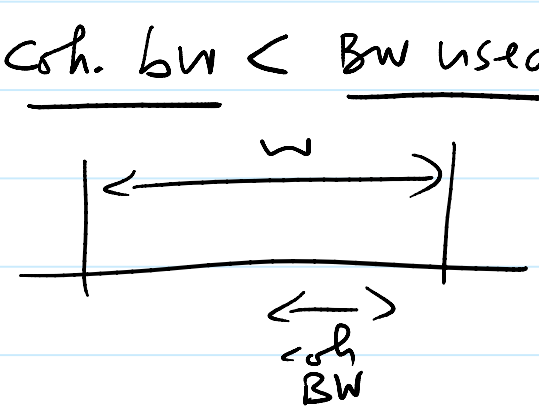
Lecture 22

07 March 2019 10:55

Frequency diversity : How to exploit it ?

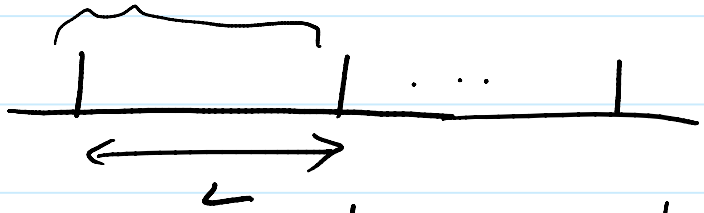
channel : Linear time varying filter with multiple taps (L)

Coh. bw < BW used



Recap:

1)



Transmit one symbol in L intervals

→ diversity L

2) Transmit one symbol in each interval

— Use ML decoding for the whole sequence/block

→ diversity L

L22

07 March 2019 11:05

Direct-sequence spread spectrum

Tx seq → $x[0]\ x[1]\ x[2]\ \ldots\ x[n-1]\ x[n]\ x[n+1]\ \ldots\ x[2n-1]\ x[2n]\ldots$

$x[0]\ldots x[n-1]$: 1st symbol

$x[n]\ldots x[2n-1]$: 2nd symbol

Spreading sequence of length n

→ $u[0]\ u[1]\ \ldots\ u[n-1]$ ✓

b_0 (symbol to be transmitted) $b_1\ldots$

BW used ↑ n

Tx seq → $b_0u[0]\ b_0u[1]\ \ldots\ b_0u[n-1]\ b_1u[0]\ b_1u[1]\ \ldots\ b_1u[n-1]$

L 22

07 March 2019 11:20

- Not very efficient use of bandwidth for a single point-to-point link.

Usually, in cellular systems, multiple transmitters will transmit at the same time in the same band. Each transmitter will use a different spreading sequence. (code division multiple access - CDMA)

(spreading code, code)

Rx signal (channel is constant for n intervals) + noise

$$h_0 \begin{pmatrix} x[0] & \dots & x[n-1] & x[n] & \dots & x[2n-1] & x[2n] \dots \end{pmatrix}^T$$
$$+ h_1 \begin{pmatrix} 0 & x[0] & \dots & x[n-1] & x[n] & \dots & x[2n-1] \dots \end{pmatrix}^T$$
$$+ h_2 \begin{pmatrix} 0 & 0 & x[0] & \dots \end{pmatrix}^T$$
$$\vdots$$
$$+ h_{L-1} \begin{pmatrix} 0 & 0 & \overbrace{0 \dots 0}^{L-1 \text{ zeros}} & x[0] & \dots \end{pmatrix}^T$$

08 March 2019 10:52

Algorithm 2 : IDFT

Algorithm steps

[In coming system based on CDMA, there
is a need to transmit/receive more
data in less time. ⇐ Interference from other users
reduces.] For that interference of one user

L 23

08 March 2019 10:28

$$
\begin{aligned}
r_0 &\triangleq \underline{u}^{(0)T} \underline{y}_\ell \approx h_0 b_\ell + n_0 \\
r_1 &\triangleq \underline{u}^{(1)T} \underline{y}_\ell \approx h_1 b_\ell + n_1 \\
&\vdots \\
r_{L-1} &\triangleq \underline{u}^{(L-1)T} \underline{y}_\ell \approx h_{L-1} b_\ell + n_{L-1}
\end{aligned}
$$

→ Diversity L ($\approx$) — subject to ideal autocorrelation assumption on spreading sequence

Simpler symbol-by-symbol detector compared to ML detector

(In ML, we have to detect all symbols jointly)

$$\boxed{h_0^* r_0 + h_1^* r_1 + \cdots + h_{L-1}^* r_{L-1}}$$ is sufficient for detection

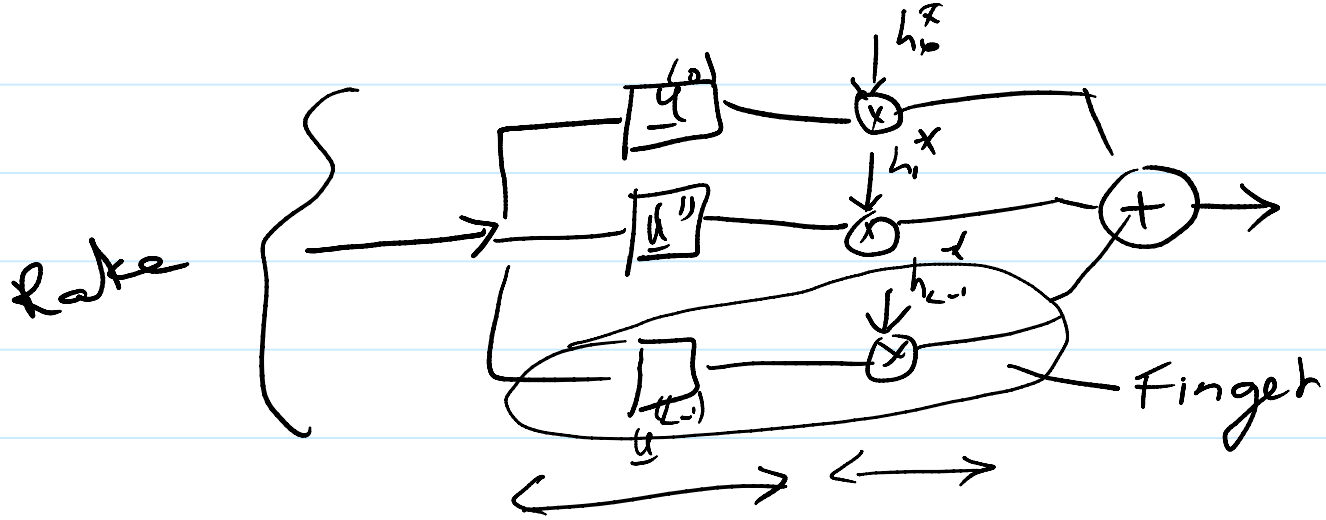
COMBINING

L23

08 March 2019 10:39

Summary

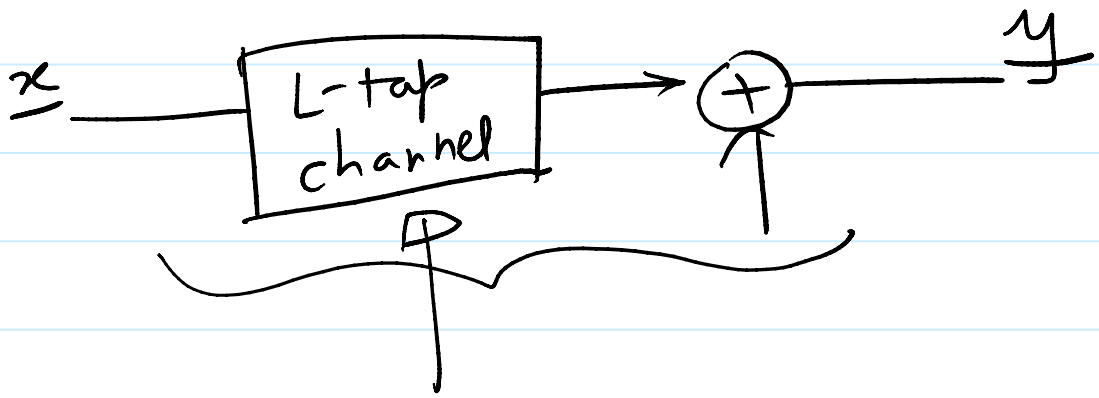
* Freq. div. is exploited by spread spectrum
  Div. gain L
  [× ISI is negligible if spreading factor is high
* Symbol-by-symbol detector sufficient only if ISI is negligible → Rake receiver



L 23

08 March 2019 10:48

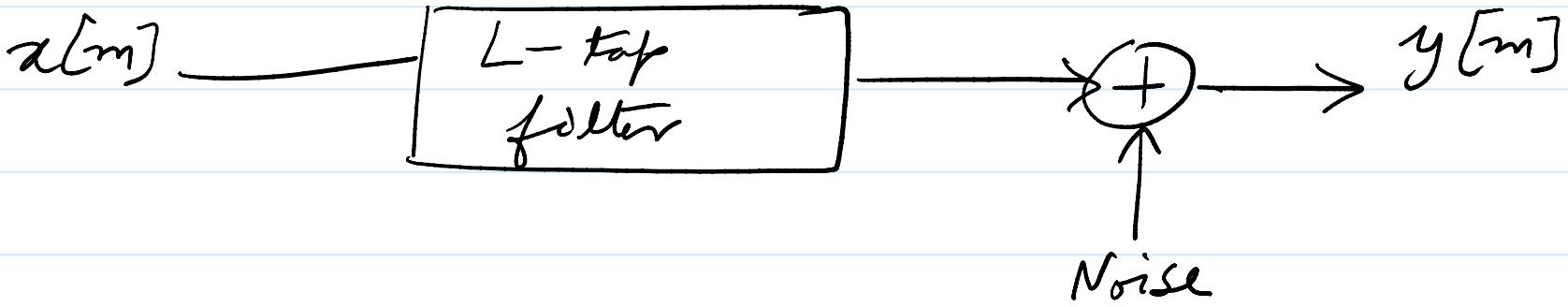
OFDM (Orthogonal frequency division multiplexing)

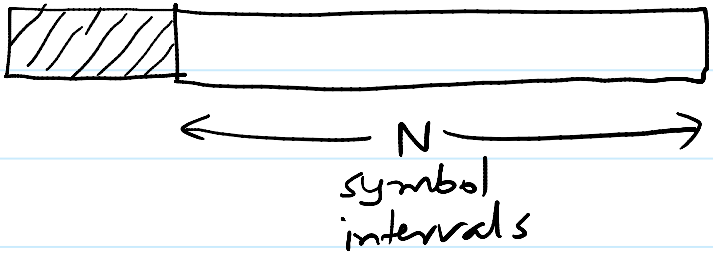


Lecture 24

11 March 2019 08:12

$$y[n] = \sum_{l=0}^{L-1} h_l[n]\, x[n-l] + w[n]$$





$$\begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} x[-L+1] \\ \vdots \\ x[-1] \\ x[0] \\ \vdots \\ x[N-1] \end{pmatrix} + \begin{pmatrix} w[0] \\ \\ \\ w[N-1] \end{pmatrix}$$

L24

11 March 2019 08:13

$$y[n] = h_0 x[n] + h_1 x[n-1] + \cdots + h_{L-1} x[n-L+1] + w[n]$$

$$\begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{pmatrix} = \begin{pmatrix} h_{L-1} & \cdots & h_1 & h_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & h_{L-1} & \cdots & h_1 & h_0 \end{pmatrix} \begin{pmatrix} x[-L+1] \\ \vdots \\ x[-1] \\ x[0] \\ \vdots \\ x[N-1] \end{pmatrix} + \begin{pmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{pmatrix}$$

L 24

11 March 2019 08:13

We want to diagonalize the channel

FACT: A circulant matrix can be diagonalized without knowing the entries of the matrix

$$\underbrace{\begin{bmatrix} \text{DFT} \\ \text{matrix} \end{bmatrix}}_{N\times N} \underbrace{\begin{bmatrix} \text{circulant} \\ \text{matrix} \end{bmatrix}}_{N\times N} = \begin{bmatrix} \diagdown \end{bmatrix} \begin{bmatrix} \text{DFT} \\ \text{matrix} \end{bmatrix}_{N\times N}$$

$(n,k)^{th}$ entry $\frac{1}{\sqrt{N}} e^{-j\frac{2\pi nk}{N}}$

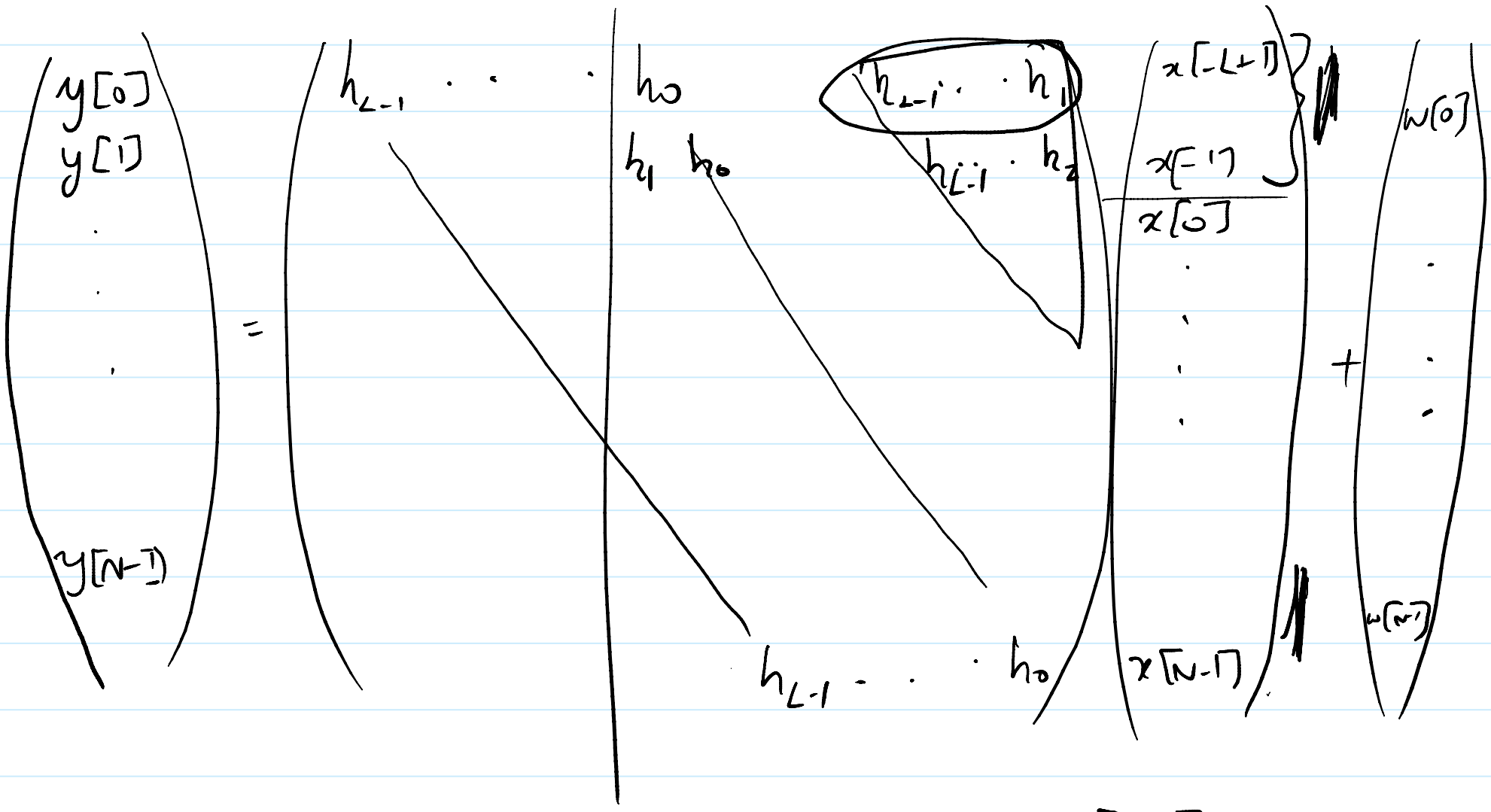
Diagonal matrix

↓

Diagonal entries are the DFT coeff of the 1st Column

L24

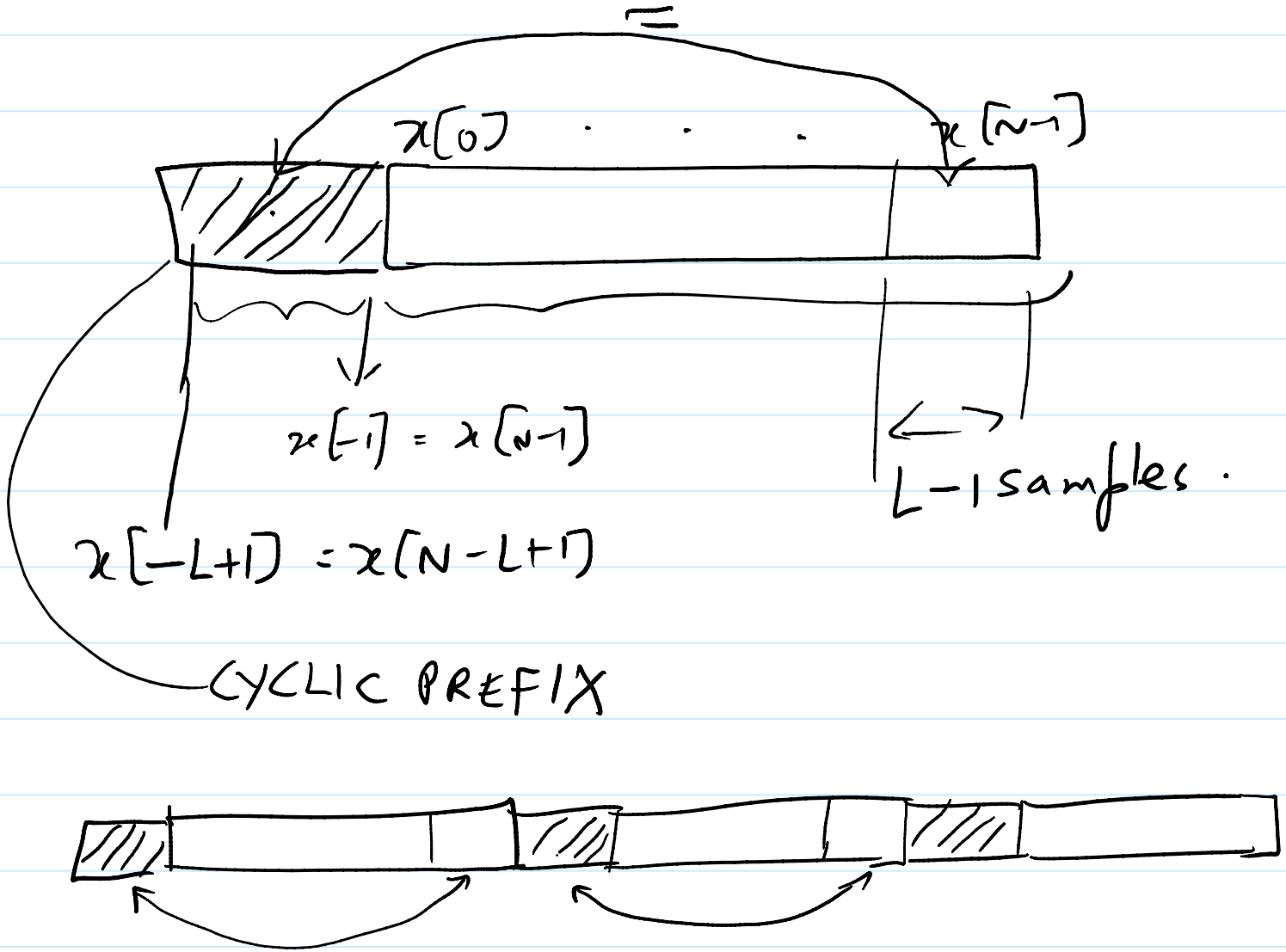
11 March 2019 08:17



$y[0]$ should be $h_0 x[0] + h_1 \underbrace{x[N-1]}_{=x[-1]} + h_2 \underbrace{x[N-2]}_{=x[-2]} + \dots + h_{L-1} x[N-L+1]$

L24

11 March 2019 | 08:21



L24

11 March 2019 08:24

$$\begin{pmatrix} y[0] \\ \vdots \\ y[N-1] \end{pmatrix} = \underbrace{\begin{pmatrix} h_0 & h_2 & h_1 \\ h_1 & h_0 & h_2 \\ h_2 & h_1 & h_0 \end{pmatrix}}_{H_c} \underbrace{\begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} w[0] \\ \vdots \\ w[N-1] \end{pmatrix}}_{w}$$

y

circulant matrix

$$\underbrace{(\text{DFT matrix})}_{F}\,\underbrace{(\text{circ matr})}_{H_c} = \underbrace{(\text{Diag})}_{H}\,(\text{DFT matrix})$$

$$(F H_c = H F)\, F^H$$

$$F H_c F^H = H$$

Let $\underline{x} = F^H \underline{d}$

$$F^H F = I$$

$$F F^H = I$$

L24

11 March 2019 08:30

$$\underline{y} = H_c \underline{x} + \underline{w}$$

$$= H_c (F^H \underline{d}) + \underline{w}$$

$$\underline{d} = \begin{pmatrix} d[0] \\ \vdots \\ d[N-1] \end{pmatrix}$$

symbols that we want to send.

$$F\underline{y} = \underbrace{F H_c F^H}_{H} \underline{d} + F\underline{w}$$

$H \rightarrow$ a diagonal matrix

$$\underline{\tilde{y}} \triangleq F\underline{y} = H\underline{d} + \underline{\tilde{w}}$$

$\underline{\tilde{w}} = F\underline{w}$

$$H = \begin{pmatrix} H_0 & & 0 \\ & H_1 & \\ & \ddots & \\ 0 & & H_{N-1} \end{pmatrix}$$

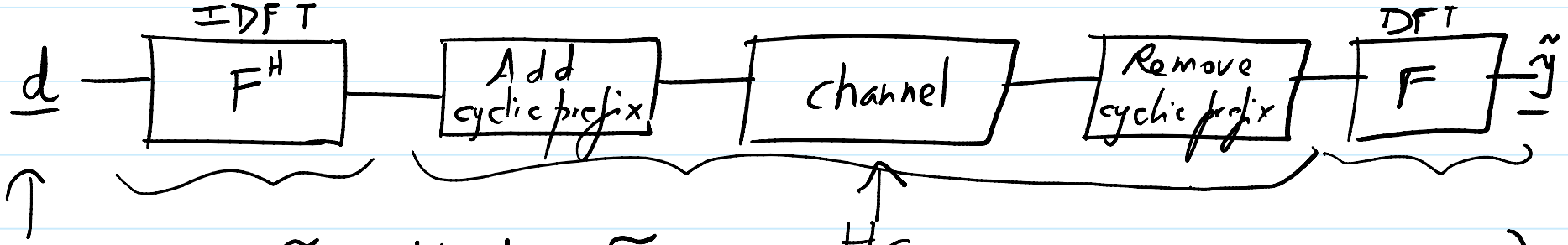
$$\tilde{y}[0] = H_0 d[0] + \tilde{w}[0]$$

$$\tilde{y}[N-1] = H_{N-1} d[N-1] + \tilde{w}[N-1]$$

$\Rightarrow$ N parallel channels

L24

11 March 2019 08:35



$$\underline{\tilde{y}} = H\,\underline{d} + \underline{\tilde{w}} \longrightarrow \sim CN(\underline{0}, \underbrace{F N_0 I F^H}_{N_0 I})$$

$\{H_0, H_1 \cdots, H_{N-1}\}$ diagonal entries of H

$$= DFT\{\underbrace{h_0, h_1, \ldots, h_{L-1}}, 0, \ldots, 0\}$$

Each channel is a Rayleigh fading channel

$$H_i \sim CN(0, ?)$$

$$\underline{H} = F \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{L-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim CN(0, I)$$

$\underline{H}$: vector

vector $\underline{H} \triangleq \begin{pmatrix} H_0 \\ H_1 \\ \vdots \\ H_{N-1} \end{pmatrix} \sim CN\left(\underline{0}, F \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} F^H\right)$

L24

11 March 2019 08:44

$$\underline{H} \sim \mathcal{CN}\left(\underline{0},\ F \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} F^H\right)$$

$\underline{H}$ → not i.i.d components

not necessarily diagonal.

Diversity for each channel = 1

$$H_i \sim \mathcal{CN}\left(0,\ \frac{1}{N} \underbrace{\sum_{l=0}^{L-1} E\left[|h_l|^2\right]}_{L}\right)$$

For diversity > 1, we need coding across the channels.

Lecture 25

12 March 2019 11:55

* Maximum possible diversity (coding over N channels) is L (not N)

* To add the correct amount of cyclic prefix we need to know L.
  - In our discussion, we had a cyclic prefix of length L−1
  - We need at least an upper bound on L. Estimate using delay spread.

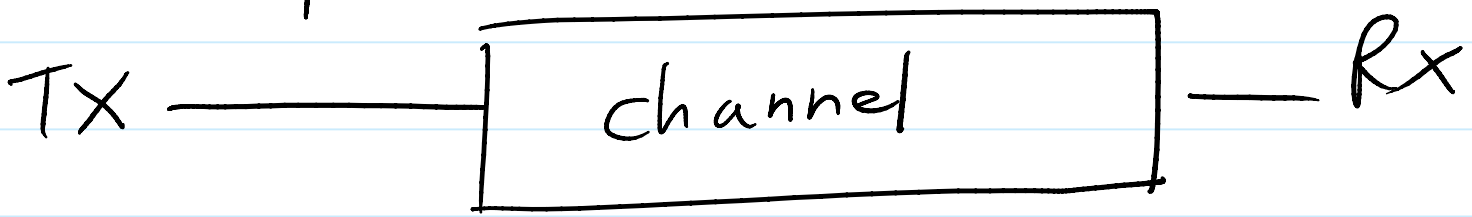
cyclic prefix
- \+ removes ISI
- − overhead

L25

12 March 2019 12:08

So far (1) channel modeling ⟶ - Linear time-varying filter model
- Rayleigh fading
- Path loss ($d^{-\alpha}$)

(2) Point-to-point communication



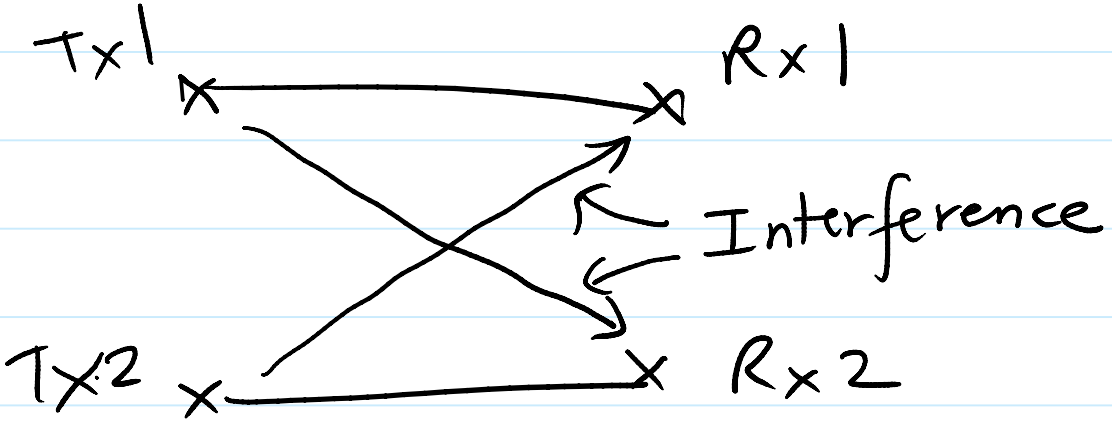
$P_e \propto \frac{1}{SNR}$ (Rayleigh) (1-tap) (e^{-SNR} AWGN)

Diversity — Time, antenna, freq ↑ div. gain
Space-time code design criteria.

L 2 5

12 March 2019 12:13

Next How to design a system with multiple wireless links? (cellular system)



Important problem : How to manage interference?

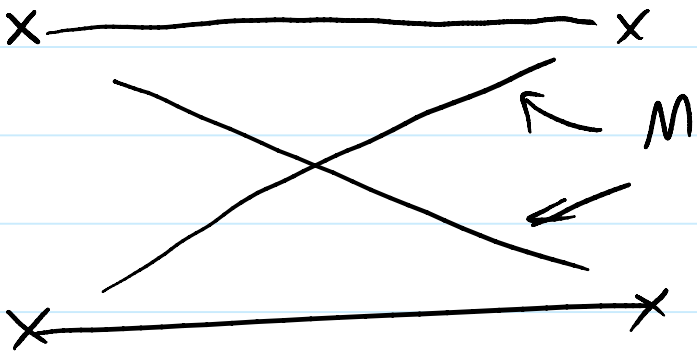
There are many ways.

L25

12 March 2019 12:17

3 systems will be discussed.

SYSTEM 1 (Interference avoidance)
using frequency planning, Time-division, ...



Make sure interference power is not too high.

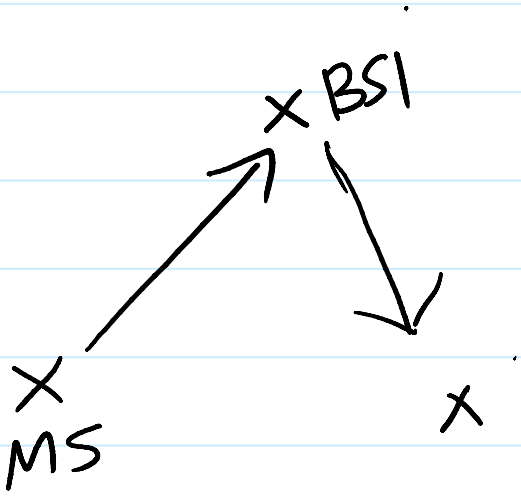
L25

12 March 2019 12:20

Cellular system : 2 links

(Uplink) (Reverse link) Mobile terminals ⟶ network (Base-station)

(Downlink) (Forward link) Network (Base-station) ⟶ Mobile terminals

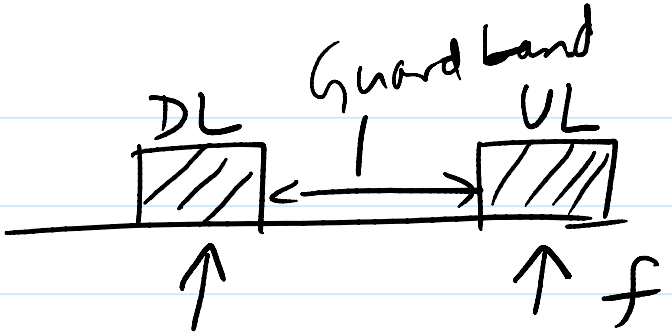


Separate uplink & downlink
— using diff. bands (or)
— using time-division

L25

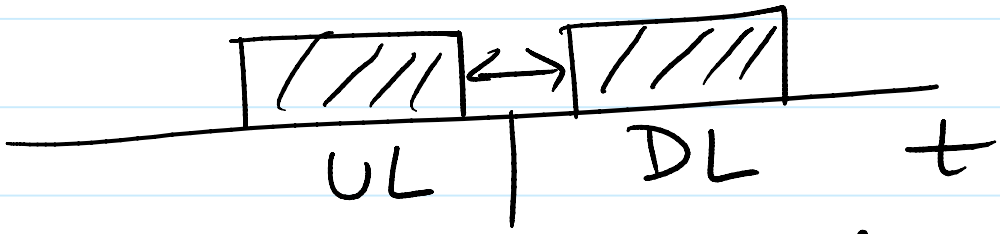
12 March 2019 12:27

✓ Frequency-division duplexing (FDD)



(or)

Time-division duplexing (TDD)



Guard interval

Let us focus on downlink (uplink design similar).

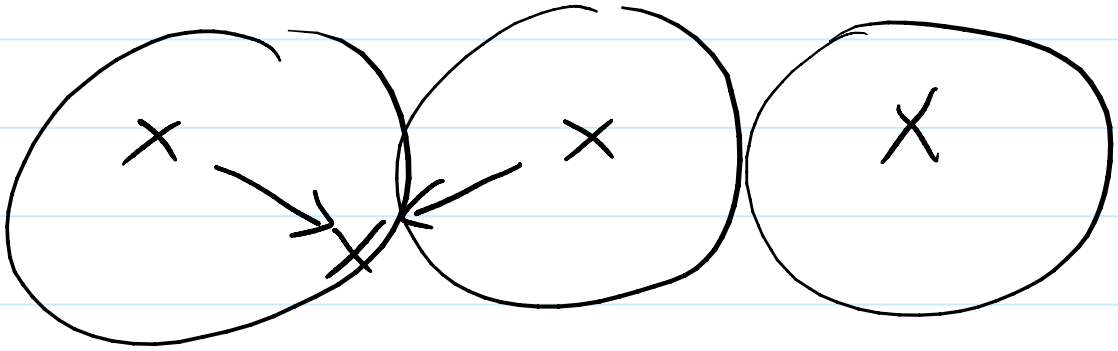
Say we have W Hz of bandwidth for the downlink.

L 25

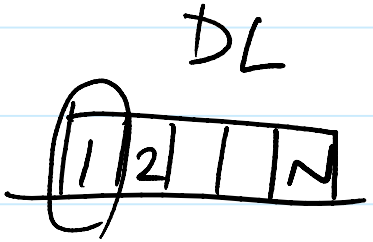
12 March 2019 12:32

Need to ensure physical distance travelled by interfering signal is large

- Each BS can communicate to only one MS at any given time.



Frequency planning to be done

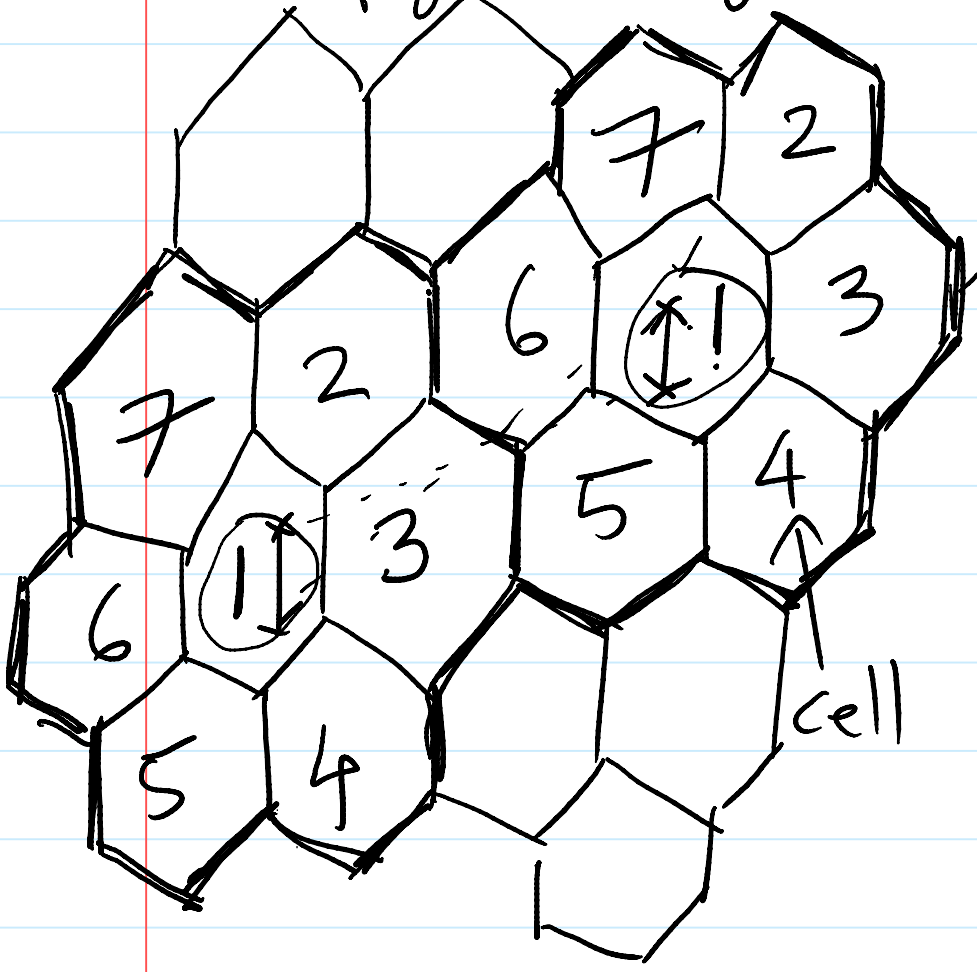


Divide W into N bands

L25

12 March 2019 12:36

- Assign n bands to each BS so that neighbouring BSs do not use the same band.



→ cluster of 7 cells $\frac{W}{7}$

Divide overall region to be covered into "cells"

BW used per cell $\frac{nW}{N}$

$N=7$

$n=1$

L25

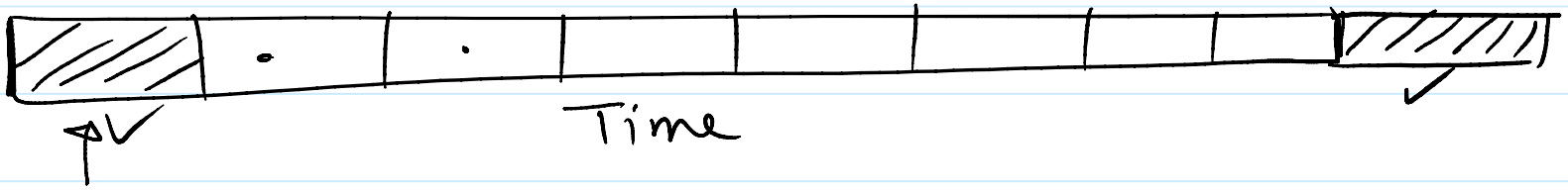
12 March 2019 12:43

Frequency reuse factor $= \frac{n}{N} = \frac{1}{7}$

As freq. reuse ↑, interference ↑.

GSM example calculation

one band $\frac{W}{N} = 200\,kHz$ → shared by 8 links using time-division



L25

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Freq. reuse factor $\frac{1}{7}$ ✓

Suppose $W = 5\ MHz \longrightarrow \frac{W}{7} = \frac{5}{7}\ MHz$

In 200 kHz → 8 users

In $\frac{5}{7}$ MHz → $\frac{5000}{7} \times \frac{1}{200} \times 8$ users $= \frac{200}{7}$ users

≈ 28

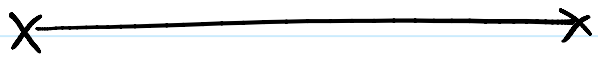
≈ 28 users/cell. "simultaneously"

Lecture 26

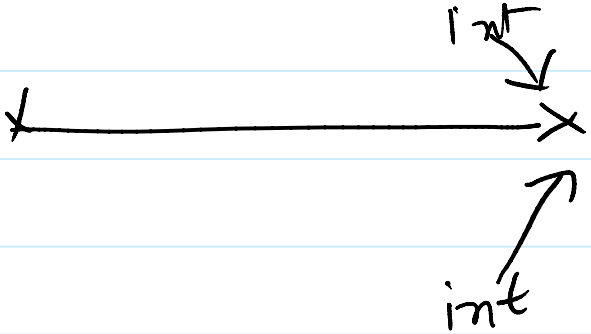
14 March 2019 11:06

* How can quantify whether the interference is at an acceptable level after frequency planning?

Acceptable : if Signal to interference power > Threshold (SIR)



For a given BER, SNR > Threshold



$$SINR = \frac{\text{Signal power}}{\text{Int. power + Noise power}} > \text{Threshold}$$

BER req. → $SINR_{req}$ (or) (SIR_{req})

L26

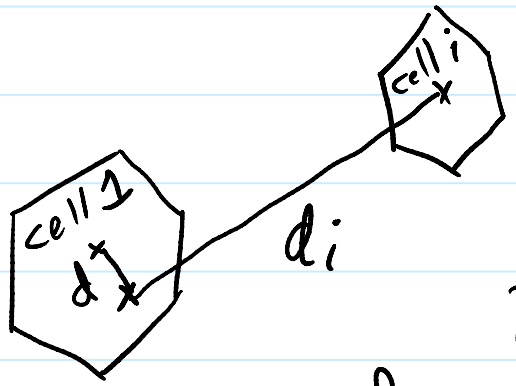
14 March 2019 11:12

Rough calculation of SIR

$K d^{-n}$

$\left(\frac{d}{d_0}\right)^{-n}$

Downlink



Consider a user in cell 1

P_0: Transmit power (for any BS)

P_{sr} = Avg Received signal power at user $= P_0 \left(\frac{d}{d_0}\right)^{-n}$ ← Path loss exponent

d: distance from BS to user d_0: reference distance

Power received from other cells using the same band $= \sum_i P_0 \left(\frac{d_i}{d_0}\right)^{-n}$

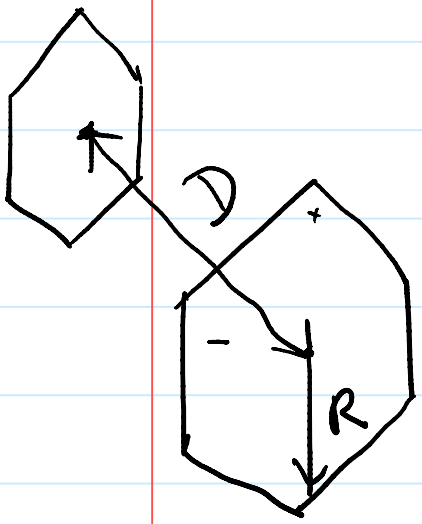
d_i = distance from Interfering BS i to user

L26

14 March 2019 11:18

For the $\frac{1}{7}$ freq. plan.

$$SIR = \frac{P_0 \left(\frac{d}{d_0}\right)^{-n}}{\sum_i P_0 \left(\frac{d_i}{d_0}\right)^{-n}} \approx \frac{P_0 \left(\frac{d}{d_0}\right)^{-n}}{\sum_{i=1}^{6} P_0 \left(\frac{d_i}{d_0}\right)^{-n}}$$



Take interference only from neighbouring clusters.

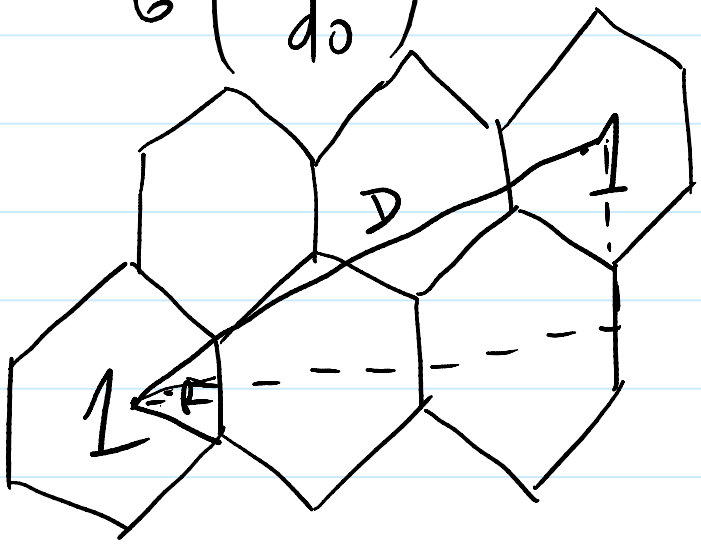
$d \leq R$

$d_i \approx D$: Distance between cluster centers

L 26

14 March 2019 11:22

$$SIR \approx \frac{\left(\frac{R}{d_0}\right)^{-n}}{6\left(\frac{D}{d_0}\right)^{-n}} = \frac{R^{-n}}{6\,D^{-n}}$$



$$\frac{R}{D} = \frac{1}{\sqrt{21}}$$

$5 \times \frac{\sqrt{3}}{2} R \quad \frac{3}{2} \times R$

$\frac{25 \times 3}{4} + \frac{9}{4}$

$= \frac{84}{4} = 21$

$$SIR \approx \frac{\left(\sqrt{21}\right)^{n}}{6} \overset{n=2}{=} \frac{21}{6} = 3.5 \approx 5\,dB$$

L26

14 March 2019 11:33

Summary

— 200kHz small BW (freq diversity = 1)

+ Interference is reduced to desired level
  Link design can be based on point-to-point communication

– Freq reuse factor low → inefficient use of spectrum (static allocation of resources)
  eg: W for each cell

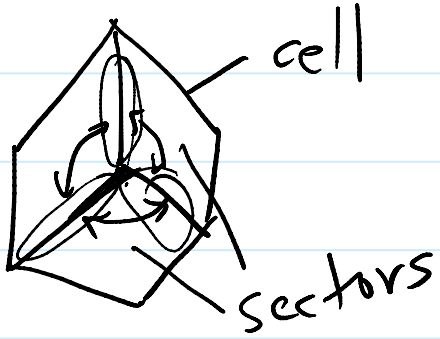
– Planning & reconfiguration are complicated

– Interference averaging is limited. May be strong from 1 or 2 sources

L26

14 March 2019 11:39

Sectorization



- Divide cell into sectors
- Each sector is served by a directional antenna to minimize interference between sectors
- Same freq in each sector (if directionality is ideal)

L26

14 March 2019 11:43

SYSTEM2 (Based on code-division multiple access (CDMA))

Common aspects (sys 1 & sys 2)

- Area divided into cells
  Each cell has a BS
- Uplink, downlink separated using FDD/TDD

What is different?

- No freq. planning. All BW will be used in all cells
- spread-spectrum signaling used.
  Different transmissions use different spreading code

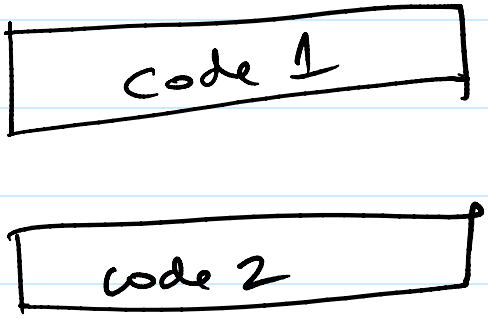
L26

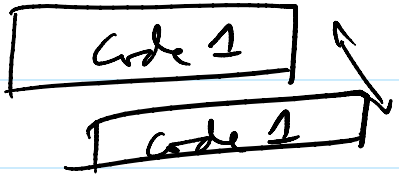
14 March 2019 11:48

— Spreading codes to be designed for

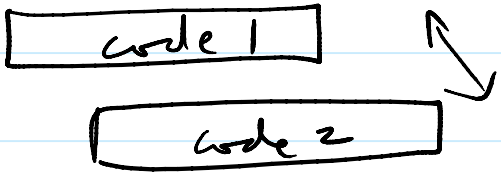
— ideal auto-correlation properties (to use RAKE receiver) — reduce ISI

— ideal cross-correlation properties — reduce interference from other transmitters





correlation should be small



correlation should be small

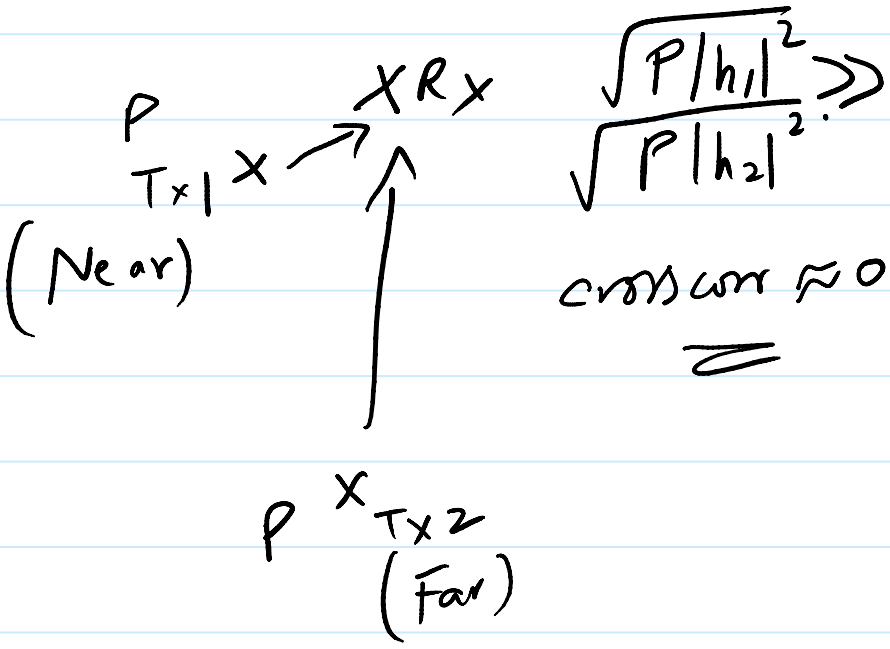
Lecture 27

18 March 2019 07:57

— Interference-limited system

* Spreading code design alone is not enough

"Near-far effect".
(UPLINK)



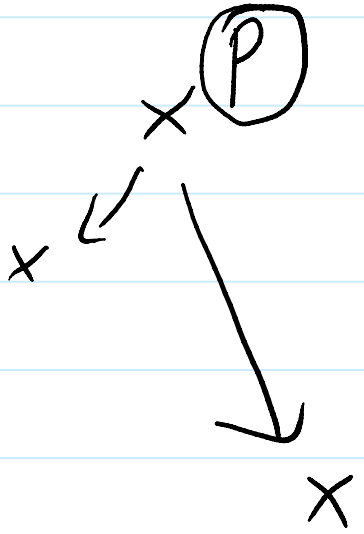
Far user's signal will see much more interference (assuming both transmit at the same power level)

Transmit power needs to be adjusted so that received power for each user is same.

L27

18 March 2019 08:10

DOWNLINK



No "near-far effect"

Power control used to reduce interference to other cells in downlink

$\underline{x}_1^H \underline{x}_2 \approx$ small δ

$(A_1 \underline{x}_1)^H (A_2 \underline{x}_2) \approx A_1 A_2 \delta$ (small)

$A_1 \gg A_2$

$A_1 \gg A_1 A_2 \delta$

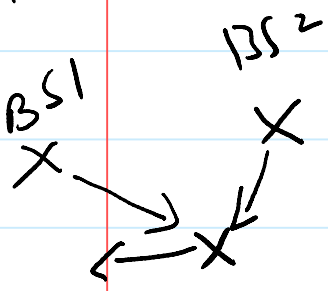
$A_2 \not\gg A_1 A_2 \delta$

L27

18 March 2019 08:16

Some other advantages:

- Frequency diversity (due to spread spectrum signaling)
- Voice activity detection

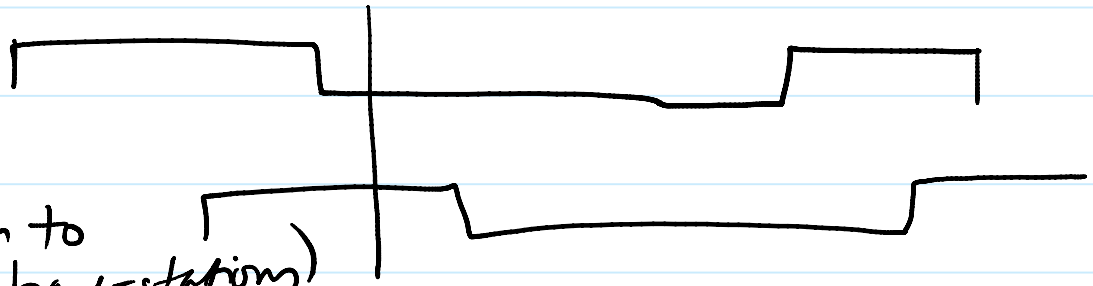


Detect activity and reduce power when not reqd → reduces interference to other transmissions

(Statistical multiplexing gain)

- Soft handoff

(Simultaneous connection to multiple base-stations)

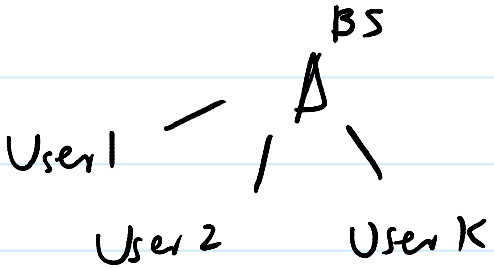


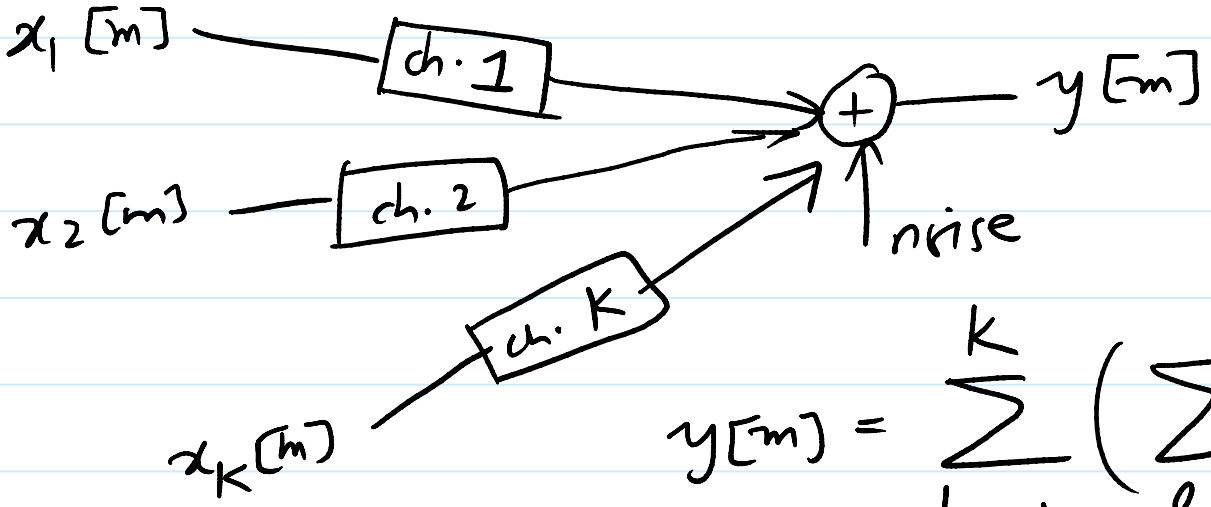
L 27

18 March 2019 08:26

CDMA uplink : (# users that can be supported)

Analysis (approximate)



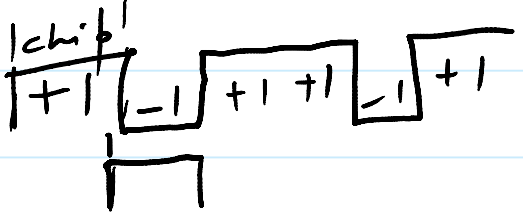


$$y[m] = \sum_{k=1}^{K} \underbrace{\left(\sum_{l} h_l^{(k)}[m]\, x_k[m-l] \right)}_{\text{Recd. signal from user } k} + \underbrace{w[m]}_{\text{Noise}}$$

Assumptions

- Users are chip-synchronous
- Random binary spreading seq. (± 1)

Sp. seq.



L27

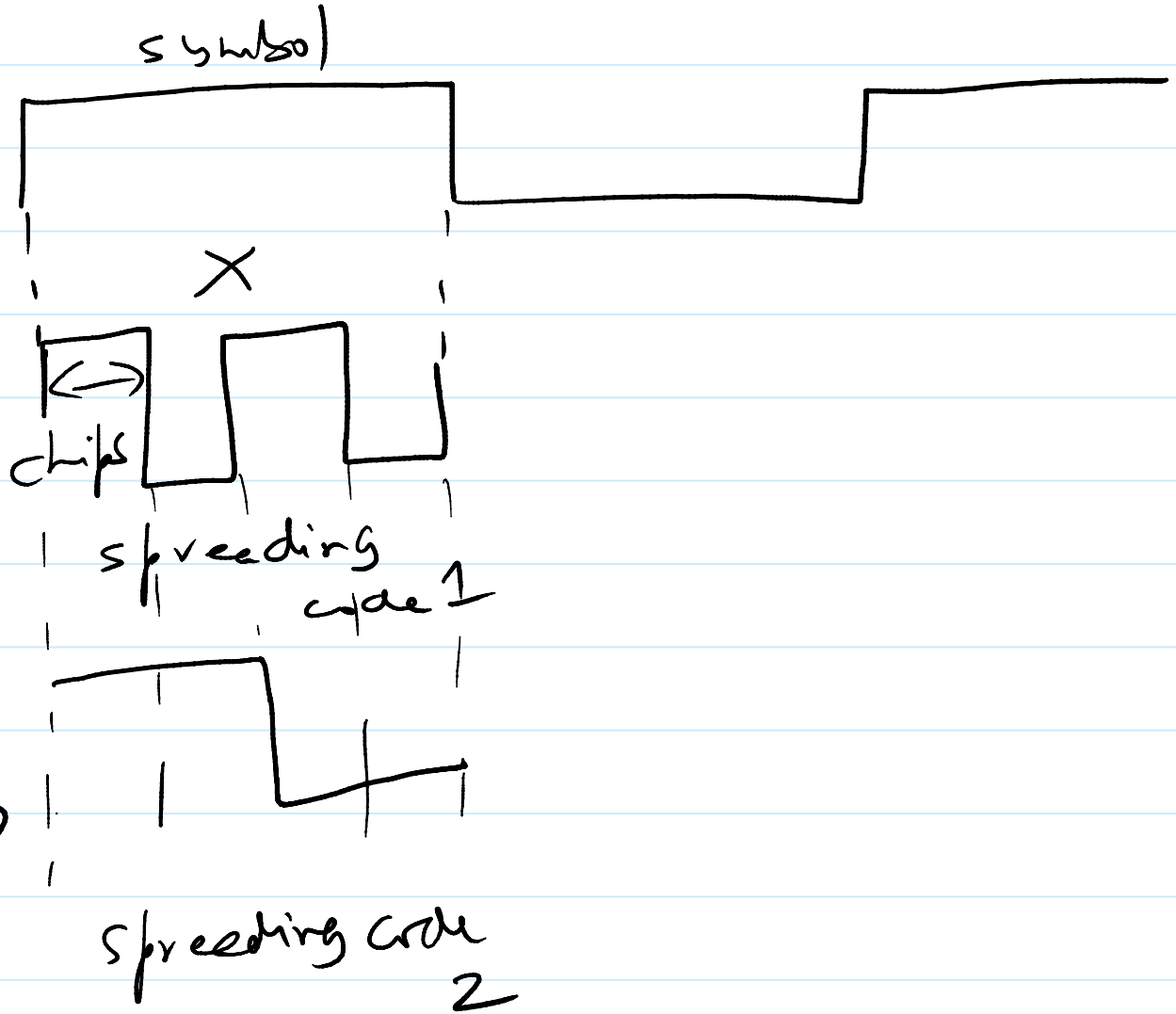
18 March 2019 08:31

- $E[x_k[m]] = 0$
- $E[x_k[m]\, x_k^*[m-l]] = 0$ for $l \neq 0$

chip synchronous

- $E[x_k[m]\, x_j[m-l]] = 0$

$k \neq j$



L27

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Calculate SINR per chip

Consider user 1 $\quad y[m] = S[m] + I[m] + w[m]$ (noise)

Desired signal: $\sum_l h_l^{(1)}[m]\, x_1[m-l]$

Interference: $\sum_{k=2}^{K}\left(\sum_l h_l^{(k)}[m] x_k[m-l]\right)$

Received signal energy per chip $E_1^c = \underbrace{E\left[|x_1[m]|^2\right]}_{P_1\left(\frac{1}{W}\right)} \underbrace{\sum_l E\left[|h_l^{(1)}[m]|^2\right]}_{g_1}$

(Duration of one chip $\frac{1}{W}$, Transmit power P_k)

$E_k^c = \underbrace{E\left[|x_k[m]|^2\right]}_{\frac{P_k}{W}} \underbrace{\sum_l E\left[|h_l^{(k)}[m]|^2\right]}_{g_k}$

L27

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$$E\left[|s[m]|^2\right] = E_1^c$$

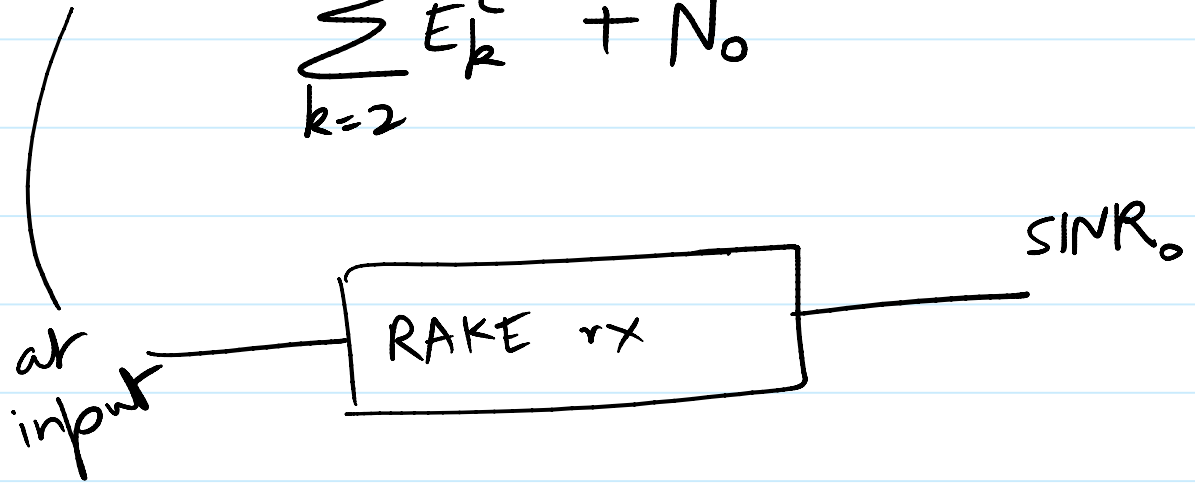
$$E\left[|I[m]|^2\right] = \sum_{k=2}^{K} E_k^c$$

$$E\left[I[m]\,I^*[m+\ell]\right] = 0 \quad \text{for } \ell \neq 0$$

(Random binary spreading seq.)

$$E\left[|w[m]|^2\right] = N_0$$

$$SINR_1 = \frac{E_1^c}{\sum_{k=2}^{K} E_k^c + N_0}.$$



L 27

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$$\sum_{m=1}^{G_f} \left[\sum_{k=1}^{K} h_l^{(k)}[m]\, x_k[m-l] \right] x_k^*(m-l)\, h_l^{*(k)}[m]$$

G_f: spreading gain. (sp. seq. length)

$$= G_f \left| h_l^{[k]}[m] \right|^2 d_k + \sum (\quad)$$

$x_k[m-l]\, x_j^*[n-l]$

either $k \neq j$ or $m \neq n$

Signal Amplitude will go up by G_f

Interference power will go up by G_f

SINR ↑ by a factor of G_f

L27

18 March 2019 08:49

After "despreading"

$$SINR_1 = \frac{G E_1^c}{\sum_{k=2}^{K} E_k^c + N_0}$$

$$= \frac{G \frac{P_1}{W} g_1}{\sum_{k=2}^{K} \frac{P_k}{W} g_k + N_0}$$

$$= \frac{G P_1 g_1}{\sum_{k=2}^{K} P_k g_k + N_0 W}$$

$$SINR_j = \frac{G P_j g_j}{\sum_{k \neq j} P_k g_k + N_0 W}$$

Lecture 28

26 March 2019 11:58

$$SINR_k = \frac{G P_k g_k}{\sum_{\ell \neq k} P_\ell g_\ell + N_0 W}$$

G: Spreading gain

P_k: Power transmitted by k^{th} user

g_k: $\sum_{\ell} E[|h_\ell^{(k)}[n]|^2]$

Suppose β is reqd. SINR for each user.

We need $SINR_k \geq \beta$ for all k.

We have to choose P_k's to satisfy this requirement.

Will we be able to choose P_k's appropriately?

$$\frac{G P_k g_k}{\sum_{\ell \neq k} P_\ell g_\ell + N_0 W} \geq \beta \quad \text{for } k = 1, 2, \ldots, K$$

L 28

26 March 2019 12:08

$$\underbrace{G P_k g_k} \geq \beta\left(\sum_{\ell \neq k} P_\ell g_\ell + N_0 W\right) \qquad \text{for } k = 1, \cdots, K$$

Necessary condition (to be able to find feasible P_k's)

Add all the K inequalities

$$G \sum_k P_k g_k \geq \beta\left(\underbrace{\sum_k \left(\sum_{\ell \neq k} P_\ell g_\ell \right)} + K N_0 W \right)$$

$\sum_{\ell \neq k} P_\ell g_\ell$

$k=1$ $\quad 0 + P_2 g_2 + \cdots + P_K g_K$

$k=2$ $\quad P_1 g_1 + 0 + \cdots + P_K g_K$

$k=K$ $\quad P_1 g_1 + P_2 g_2 + \cdots + 0$

} sum $(K-1) \sum_k P_k g_k$

L28

26 March 2019 12:13

$$G\left(\sum_k P_k g_k\right) \geq \beta\left((k-1)\left(\sum_k P_k g_k\right) + IkN_0W\right)$$

$$\underbrace{\left[G - \beta(k-1)\right]}_{\geq 0 \text{ if } G \geq \beta(k-1)}\underbrace{\left(\sum_k P_k g_k\right)}_{\geq 0} \geq \underbrace{\beta IkN_0W}_{>0}$$

If $G < \beta(k-1)$, necessary condition cannot be satisfied.

L28

26 March 2019 12:15

$\Rightarrow \beta(K-1) < G_T$ is necessary

$\Rightarrow \boxed{K < \frac{G_T}{\beta} + 1}$ (# users cannot be larger than $\frac{G_T}{\beta}+1$)

Let $K < \frac{G_T}{\beta} + 1$.

Find a set of P_k's such that $SINR_k \geq \beta$ for all k.

SOLUTION $P_k g_k = \frac{\beta N_0 W}{G_T - \beta(K-1)}$

L28

26 March 2019 12:19

Is $G_7 P_k g_k \geq \beta\left(\sum_{\ell \neq k} P_\ell g_\ell + N_0 W\right)$? YES

$$G_7\left(\frac{\beta N_0 W}{G_7 - \beta(K-1)}\right) \quad \beta\left((K-1)\frac{\beta N_0 W}{G_7 - \beta(K-1)} + N_0 W\right)$$

LHS RHS

$$= \beta\left(\frac{G_7 N_0 W}{G_7 - \beta(K-1)}\right)$$

L 28

26 March 2019 12:24

$\Rightarrow$ $\boxed{K < \frac{G}{\beta} + 1}$ is also a sufficient condition

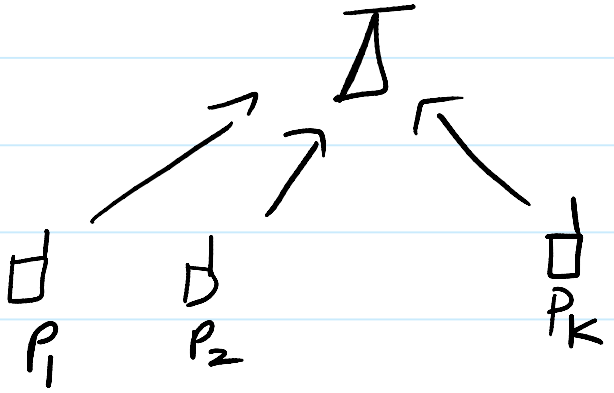
$$\begin{bmatrix} Gg_1 & -\beta g_2 & -\beta g_3 & \cdots & -\beta g_K \\ -\beta g_1 & Gg_2 & & & \\ \vdots & & \ddots & & \\ -\beta g_1 & \cdots & & -\beta g_{K-1} & Gg_K \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{bmatrix} \geq \begin{bmatrix} \beta N_0 W \\ \beta N_0 W \\ \vdots \\ \beta N_0 W \end{bmatrix}$$

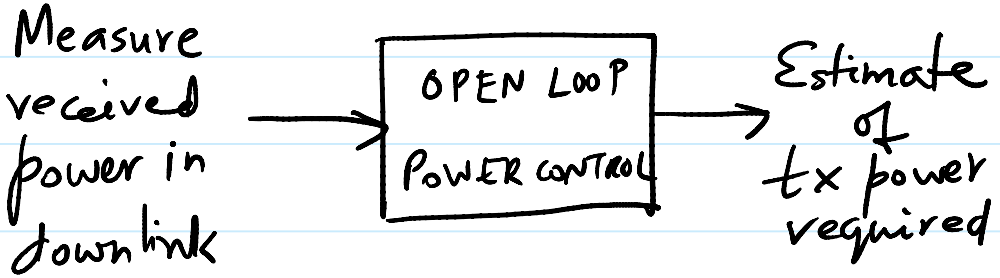
channel knowledge used here

L 28

26 March 2019 12:31

Uplink power control in 3G:

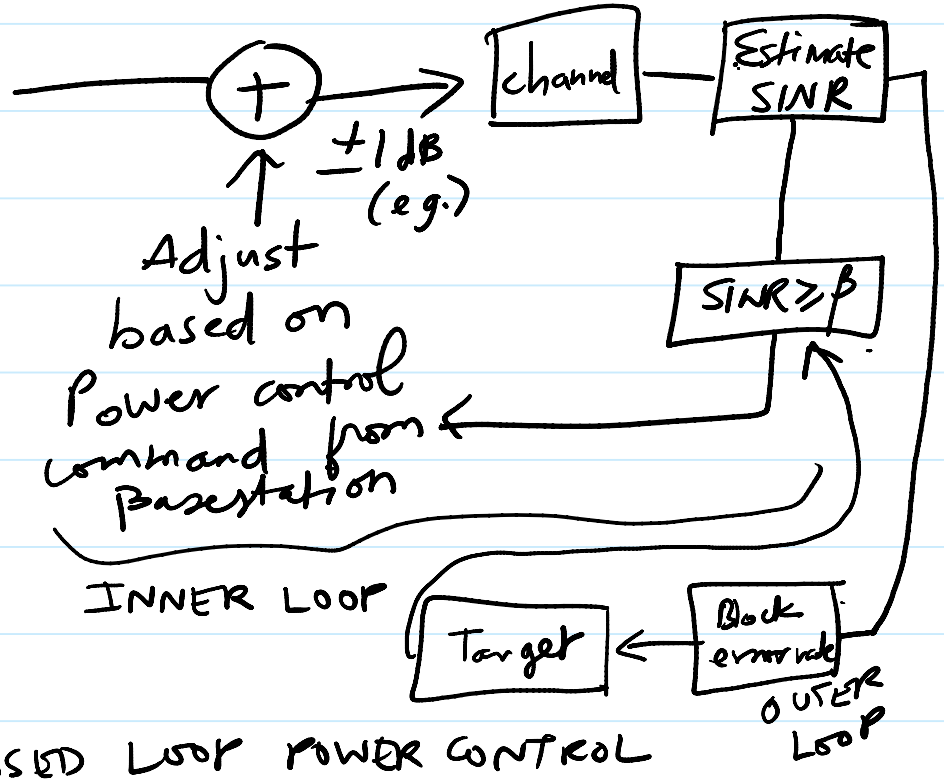




OPEN LOOP: Initialization

INNER LOOP: (FAST) small time-scale
(update P_k) ($\approx$ once a ms)

OUTER LOOP: (SLOW) larger time-scale
(update β) ($\approx$ 100's of ms)



CLOSED LOOP POWER CONTROL

L28

26 March 2019 12:45

Voice activity detection: If a user is active only 'p' fraction of time, then interference reduces by the same factor

Now $K < \frac{G}{p\beta} + 1$ is sufficient

- Sectorization useful in CDMA too (Reduces interference)



Lecture 29

28 March 2019 11:02

Two designs so far

1) Based on frequency planning and time-division multiple access
   - Interference avoidance
   - BW per user is not large (frequency diversity not available)
   - Frequency reuse factor is low

   (eg. GSM)

2) Based on CDMA
   - Freq. reuse factor is high
   - Intra-cell and inter-cell interference
   - Full BW used by each tx (frequency diversity)

   (eg. IS-95, WCDMA)

L 29

28 March 2019 11:08

Flarion

3) Based on OFDM (eg. Flash-OFDM)

- OFDM used to split BW into smaller subchannels
- Multiple subchannels to each user (freq diversity)
- Intra-cell interference avoided (by allocation of subchannels)
- Inter-cell interference is 'averaged' by 'frequency hopping'

L29

28 March 2019 11:13

* Thumb rule :

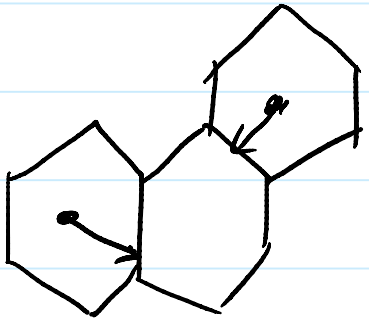
Desired signal + Interference + noise

Sum of many low power interferers (better than) Small # of high power interferers

'interference averaged'

When we treat interference as noise

Freq. planning



↑ freq. reuse factor
few high power int.

CDMA : 'interference averaging'

L29

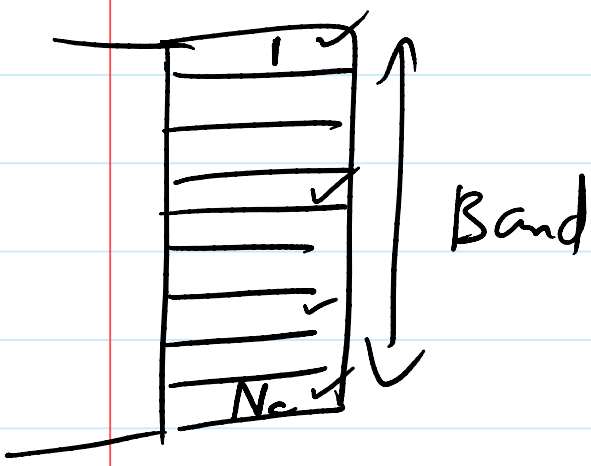
28 March 2019 11:18

Flash-OFDM example

BW → split into N_c subchannels

each user is allocated n subchannels

($\frac{N_c}{n}$ users each with n subchannels)



These n subchannels are

(diversity) — spread out over the whole BW

(interference averaging) — hopped over time (different hopping pattern in diff. cells)

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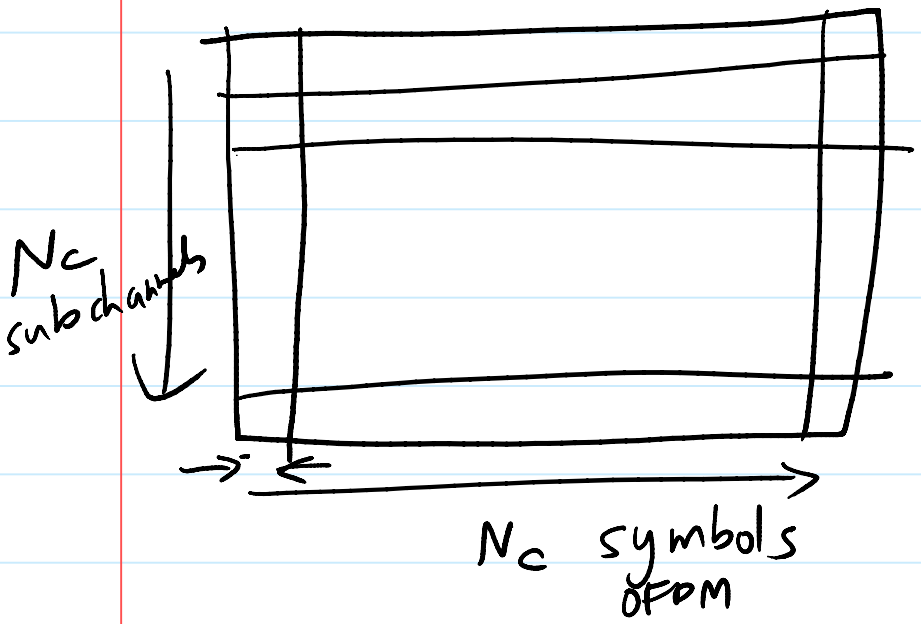
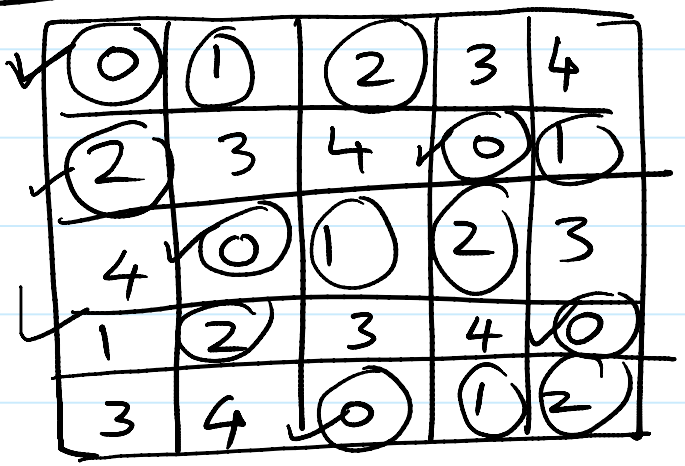
Example Hopping pattern is over N_c OFDM symbols
N_c subchannels are used.



$N_c = 5$



5 subchannels
×
5 symbols
⟶ 5 virtual channels (0, 1, 2, 3, 4)

L29

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— Each virtual channel hops over many subchannels

VC 0: at time 0, subchannel 0
1, 2
2, 4
3, 1
4, 3

hops over all subchannels

VC 1: at time 0, 3
1, 0
2, 2
3, 4
4, 1

— Each user allocted n virtual channels
— VCs do not interfere with each other, allow freq. diversity

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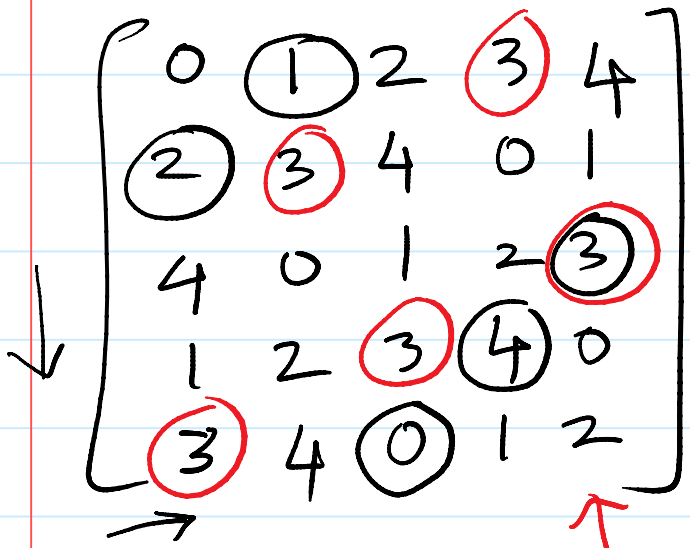
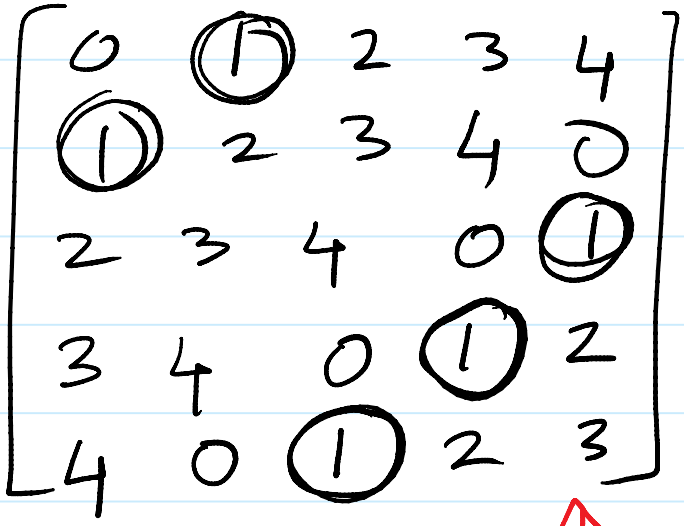
- Each cell will use a different hopping pattern



BS 1



BS 2

VC3 VC1

collisions = 1

VC1:

interference from cell 1

at time 0, VC2
1, VC1
2, VC0
3, VC4
4, VC3

Lecture 30

01 April 2019 12:03

- Each virtual channel hops over the entire band (frequency diversity)
- Each virtual channel sees interference only once from any of the N_c virtual channels in another cell during a period of N_c symbols

  * Any pair of virtual channels from 2 different hopping patterns collide only <u>once</u>.

eg <u>$N_c = 5$</u> We saw 2 patterns. How many patterns can we design with this property? <u>ANS</u> $N_c - 1$ (if N_c is prime)

L30

01 April 2019 12:10

$N_c \times N_c$ matrices → called a Latin square

$$R^a = \begin{bmatrix} & \times & \\ & & \\ & & \end{bmatrix}_{N_c \times N_c}$$

for $a = 1, 2, \ldots, N_c - 1$

$(i,j)^{th}$ entry $R^a_{ij} = (ai + j) \bmod N_c$.

$i, j \in \{0, 1, \ldots, N_c - 1\}$

L30

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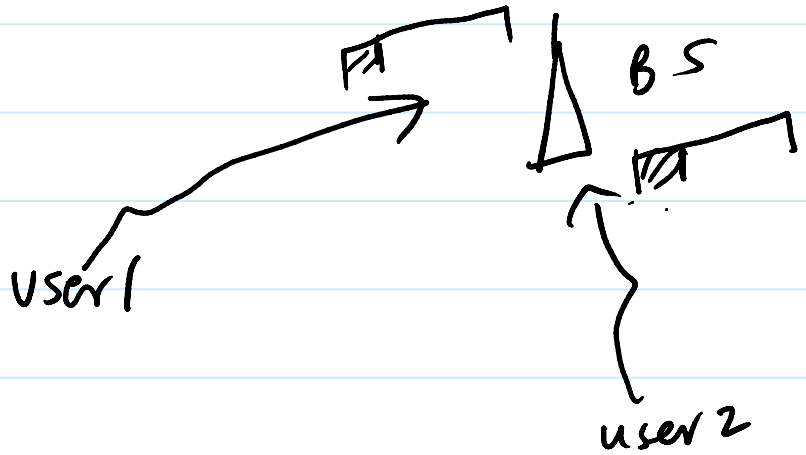
Above discussion assumes

— Synchronization of users within a cell



— Synchronization of transmissions of neighbouring cells at the level of OFDM symbols

— Coding over several OFDM symbols allows freq div & interference averaging

L30

01 April 2019 12:23

So far

- Channel model
- Point-to-point communication over this model → Error rate for given transmission rate

Design of cellular system with multiple links

1) GSM (TDMA)
2) CDMA
3) OFDM

Next Capacity limits

1) point-to-point channels
2) multiuser channels

(INFORMATION THEORY)

L 30

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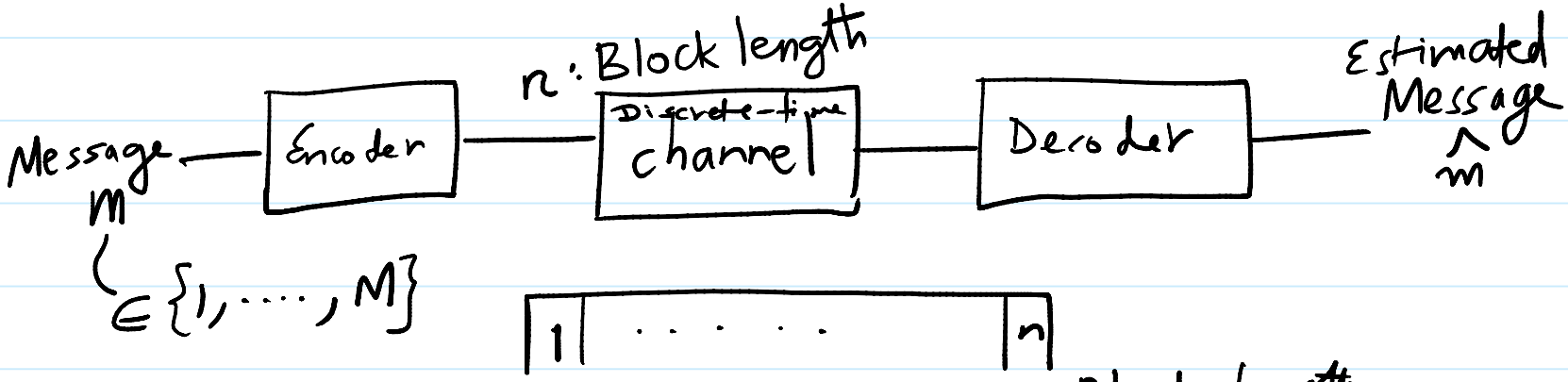

Block length
'#times channel is used to send one message'

$$\text{Rate} = \frac{\log_2 M}{n} \text{ bits/channel use}$$

bits/transmission

$$P_e^{(n)} = Pr(m \neq \hat{m})$$

[For any $\epsilon > 0$, $\exists\, n_0$ such that $P_e^{(n)} < \epsilon \; \forall n > n_0$.]

A rate R is achievable if $P_e^{(n)} \to 0$ as $n \to \infty$ for some encoder/decoder pair.

L 30

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Capacity C $\Rightarrow$ Any $R < C$ is achievable
$R > C$ is not achievable

Discrete-time real AWGN channel: $C = ?$

$$y[n] = x[n] + w[n]$$

$w[n] \sim$ i.i.d $N(0, \sigma^2)$

Average transmit power constraint P

$$E[x^2[n]] \leq P$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma^2}\right) \text{ bits/channel use}$$

L30

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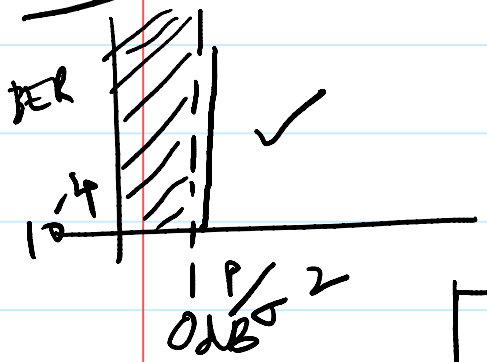
Info thy says "There exists some encoder/decoder pair that ensures $P_e^{(n)} < \epsilon$ for $n > n_0$ as long as $R < C$".

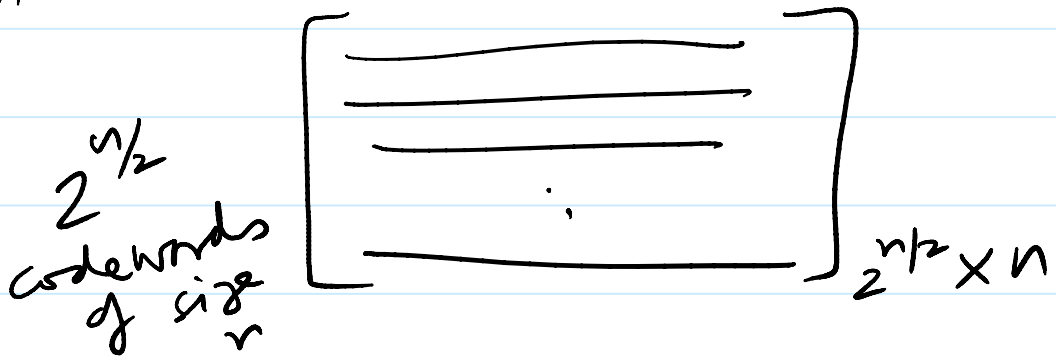
$R = 1/2$ eg $\frac{P}{\sigma^2} = 1$ (0 dB) $C = \frac{1}{2}\log_2(1+1) = \frac{1}{2}$ bits/channel use



$\frac{n}{2}$ bits in → n

$R = \frac{1}{2} = \frac{\log_2 M}{n}$

$M = 2^{n/2}$

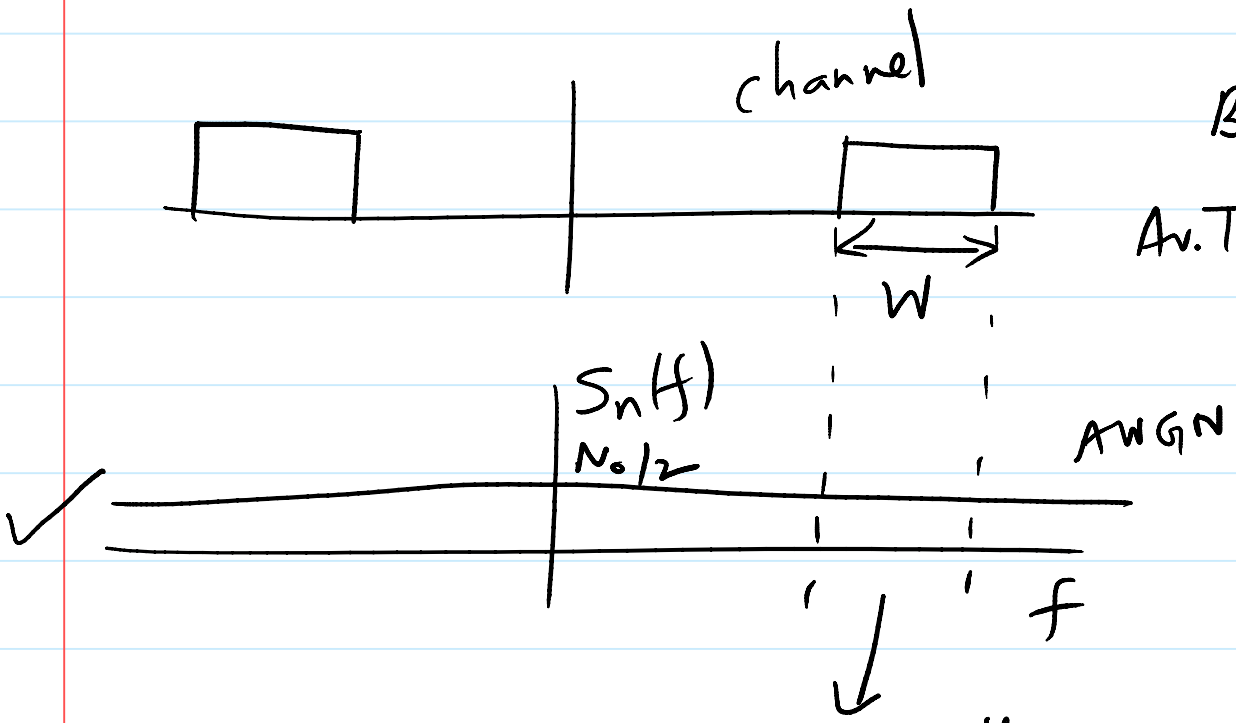


½ LDPC code
10^{-4} BER at ≈ 0.1dB for $n = 8000$

Lecture 31

02 April 2019 12:02

Continuous-time AWGN channel



Flat in the band of interest

Bandpass channel BW W Hz

Av. Transmit power constraint $\bar{P}$

$$y(t) = x(t) + w(t)$$

↓ Passband → baseband

↓ Sample

W complex samples/sec

$$y[m] = x[m] + w[m]$$

$w[m] \sim CN(0, N_0)$ i.i.d.

L31

02 April 2019 12:15

Transmit power constraint per sample $\bar{P}\left(\frac{1}{W}\right)$

2 real channels with $P = \frac{\bar{P}}{2W}$, $\sigma^2 = \frac{N_0}{2}$

$$C_{awgn}(\bar{P}, W) = 2 \times \frac{1}{2} \log_2\left(1 + \frac{\bar{P}}{2W\left(\frac{N_0}{2}\right)}\right) \times W$$

$$= \left[\log_2\left(1 + \frac{\bar{P}}{N_0 W}\right)\right] \times W \quad \text{— \# samples/sec}$$

SPECTRAL EFFICIENCY

bits/complex sample

$$= W \log_2\left(1 + \frac{\bar{P}}{N_0 W}\right) \text{ bits/sec}$$

$\div W$

bits/sec/Hz

L 31

02 April 2019 12:20

Let us study how C depends on power $\overline{P}$ and bandwidth W.

(1) $C = W \log_2\left(1 + \frac{\overline{P}}{N_0 W}\right)$ bits/s

$\log_e(1+x) \approx x$ for small x

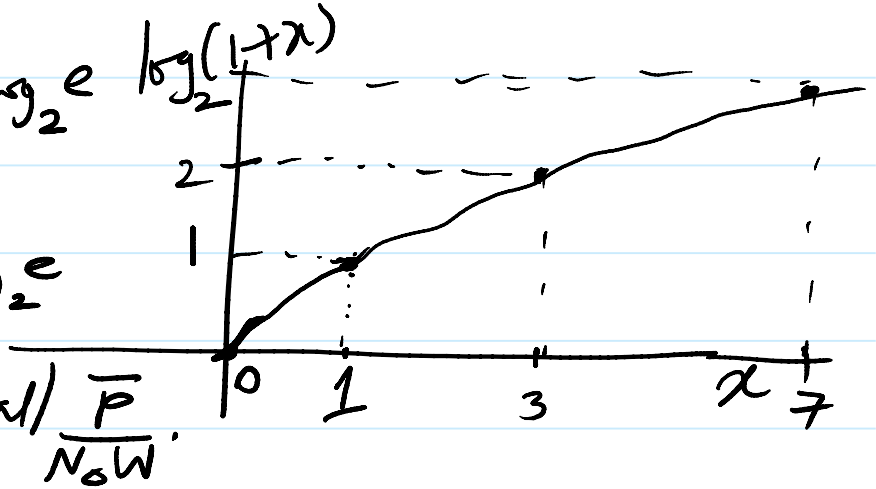
$\log(1+x) \approx \log x$

Suppose W is fixed.

As we increase $\overline{P}$, C increases.

For small $\frac{\overline{P}}{N_0 W}$, $C \approx W \cdot \frac{\overline{P}}{N_0 W} \log_2 e$

$= \frac{\overline{P}}{N_0} \log_2 e$



C increases linearly with $\overline{P}$ for small $\frac{\overline{P}}{N_0 W}$.

L31

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For large $\frac{\bar{P}}{N_0 W}$, $C \approx W \log_2 \frac{\bar{P}}{N_0 W}$

C increases logarithmically with $\bar{P}$

To increase C by 1 bits/s/Hz (or W bits/s), $\bar{P}$ should double.

$$W \log_2 \frac{2\bar{P}}{N_0 W} = W \log_2 \frac{\bar{P}}{N_0 W} + \underbrace{W \log_2 2}_{W}$$

2) Suppose $\bar{P}$ is fixed. What happens as we increase W?

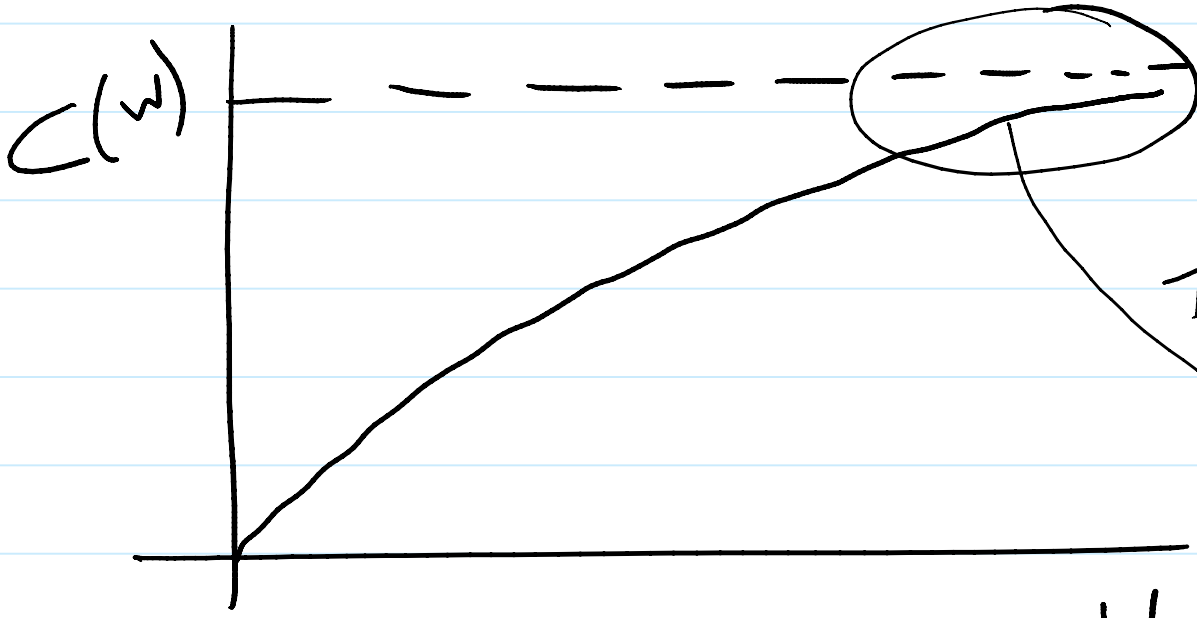
$$C = W \log_2 \left(1 + \frac{\bar{P}}{N_0 W}\right) \text{ bits/sec}$$

L31

02 April 2019 12:30

For large W $\quad W \log\left(1+\frac{\bar{P}}{N_0 W}\right) \approx W \cdot \frac{\bar{P}}{N_0 W} \cdot \log_2 e$.

$= \frac{\bar{P}}{N_0} \log_2 e$



for a fixed $\bar{P}$

Limited by power

(Power limited region)

$C(W) \longrightarrow \frac{\bar{P}}{N_0} \log_2 e$ bits/sec.

L31

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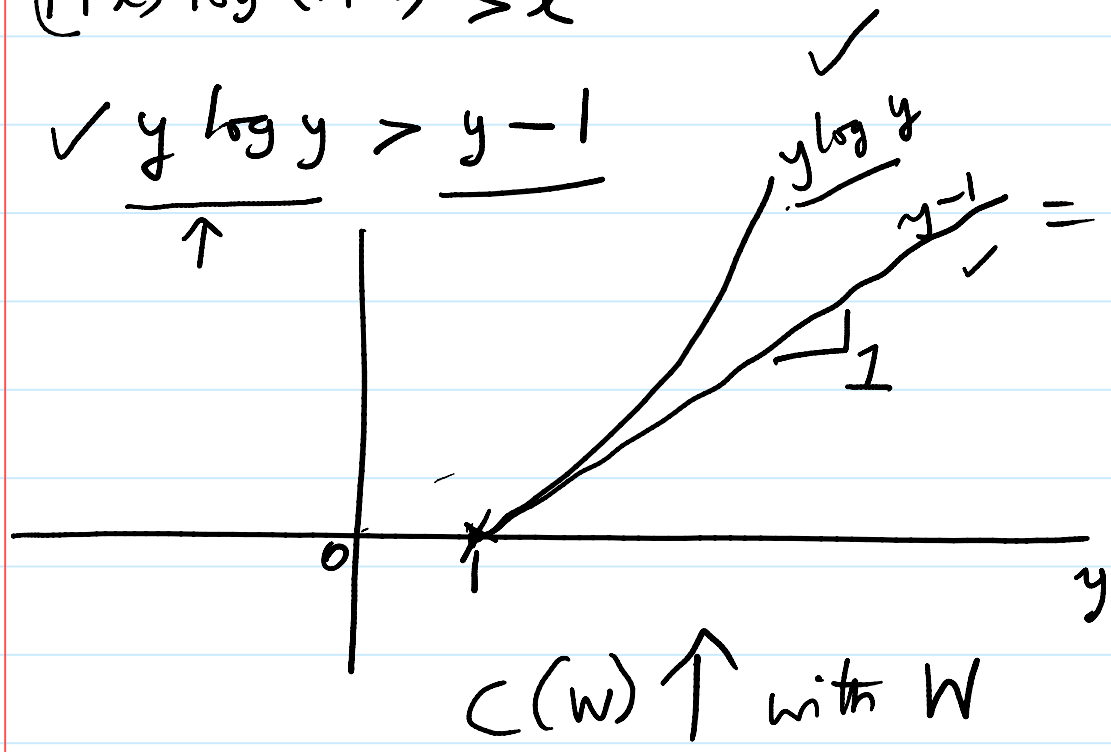
$$\log(1+x) > \frac{x}{1+x}$$

$$(1+x)\log(1+x) > x$$

$$y\log y > y-1$$



$C(W) \uparrow$ with W

$$\frac{d}{dW}\left(W\log_e\left(1+\frac{\overline{P}}{N_0W}\right)\right)$$

$$= W\cdot\frac{1}{1+\frac{\overline{P}}{N_0W}}\cdot\frac{\overline{P}}{N_0}\cdot\left(\frac{-1}{W^2}\right)+\log_e\left(1+\frac{\overline{P}}{N_0W}\right)$$

$$= \log_e\left(1+\frac{\overline{P}}{N_0W}\right)-\frac{\frac{\overline{P}}{N_0W}}{1+\frac{\overline{P}}{N_0W}}$$

$$\frac{y}{y}+\log y = 1+\log y.$$

$$> 1$$

L31

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Min Energy per bit $= \dfrac{\bar{P}}{C_{awgn}(\bar{P}, W)}$ ← energy/sec, bits/sec

(Measure of Energy efficiency)

$E_{b\,min}$

$$= \frac{\bar{P}}{W \log\left(1 + \frac{\bar{P}}{N_0 W}\right)}$$

$$\boxed{\left(\frac{E_b}{N_0}\right)_{min}} = \frac{\frac{\bar{P}}{N_0 W}}{\underbrace{\log\left(1 + \frac{\bar{P}}{N_0 W}\right)}_{\text{as } \bar{P} \to 0.}} \longrightarrow \frac{\frac{\bar{P}}{N_0 W}}{\frac{\bar{P}}{N_0 W} \log_2 e} = \frac{1}{\log_2 e}$$

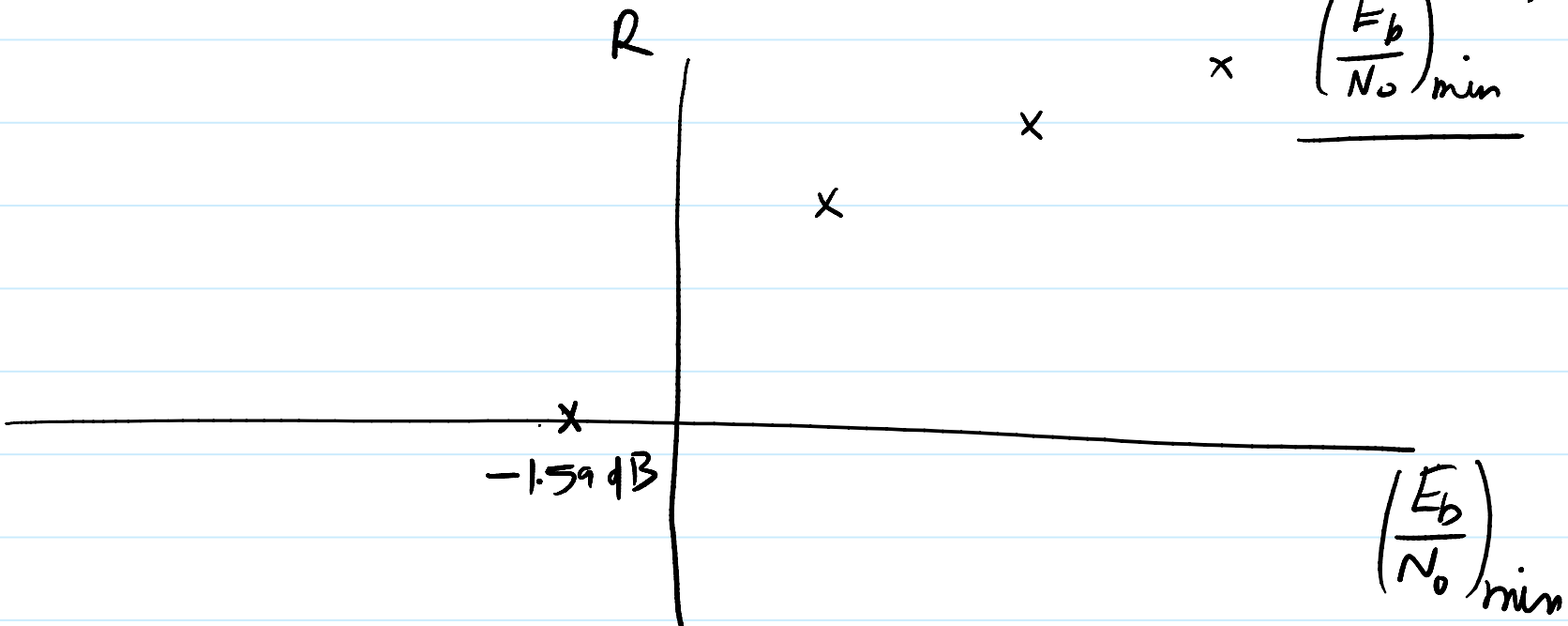
$\log_2 e \approx -1.59$ dB

Lecture 32

04 April 2019 11:03

$\left(\frac{E_b}{N_0}\right)_{min} = -1.59\,dB$ as $\bar{P} \to 0$.

$\frac{\text{Rate } R}{\left(\frac{E_b}{N_0}\right)_{min}}$.



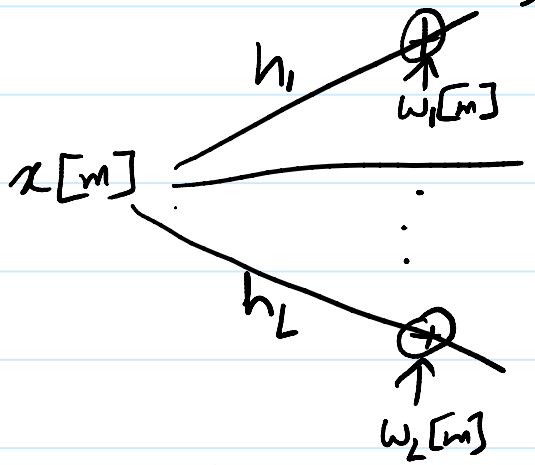
$\left(\frac{E_b}{N_0}\right)_{min}$ is higher for higher rate transmission

L 32

04 April 2019 11:14

Linear time-invariant Gaussian channels

Examples 1) SIMO channel



$$y_1[m] = h_1 x[m] + w_1[m]$$

$$y_L[m] = h_L x[m] + w_L[m]$$

$w_\ell[m] \sim \mathcal{CN}(0, N_0)$ i.i.d.

h_ℓ: fixed complex channel coefficients (known at the receiver)

$$\underline{y}[m] = \underline{h}\, x[m] + \underline{w}[m]$$

$\tilde{y}[m] = \underline{h}^H \underline{y}[m]$ is a sufficient statistic

L32

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$$\tilde{y}[m] = \underbrace{\underline{h}^H \underline{h}}\, x[m] + \underline{h}^H \underline{w}[m]$$

$$= \boxed{\|\underline{h}\|^2}\, x[m] + \tilde{w}[m]$$

Capacity $C_{SIMO} = \log\left(1 + \frac{P\|\underline{h}\|^2}{N_0}\right)$ bits/s/Hz (or) bits/channel use

Multiple receive antennas

- Power gain
- $\underline{h}^H \underline{y}$: Combining scheme called <u>receive beamforming</u>

$y_1[m] \quad \cdots \quad y_L[m]$

$\tilde{y}[m]$

$$\tilde{y}[m] = \sum_{l=1}^{L} h_l^* \underbrace{y_l[m]}_{h_l x[m] + w_l[m]}$$

L 32

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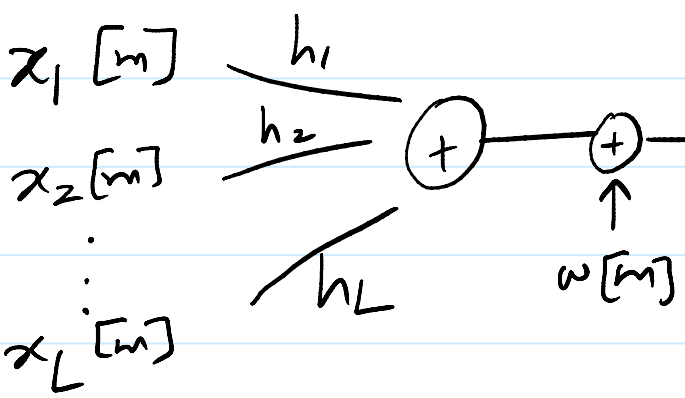
2) MISO channel



$$= \underline{h}^H \underline{x} + w[n]$$

$$\underline{h} = \begin{pmatrix} h_1^* \\ \vdots \\ h_L^* \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1[n] \\ \vdots \\ x_L[n] \end{pmatrix}$$

Assume $\underline{h}$ is known at both tx and rx.

Total power constraint P $\sum_{l=1}^{L} E[|x_l[n]|^2] \leq P$

L32

04 April 2019 11:31

If $\underline{x} \perp \underline{h}$, no signal is received (only noise)

$\underline{x}$ = component along $\underline{h}$ + component orth to $\underline{h}$

Power allocated to this component is useless

$\underline{x}$ should be along $\underline{h}$.

optimal $\underline{x}[m] = \frac{\underline{h}}{\|\underline{h}\|} \tilde{x}[m]$

Power constraint P

$$y[m] = \underline{h}^H \underline{x}[m] + w[m] = \|\underline{h}\| \tilde{x}[m] + w[m]$$

Scalar AWGN channel

L32

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Capacity $C_{MISO} = \log\left(1 + \frac{P\|\underline{h}\|^2}{N_0}\right)$ bits/channel use

C_{SIMO} (with channel known at rx) $\equiv C_{MISO}$ (with channel known at tx and rx)

3) Frequency-selective channel

$$y[m] = \sum_{l=0}^{L-1} h_l x[m-l] + w[m]$$

→ L-tap time-invariant channel

→ known at tx and rx

L 32

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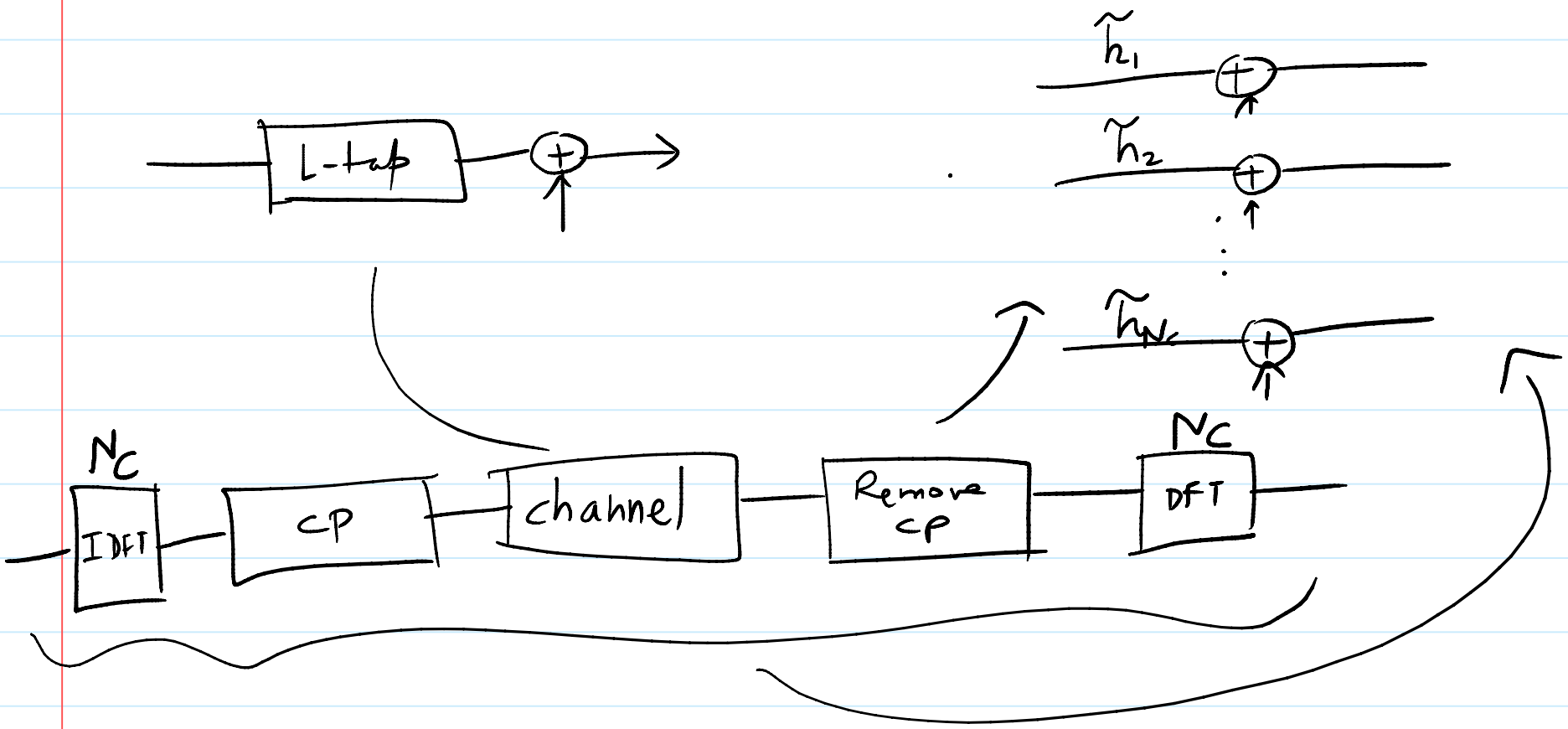

CP overhead should be at least $L-1$

$\frac{L-1}{L-1+N_c}$: Fraction of time used for CP overhead

$N_c \uparrow$ overhead $\downarrow$

L32

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Capacity of the original freq-sel channel

= capacity of the transformed N_c channels as $N_c \to \infty$

Capacity of the transformed parallel channels

N_c AWGN channels with total power constraint P

P_n : Power allocated to the nth channel

$$C = \max_{\substack{P_1, P_2, \ldots, P_{N_c} \\ \sum_{n=1}^{N_c} P_n \le P}} \sum_{n=1}^{N_c} \log\left(1 + \frac{P_n |\tilde{h}_n|^2}{N_0}\right)$$

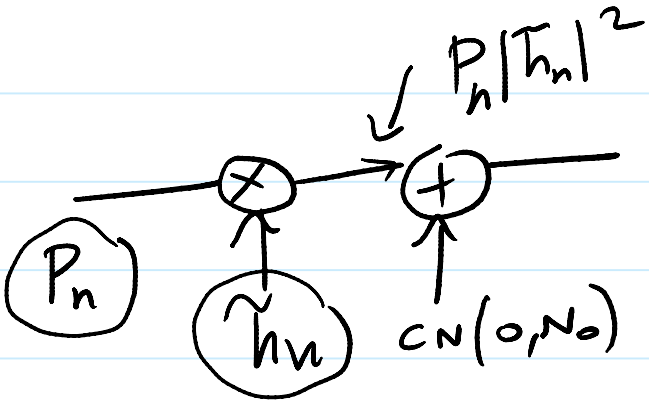
Lecture 33

05 April 2019 09:59

$$\max_{\{P_n\}} \underbrace{\sum_{n=1}^{N_c} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right)}_{g(P_1,\ldots,P_{N_c})}$$

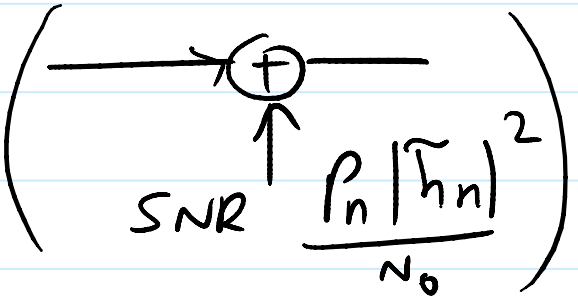
$$\sum_{n=1}^{N_c} P_n \le P$$

$$P_n \ge 0$$



Let $P_1^*, P_2^*, \ldots, P_{N_c}^*$ be the optimal soln.

observe that $\sum_{n=1}^{N_c} P_n^* = P$



(Argue by contradiction: If $\sum_{n=1}^{N_c} P_n^* < P$, we can find a better soln & ∴ we have a contradiction)



$$\frac{\partial g(P_1,\ldots,P_{N_c})}{\partial P_n} = \frac{1}{1+\frac{P_n|\tilde{h}_n|^2}{N_0}} \cdot \frac{|\tilde{h}_n|^2}{N_0} = \frac{1}{\frac{N_0}{|\tilde{h}_n|^2}+P_n}$$

L33

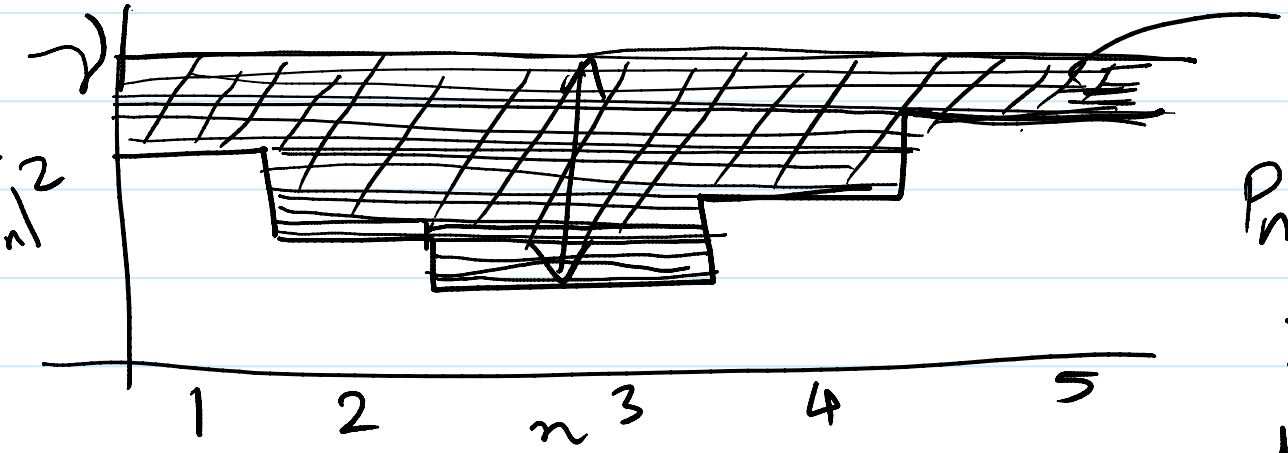
05 April 2019 10:12

$$\Delta R = \left(\frac{\partial g(\ )}{\partial P_n}\right) \Delta P$$

↳ choose n such that this is max

$$\frac{1}{\frac{N_0}{|\tilde{h}_1|^2} + P_1}, \quad \frac{1}{\frac{N_0}{|\tilde{h}_2|^2} + P_2}, \quad \ldots, \quad \frac{1}{\frac{N_0}{|\tilde{h}_{NC}|^2} + P_{NC}}$$

"water filling" solution $\frac{N_0}{|\tilde{h}_n|^2}$



$$P_n = \nu - \frac{N_0}{|\tilde{h}_n|^2}$$

if $\nu > \frac{N_0}{|\tilde{h}_n|^2}$.

else $P_n = 0$

L33

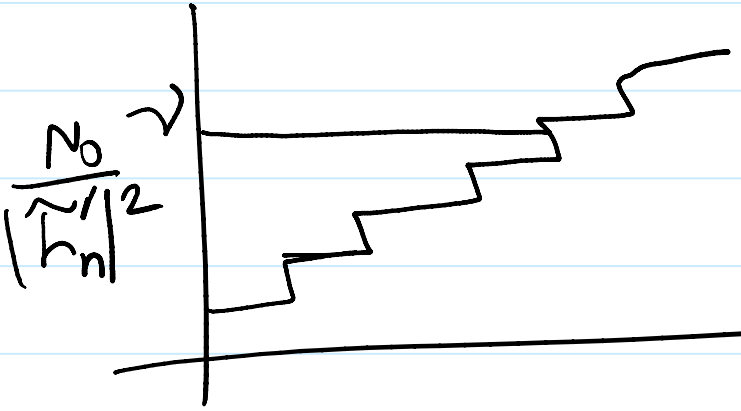
05 April 2019 10:20

$$p_n^* = \left(\nu - \frac{N_0}{|\tilde{h}_n|^2}\right)^+$$

$$\sum_{n=1}^{N_c} p_n^* = P$$

$$(x)^+ = \begin{cases} x & x > 0 \\ 0 & \text{else} \end{cases}$$

Sort channels according to $\frac{N_0}{|\tilde{h}_n|^2}$ in ascending order



$$\nu = \frac{P + (\ )}{k}$$

(after sorting) ↑

Solve $\sum_{n=1}^{k} \left(\nu - \frac{N_0}{|\tilde{h}_n'|^2}\right) = P$

check if $\nu < N_0 / |\tilde{h}_{k+1}'|^2$

L33

05 April 2019 10:33

$$\max_{\{P_n\}} \sum_{n=1}^{N_c} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right)$$

$$\sum_{n=1}^{N_c} P_n \le P, \quad P_n \ge 0$$

concave

$$\min_{\{P_n\}} \left(-\sum_{n=1}^{N_c} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right)\right)$$

$$\sum_{n=1}^{N_c} P_n \le P, \quad P_n \ge 0$$

convex

$-P_n \le 0 \Leftarrow P_n \ge 0$

Define $\mathcal{L} = -\sum_{n=1}^{N_c} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right) + \lambda\left(\sum_{n=1}^{N_c} P_n - P\right) + \sum_{n=1}^{N_c} \gamma_n(-P_n)$

λ, γ_n's Lagrange multipliers ≥ 0.

convex optimization problem

$$\min_{\underline{x}} f_0(\underline{x})$$

subject to $f_i(\underline{x}) \le 0$

$h_i(\underline{x}) = 0$

f_0, f_i, h_i convex fns

L 33

05 April 2019 10:39

Optimal $\{P_n^*\}$ satisfy the following KKT conditions

(Karush-Kuhn-Tucker)

1) $f_i(\underline{x}^*) \leq 0$
2) $h_i(\underline{x}^*) = 0$

$\Big\}$ $\underline{x}^*$ is feasible $\Big\} \rightarrow \sum_{n=1}^{N_c} P_n^* \leq P$, $P_n^* \geq 0 \; \forall n$

3) $\lambda_i \geq 0 \} \longrightarrow \lambda \geq 0$, $\gamma_n \geq 0 \; \forall n$

4) $\lambda_i f_i(\underline{x}^*) = 0 \longrightarrow \lambda \left(\sum_{n=1}^{N_c} P_n^* - P \right) = 0$, $\gamma_n P_n^* = 0 \; \forall n$

5) $\nabla \mathcal{L}$ at $\underline{x}^* = 0 \longrightarrow \dfrac{\partial \mathcal{L}}{\partial P_n} = 0$ at $P_n = P_n^* \; \forall n$

Solve the above eqns to find the soln.

L33

05 April 2019 10:46

$$p_n^* \geq 0, \quad \sum_n p_n^* \leq P, \quad \lambda \geq 0, \gamma_n \geq 0 \ \forall n$$

$$\left.\frac{\partial \mathcal{L}}{\partial p_n}\right|_{p_n^*} = -\frac{1}{\frac{N_0}{|\tilde{h}_n|^2} + p_n^*} + \lambda - \gamma_n = 0 \quad \forall n$$

$$\lambda\left(\sum_{n=1}^{N_c} p_n^* - P\right) = 0$$

$$\gamma_n p_n^* = 0 \ \forall n.$$

$$\gamma_n = \lambda - \frac{1}{\frac{N_0}{|\tilde{h}_n|^2} + p_n^*} \Rightarrow$$

either $p_n^* = 0$

or

$$\lambda = \frac{1}{\frac{N_0}{|\tilde{h}_n|^2} + p_n^*}$$

$$p_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+$$

$$p_n^* = \frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}$$

$$\lambda \neq 0$$

$$\sum_{n=1}^{N_c} p_n^* = P$$

Lecture 34

08 April 2019 07:55

$$\tilde{h}_n = \frac{1}{\sqrt{N_c}} \sum_{l=0}^{L-1} h_l \, e^{-\frac{j2\pi ln}{N_c}}$$

As $N_c \to \infty$

$$C = \int_0^W \log\left(1 + \frac{P^*(f)\,|H(f)|^2}{N_0}\right) df$$

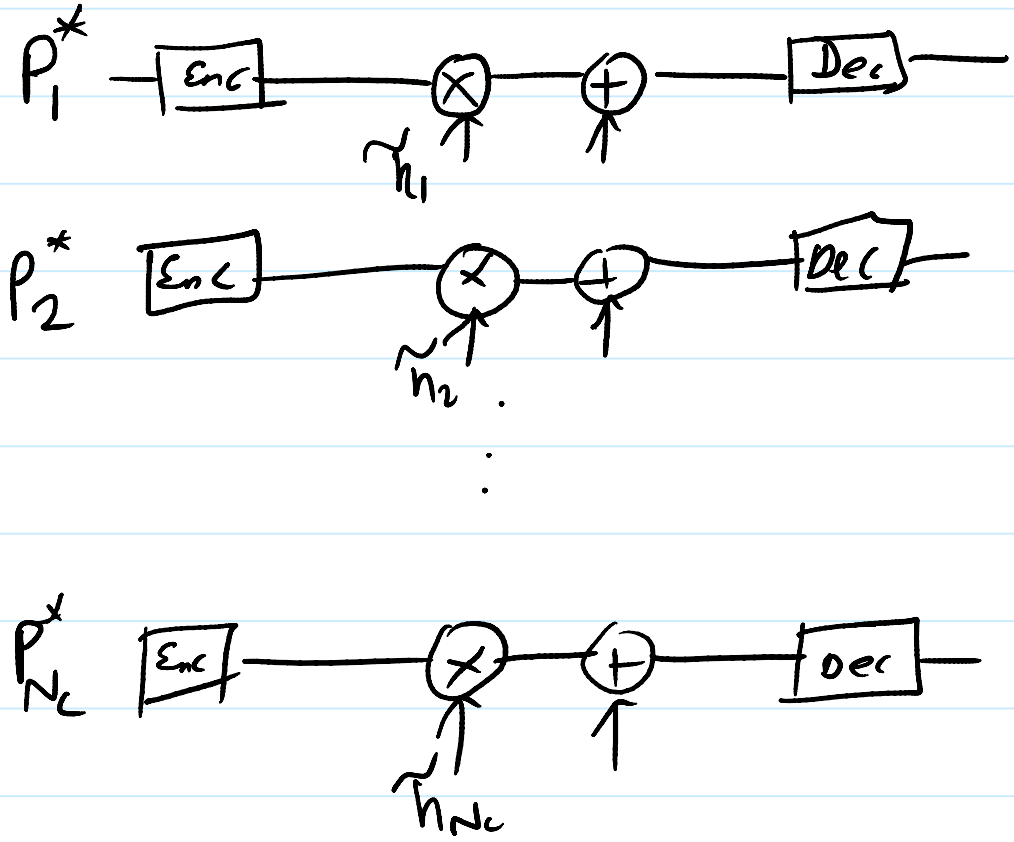
where $P^*(f) = \left(\nu - \frac{N_0}{|H(f)|^2}\right)^+$ (Waterfilling power allocation)

and $\int_0^W P^*(f)\, df = P$ (Power constraint)

L34

08 April 2019 08:06

Parallel AWGN channels $\equiv$ Waterfilling power allocation + Scalar AWGN channel coding for each channel.

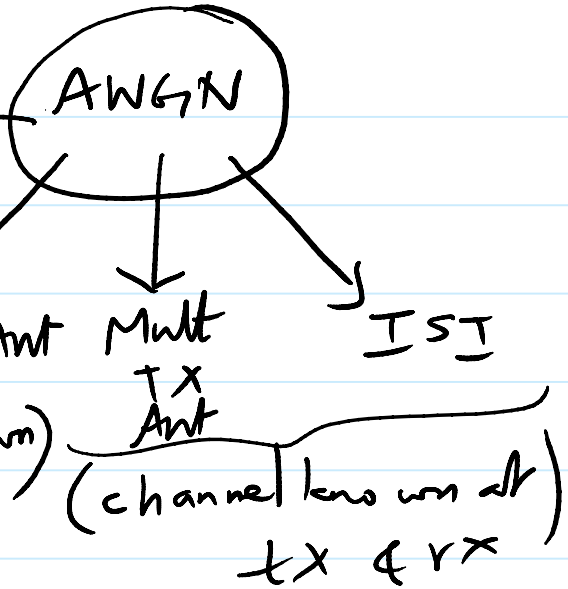


- Coding across subchannels is not required to achieve capacity (Block length $\to \infty$)
- For finite blocklength, performance will improve by coding across subchannels.

L 34

08 April 2019 08:13

Fading channel capacity:



Flat fading

$$y[m] = h[m]\,x[m] + w[m]$$

$w[m] \sim CN(0, N_0)$ i.i.d.

$h[m]$: Fading process ($h[m]$ known at the receiver)

$E[|x[m]|^2] \le P$

Assume tx knows only the statistical model for $\{h[m]\}$.

L34

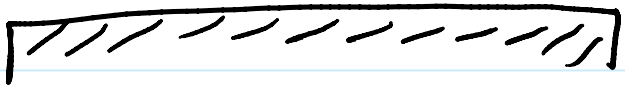
08 April 2019 08:19

2 cases

Slow fading

Fast fading





coding blocklength $\ll$ coherence time

coding blocklength $\gg$ coh. time

Slow fading channel: Rayleigh model

$$y[m] = h\,x[m] + w[m]$$

$h \sim CN(0,1)$

$w[m] \sim CN(0, N_0)$

$$SNR \triangleq \frac{P}{N_0}$$

L34

08 April 2019 08:27

For a given realization h, rates above

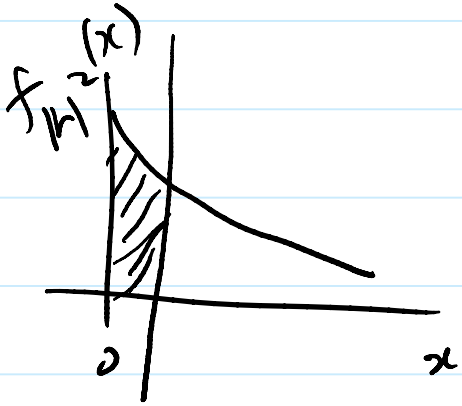
$$\log(1+|h|^2 SNR)$$

are not achievable.

For any rate R, $P\left(\underbrace{\log(1+|h|^2 SNR) < R}_{R \text{ not achievable}}\right)$

$$= P\left(|h|^2 < \frac{2^R - 1}{SNR}\right) > 0 \text{ for any } R$$

Rayleigh



Can be evaluated based on the fading model

$\Rightarrow$ Decoding error probability cannot be made arbitrarily small for any $R > 0$.

L34

08 April 2019 08:32

Capacity according to the traditional defn = 0.

Define "outage event" = " $\log(1+|h|^2 SNR) < R$ "

$P_{out}(R)$: Probability of outage for a rate R

Block Error prob for any scheme $\geq P_{out}(R)$

ϵ-outage capacity (C_ϵ) : The maximum rate of transmission R such that $P_{out}(R) \leq \epsilon$.

As $R \uparrow$, $P_{out}(R) \uparrow$ (non decreasing) (make $P_{out}(C_\epsilon) = \epsilon$)

L 34

08 April 2019 08:40

$$P_{out}(R) = P\left[\log(1+|h|^2 SNR) < R\right]$$

$$= P\left[|h|^2 < \frac{2^R - 1}{SNR}\right]$$

$$= 1 - F_{|h|^2}\left(\frac{2^R - 1}{SNR}\right)$$

$$F(x) \triangleq P(|h|^2 > x)$$

$$P_{out}(C_\epsilon) = \epsilon = 1 - F_{|h|^2}\left(\frac{2^{C_\epsilon} - 1}{SNR}\right)$$

$$1 - \epsilon = F\left(\frac{2^{C_\epsilon} - 1}{SNR}\right)$$

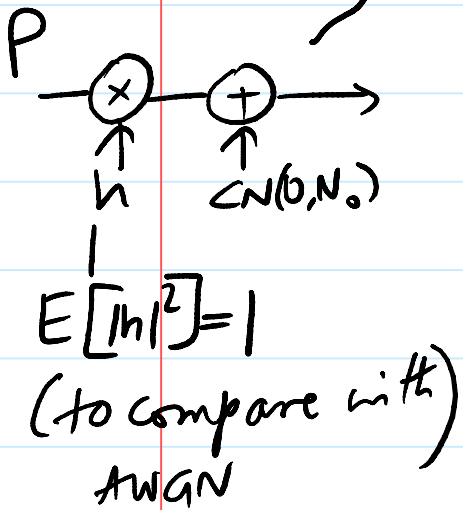
$$F^{-1}(1-\epsilon) = \frac{2^{C_\epsilon} - 1}{SNR}$$

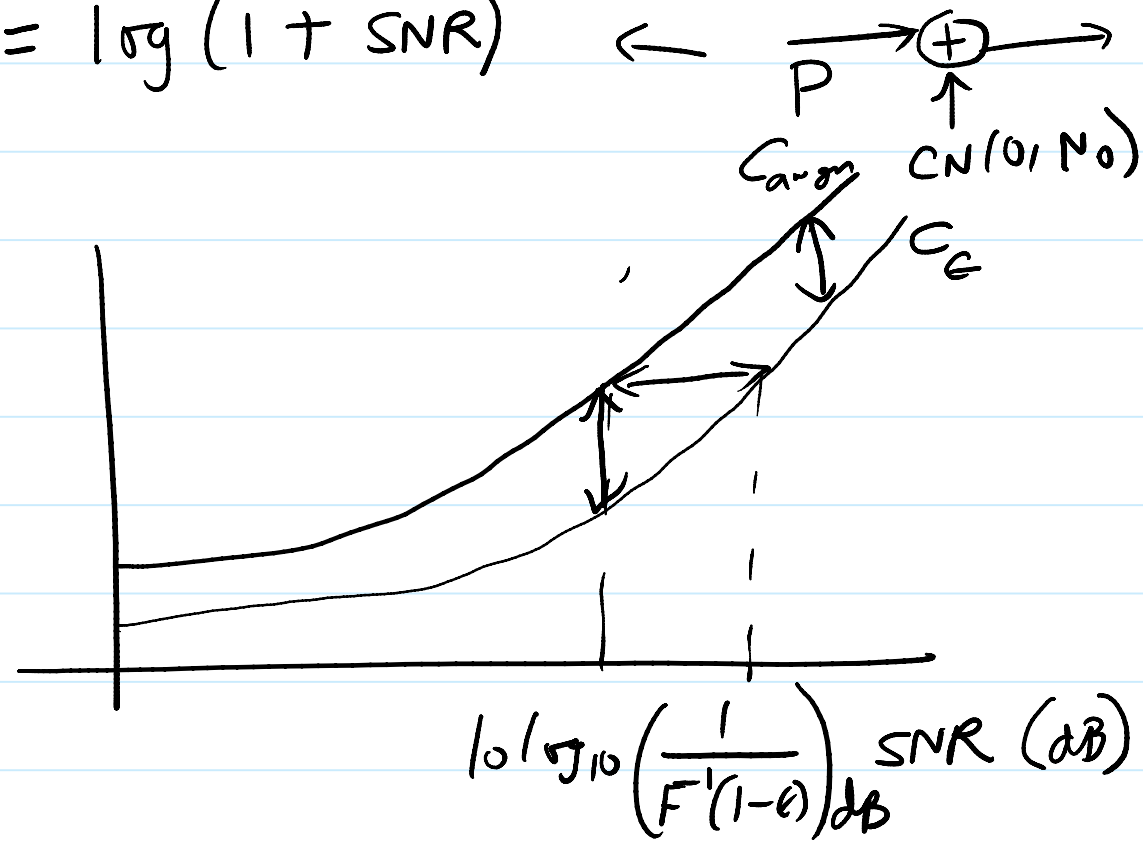
L 34

08 April 2019 08:43

$$C_\epsilon = \log\left(1 + F^{-1}(1-\epsilon)\,SNR\right) \quad \text{bits/transmission}$$

$$C_{awgn} = \log(1 + SNR)$$





Lecture 35

09 April 2019 12:05

* For the same rate ($C_\epsilon = C_{awgn}$), $10 \log_{10}\left(\frac{1}{F^{-1}(1-\epsilon)}\right)$ dB more power is needed in the slow fading channel called "Fade margin".

* For the same SNR, $C_\epsilon < C_{awgn}$

$$C_{awgn} - C_\epsilon = \log(1+SNR) - \log\left(1+F^{-1}(1-\epsilon)SNR\right)$$

$$\left(\text{at high SNR}\right) \approx \log\left(\frac{1}{F^{-1}(1-\epsilon)}\right) = \text{constant}$$

$$\left(\text{at low SNR}\right) \approx SNR\left(1 - F^{-1}(1-\epsilon)\right)$$

L 35

09 April 2019 12:11

Rayleigh fading : $F(x) = P(|h|^2 > x)$

$$= \int_x^\infty e^{-y}\, dy$$

$$= \frac{e^{-y}}{-1}\Big|_x^\infty = e^{-x}$$

$e^{-x} = 1-\epsilon$

$$\boxed{F^{-1}(1-\epsilon) = \log \frac{1}{1-\epsilon}}$$

$$P_{out}(R) = P\left(|h|^2 < \frac{2^R - 1}{SNR}\right) = 1 - e^{-\frac{2^R-1}{SNR}}$$

$$\approx \frac{2^R - 1}{SNR} \text{ for high } SNR$$

$$\boxed{P_{out}(R) \propto \frac{1}{SNR}}$$

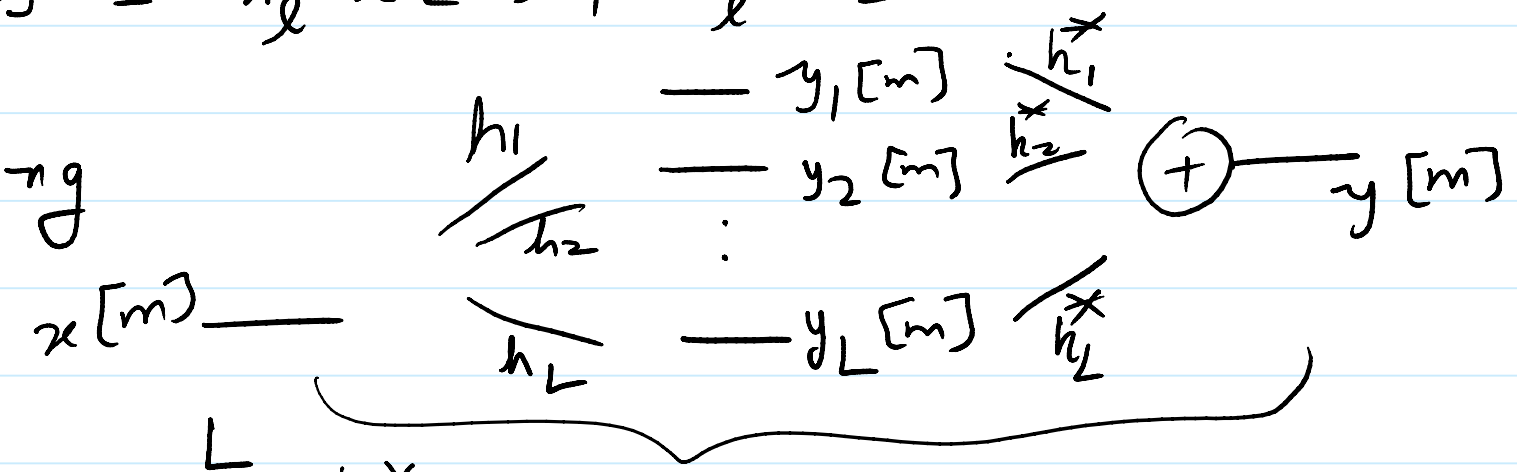
L 35

09 April 2019 12:19

What if we have multiple receive antennas?

$$y_\ell[m] = h_\ell x[m] + w_\ell[m] \to CN(0, N_0)$$

'slow' fading



$$y[m] = \sum_{\ell=1}^{L} h_\ell^* y_\ell[m]$$

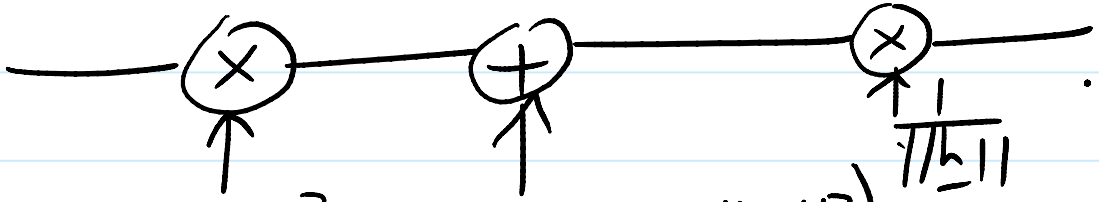
$$= \|\underline{h}\|^2 x[m] + \underline{h}^H \underline{w}$$

$\underline{w} \sim CN(\underline{0}, N_0 \underline{I})$

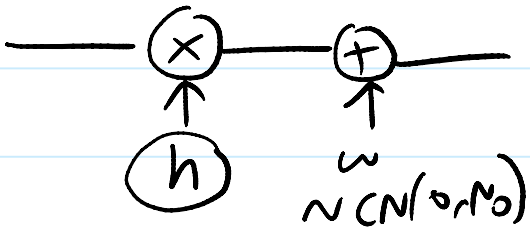
$\underline{h}^H \underline{w} \sim CN(0, N_0 \|\underline{h}\|^2)$

L35

09 April 2019 12:24



$\|\underline{h}\|$ $\quad CN(0, N_0)$



$$P_{out}(R) = P\left[\log\left(1+\|\underline{h}\|^2 SNR\right) < R\right]$$

$$= P\left[\|\underline{h}\|^2 < \frac{2^R-1}{SNR}\right] = P\left[\sum_{l=1}^{L} |h_l|^2 < \frac{2^R-1}{SNR}\right]$$

(Rayleigh model for each h_l) $\approx \frac{(2^R-1)^L}{L!\, SNR^L} \propto \frac{1}{SNR^L}$ at high SNR

L 35

09 April 2019 12:29

$$C_\epsilon = \log\left(1 + F^{-1}(1-\epsilon)\,SNR\right)$$

$$\simeq F^{-1}(1-\epsilon)\,SNR \quad (\text{at low } SNR)$$

$C_{awgn} \simeq SNR$ (at low SNR)

$L = 1, \epsilon = 0.01$

$C_\epsilon \approx 1\%$ of C_{awgn}

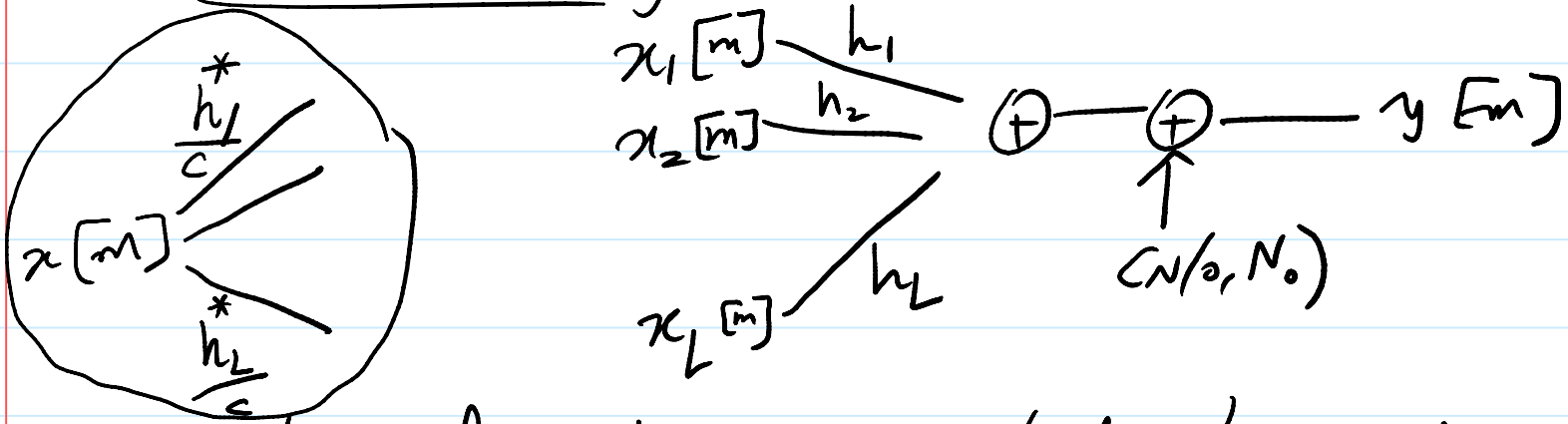
$L = 2, \epsilon = 0.01$

$C_\epsilon \approx 14\%$ of C_{awgn}

L 35

09 April 2019 12:33

Transmit diversity :



If channel is known at both tx and rx

$$\equiv \quad y[n] = \left(\frac{\sum_{\ell=1}^{L} |h_\ell|^2}{c}\right) x[n] + w[n]$$

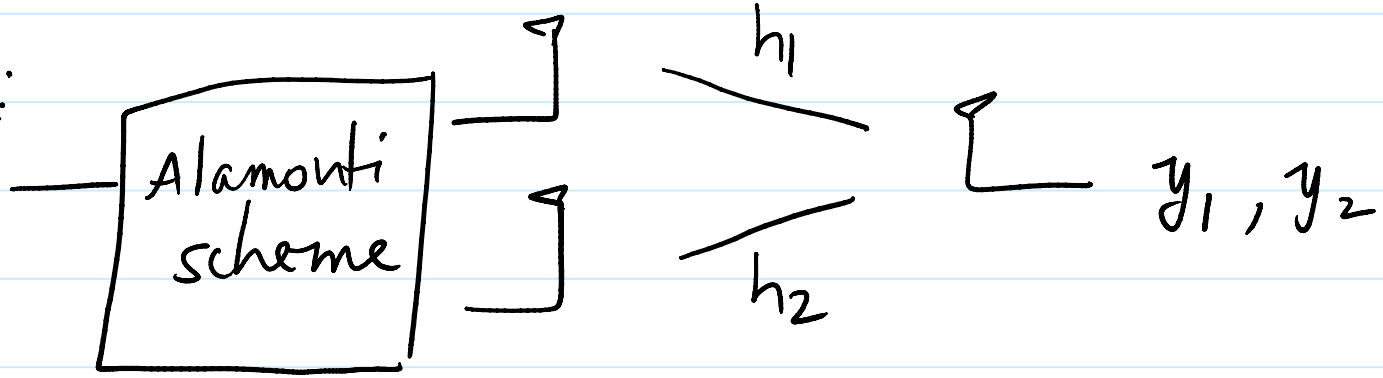
Same results as for receive diversity with channel knowledge at rx

L 35

09 April 2019 12:38

Transmit diversity, channel not known at tx, channel known at rx

Eg:



code over 2 symbol times, 2 antennas

$$\begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$ 2×2

L 35

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$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2^* \\ h_2 & -h_1^* \end{bmatrix}}_{H} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$

$$\begin{bmatrix} h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix} = 0$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \triangleq H^H \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = (|h_1|^2 + |h_2|^2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + H^H \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$

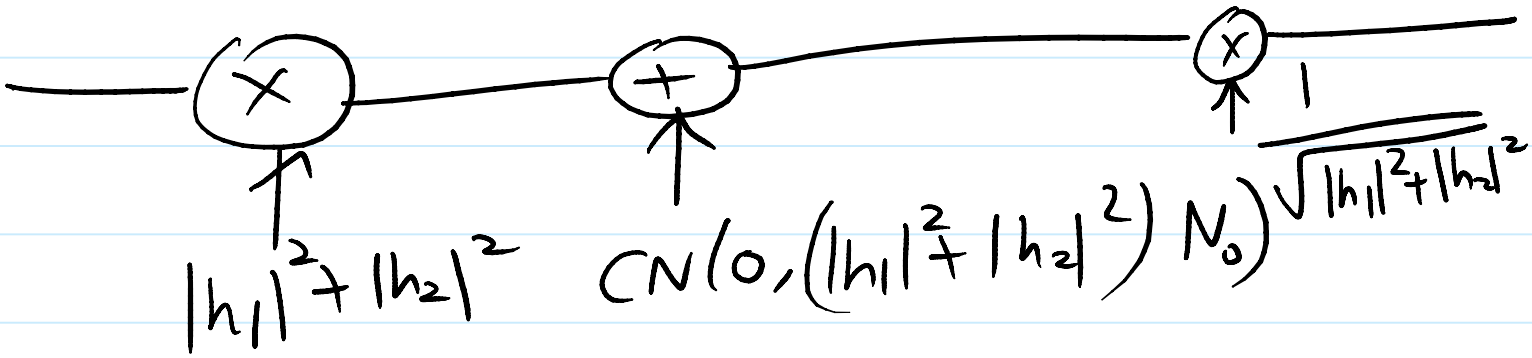
$$r_1 = (|h_1|^2 + |h_2|^2) x_1 + n_1 \quad \leftarrow \frac{1}{2} \text{ power}$$

$$r_2 = (|h_1|^2 + |h_2|^2) x_2 + n_2 \quad \uparrow \frac{1}{2} \text{ power}$$

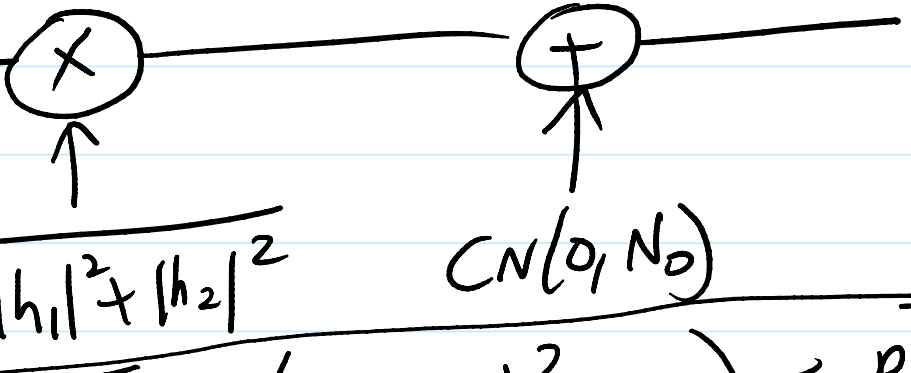
$$\sim CN(0, \underbrace{H^H N_0 H}_{(|h_1|^2 + |h_2|^2) N_0 I})$$

L 35

09 April 2019 12:46



$(=)$ $\frac{P}{2}$



$SNR \triangleq \frac{P}{N_0}$

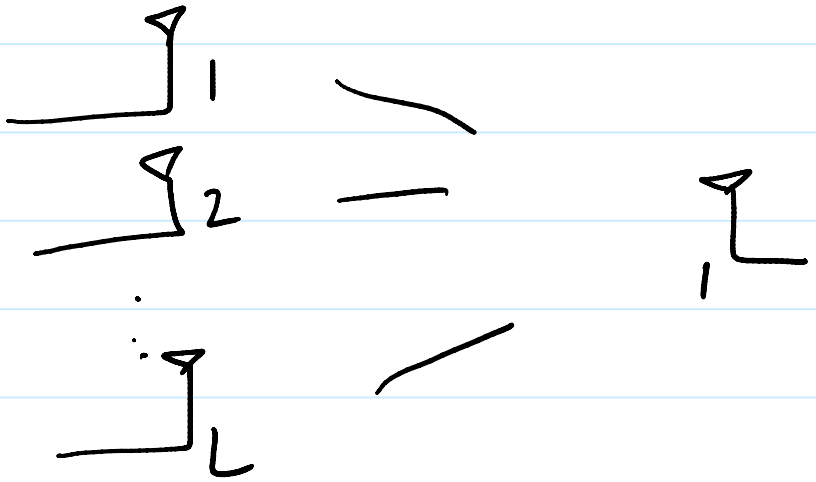
$$P_{out}(R) = P\left[\log\left(1+\frac{\|h\|^2}{2}SNR\right) < R\right]$$

Lecture 36

11 April 2019 10:57

Eg: L transmit antennas, channel not known at the tx channel known at the rx.

Repetition code:

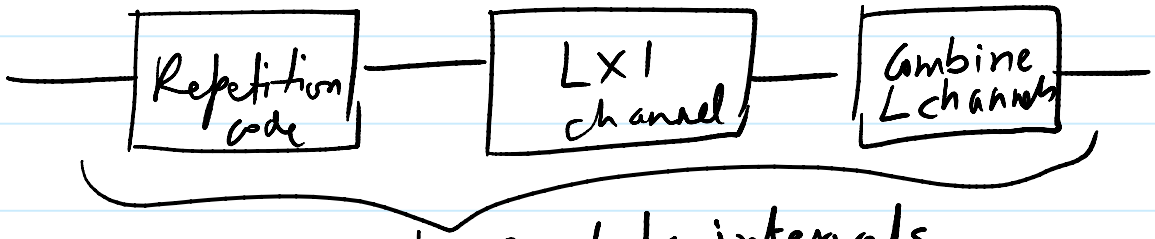


$$\begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix}_{L \times L} \quad x$$

$P_{out}(R) = ?$

L36

11 April 2019 11:07



over L symbols intervals we have the equivalent channel

$$SNR = \frac{P}{N_0}.$$

$$\frac{(\|\underline{h}\|^2)^2 P}{\|\underline{h}\|^2 N_0}$$

$$y = \|\underline{h}\|^2 x + \underbrace{\underline{h}^H \underline{w}}_{\sim \mathcal{CN}(0,\ \|\underline{h}\|^2 N_0)}$$

$$Pr\left(\frac{1}{L}\log\left(1+\|\underline{h}\|^2 SNR\right) < R\right) = P_{out}(R)$$

rate achievable per symbol interval for a given $\underline{h}$

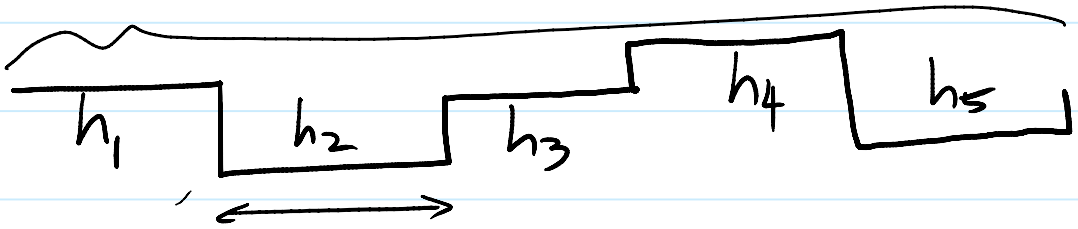
$$Pr\left(\|\underline{h}\|^2 < \frac{2^{LR}-1}{SNR}\right) \propto \frac{1}{SNR^L}$$

L 36

11 April 2019 11:13

Block fading channels

slow Block fast



coding over several coherence intervals (say L)

(

- channel constant over one block (coherence interval)
- i.i.d. from block to block

Power constraint P for each block

Suppose channel is known at tx and we use power P for each block

Rate of $\sum_{\ell=1}^{L} \log(1+|h_\ell|^2 SNR)$ can be achieved over L blocks.

L36

11 April 2019 11:19

✓ Average rate of transmission = $\frac{1}{L}\sum_{l=1}^{L}\log\left(1+|h_l|^2 SNR\right)$

(Here, coding is not across blocks)

* It turns out that this average rate can also be achieved using a single code over L blocks without channel knowledge at the transmitter.

$$P_{out}(R) = Pr\left(\frac{1}{L}\sum_{l=1}^{L}\log\left(1+|h_l|^2 SNR\right) < R\right)$$

* Can also be thought of as L parallel channels. The L parallel channels can be from time diversity / freq diversity.
* Coding across parallel channels is necessary if channel is not known at the tx

Lecture 34

21 April 2016 09:30

Oxford Cup

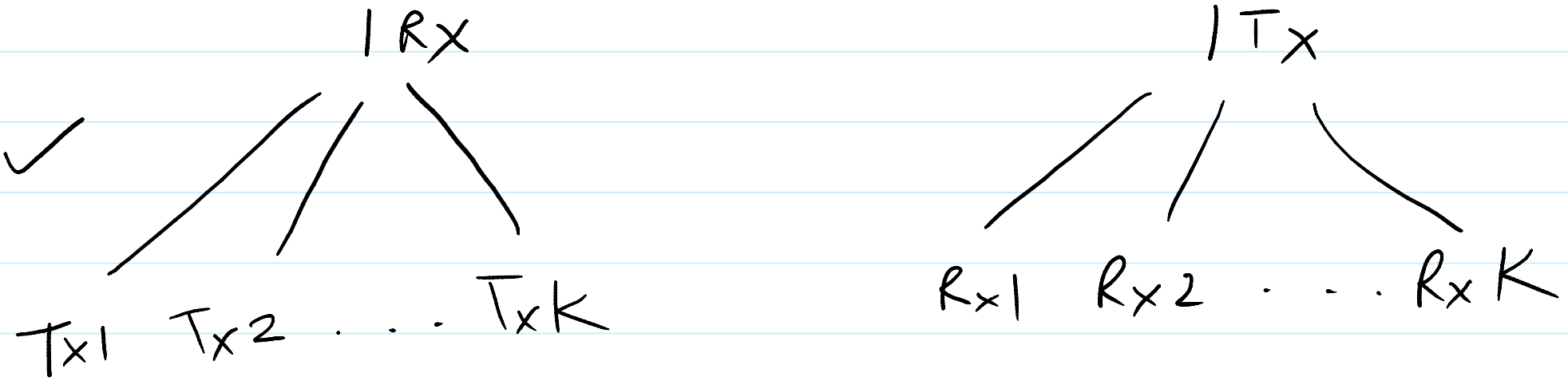
Some examples of the form

L37

15 April 2019 08:09

Multiuser channels : Capacity

Eg: Multiple access channel (MAC) Broadcast channel (BC)



Important in cellular systems

L37

15 April 2019 08:12

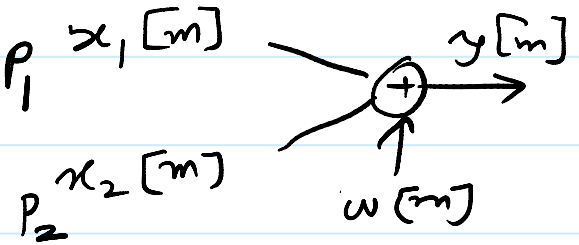
1) Gaussian multiple access channel
   - 2-user case
   - K-user case
2) Slow and fast fading MAC (briefly)

Two-user Gaussian MAC:

$$y[m] = x_1[m] + x_2[m] + w[m]$$

$x_1[m]$: Power constraint P_1

$x_2[m]$: Power constraint P_2



Let R_1 is the rate for user 1 (i.e. from Tx 1 to Rx)
R_2 ——"—— user 2 (i.e from Tx 2 to Rx)

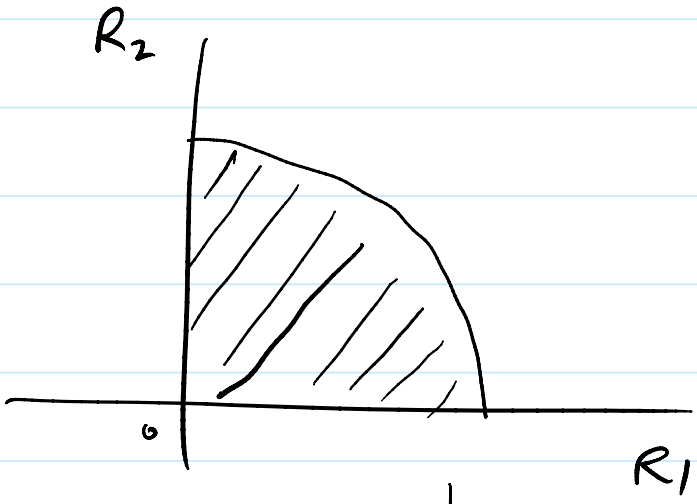
L 37

15 April 2019 08:18

Set of all (R_1, R_2) such that users 1 and 2 can simultaneously communicate <u>reliably</u> at rates R_1 and R_2 respectively

↓

CAPACITY REGION

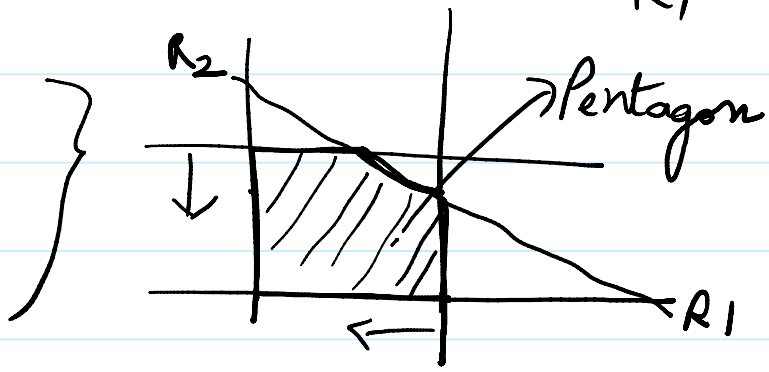


<u>SOLUTION</u>

$$C = \{(R_1, R_2) : R_1 < \log\left(1 + \frac{P_1}{N_0}\right)$$

$$R_2 < \log\left(1 + \frac{P_2}{N_0}\right)$$

$$R_1 + R_2 < \log\left(1 + \frac{P_1 + P_2}{N_0}\right) \}$$



L 37

15 April 2019 08:24

Remarks

1) The first 2 conditions say that the rate of each user cannot exceed the capacity of the point-to-point link with the other user being absent from the system. (single-user bounds)

2) The third condition says that the sum rate of both users together cannot exceed the capacity of a point-to-point AWGN channel with power constraint $P_1 + P_2$.

(If 2 users send independent signals, the aggregate received power will be the sum of the individual powers)

L 37

15 April 2019 08:29

$$R_1 + R_2 < \log\left(1 + \frac{P_1 + P_2}{N_0}\right)$$

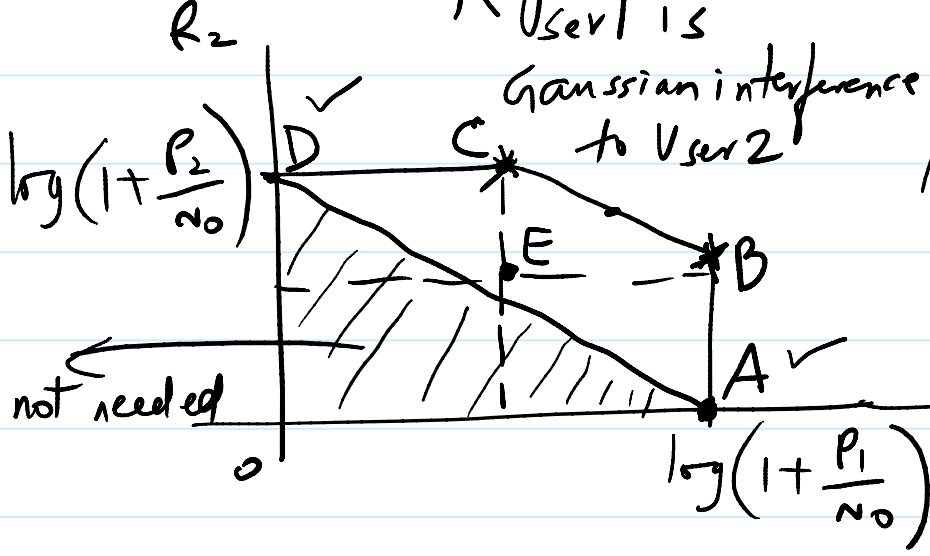
User 2 is Gaussian interference to user 1.

$$\log\left(1 + \frac{P_1}{N_0}\right) + \log\left(1 + \frac{P_2}{N_0 + P_1}\right)$$

pt (B)

$$\log\left(1 + \frac{P_2}{N_0}\right) + \log\left(1 + \frac{P_1}{N_0 + P_2}\right)$$

pt (C)



↖ User 1 is Gaussian interference to User 2

Simultaneous transmission not needed

$$\log\left(\frac{N_0 + P_1 + P_2}{N_0} \cdot \frac{N_0}{N_0 + P_1}\right) = \log\left(1 + \frac{P_2}{N_0 + P_1}\right)$$

L37

15 April 2019 08:36

Pt A: User 2 does not transmit

User 1 transmits at rate $\log\left(1+\frac{P_1}{N_0}\right)$

Pt D: User 1 does not transmit

User 2 transmits at rate $\log\left(1+\frac{P_2}{N_0}\right)$

Pt B: User 2 transmits at rate $\log\left(1+\frac{P_2}{N_0+P_1}\right)$

Successive interference cancellation (SIC) decoder ←

- Receiver will first decode user 2 treating user 1 as noise
- Cancel user 2's signal
- Decode user 1

User 1 transmits at rate $\log\left(1+\frac{P_1}{N_0}\right)$

Pt C: Similar to Pt B (User 1 ⟺ User 2)

L 37

15 April 2019 08:40

Pts. on the line joining A and D : Time-sharing between strategies for pts A & D.

Pts —"— B and C : —"— pts B & C

Pt E : $\left(\log\left(1+\frac{P_1}{N_0+P_2}\right), \log\left(1+\frac{P_2}{N_0+P_1}\right)\right)$

Treat interference as noise for both users.

3) Pts. in Δ^{le} OAD are achievable without simultaneous transmission.

User 1 transmits for α fraction of time (at power P_1')

2 $1-\alpha$ time (at Power P_2')

L37

15 April 2019 08:46

$$\underbrace{\alpha \log\left(1+\frac{P_1'}{N_0}\right)}_{R_1} \qquad \underbrace{(1-\alpha)\log\left(1+\frac{P_2'}{N_0}\right)}_{R_2}$$

User 1 avg power: $\alpha P_1'$ User 2 avg power: $(1-\alpha)P_2'$

If user 1 can transmit such that avg tx power is P_1, then

$$R_1 = \alpha \log\left(1+\frac{P_1}{\alpha N_0}\right) \qquad \left(P_1' = \frac{P_1}{\alpha}\right)$$

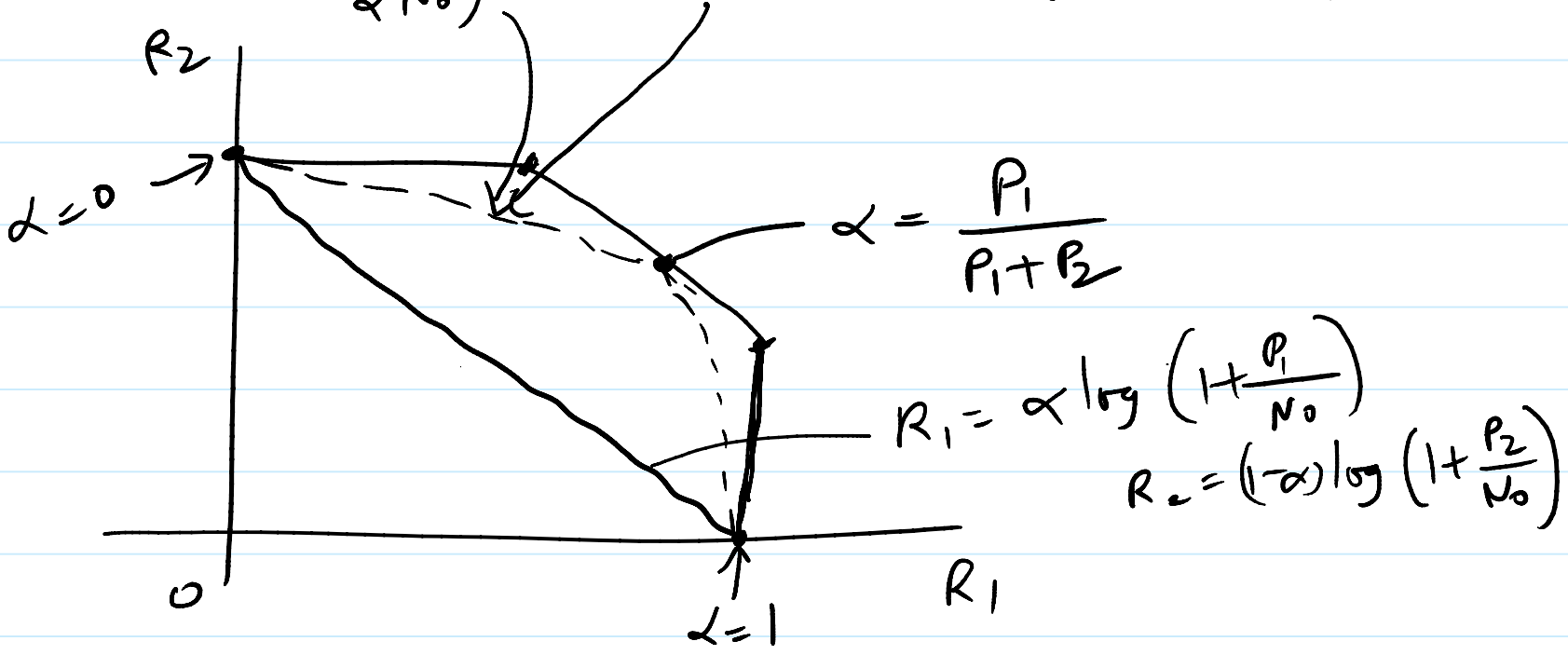
$$R_2 = (1-\alpha)\log\left(1+\frac{P_2}{(1-\alpha)N_0}\right) \qquad \left(P_2' = \frac{P_2}{1-\alpha}\right)$$

Lecture 38

16 April 2019 12:01

Time sharing with power control

$$R_1 = \alpha \log\left(1 + \frac{P_1}{\alpha N_0}\right) \qquad R_2 = (1-\alpha)\log\left(1 + \frac{P_2}{(1-\alpha)N_0}\right)$$



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$$R_1+R_2 = \alpha \log\left(1+\frac{P_1}{\alpha N_0}\right) + (1-\alpha)\log\left(1+\frac{P_2}{(1-\alpha)N_0}\right)$$

$$\frac{d(R_1+R_2)}{d\alpha} = \log\left(1+\frac{P_1}{\alpha N_0}\right) + \frac{\alpha}{1+\frac{P_1}{\alpha N_0}}\cdot\left(\frac{-P_1}{\alpha^2 N_0}\right)$$

✓

$$-\log\left(1+\frac{P_2}{(1-\alpha)N_0}\right) + \frac{(1-\alpha)}{1+\frac{P_2}{(1-\alpha)N_0}}\left(\frac{+P_2}{(1-\alpha)^2 N_0}\right) = 0$$

✓

$$\boxed{\alpha = \frac{P_1}{P_1+P_2}}$$

check $\frac{d^2(R_1+R_2)}{d\alpha^2} < 0$

$$\left(\frac{P_1}{\alpha N_0} = \frac{P_1+P_2}{N_0} \qquad \frac{P_2}{(1-\alpha)N_0} = \frac{P_1+P_2}{N_0}\right)$$

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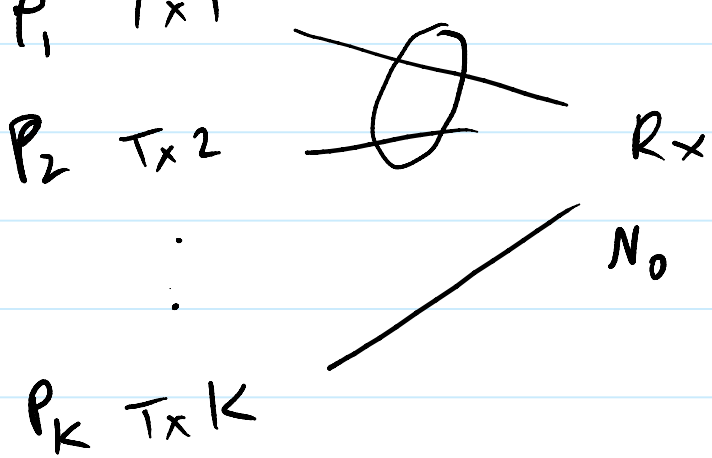
K-user MAC

Capacity region $\left\{(R_1, R_2, \cdots, R_K) : \right.$

$$\sum_{k \in S} R_k \le \log\left(1 + \frac{\sum_{k \in S} P_k}{N_0}\right) \Bigg\}$$

$2^K - 1$ inequalities

for all $S \subseteq \{1, 2, \cdots K\}$



S is any subset of $\{1, 2, \cdots, K\}$

$$\text{Sum capacity} = \max_{\{R_1, \cdots, R_K\} \in \text{Cap region}} R_1 + R_2 + \cdots + R_K$$

$$= \log\left(1 + \frac{\sum_{k=1}^{K} P_k}{N_0}\right)$$

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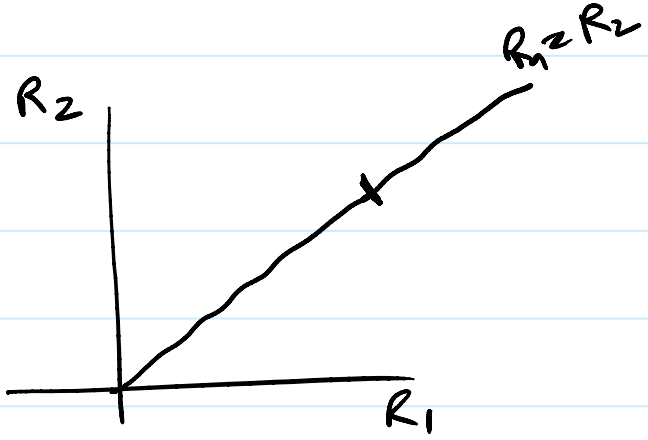
Fading MAC

- Slow (outage capacity region)
- Fast (Ergodic capacity region)



special case

Symmetric MAC + symmetric rates $\Rightarrow R_1 = R_2 = \cdots = R_K = R$

$$y[m] = \sum_{k=1}^{K} h_k[m]\, x_k[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0, N_0)$$

$h_k[m]$: fading

- $h_k[m]$ i.i.d. $E[|h_k[m]|^2] = 1$
- Power constraint P for each user

slow: $h_k[m] = h_k$

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For a given channel realization, the capacity region is

$$\left\{(R_1,\cdots,R_K) : \sum_{k\in S} R_k < \log\left(1+\frac{\sum_{k\in S} P|h_k|^2}{N_0}\right) \ \forall\, S\subseteq\{1,\cdots,K\}\right\}$$

Symmetric rate $(R,\cdots,R)$:

$$\boxed{R|S| < \log\left(1+\frac{P\sum_{k\in S}|h_k|^2}{N_0}\right) \ \forall\, S\subseteq\{1,\cdots,K\}}$$

$|S|$: # elements in S

$$P_{out}(R) = Pr\left[\log\left(1+\frac{P\sum_{k\in S}|h_k|^2}{N_0}\right) < R|S| \text{ for some } S\right] \qquad SNR \triangleq \frac{P}{N_0}.$$

$$= Pr\left[\sum_{k\in S}|h_k|^2 < \frac{2^{R|S|}-1}{SNR} \text{ for some } S\right]$$

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C_ϵ^{sym} : Largest symmetric rate R such that $Pr(\text{outage}) \leq \epsilon$.

Asymmetric case : outage capacity region

↳ $\{(R_1, \cdots, R_K) : P_{out} \leq \epsilon\}$

Fast fading

symmetric case $C_{sim}^{sym} = E\left[\log\left(1 + \frac{\sum_{k=1}^{K} |h_k|^2 P}{N_0}\right)\right]$
