Lecture 1: Course Overview

11 July 2017 11:10

EE3005 Communication Systems — Srikrishna

http://www.ee.iitm.ac.in/~skrishna/ee3005/

"Introduction to Communication Systems"

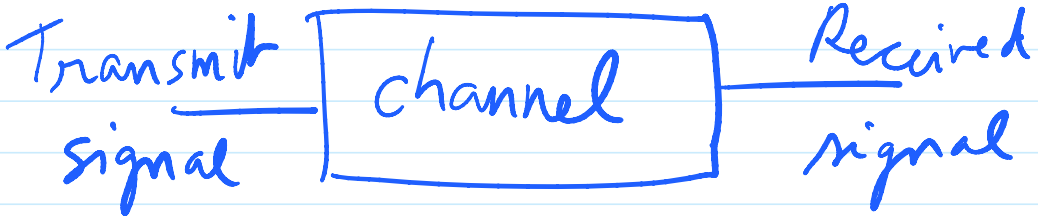
U. Madhow, Cambridge Univ. Press, 2014.

Communication: The process of information transfer across space or time.

Eg across space — Radio (AM, FM)
Television

Cell phones
Phones
⋮
across time — Storage systems
CDs DVDs Hard drives

Key: Channel (over which we communicate)



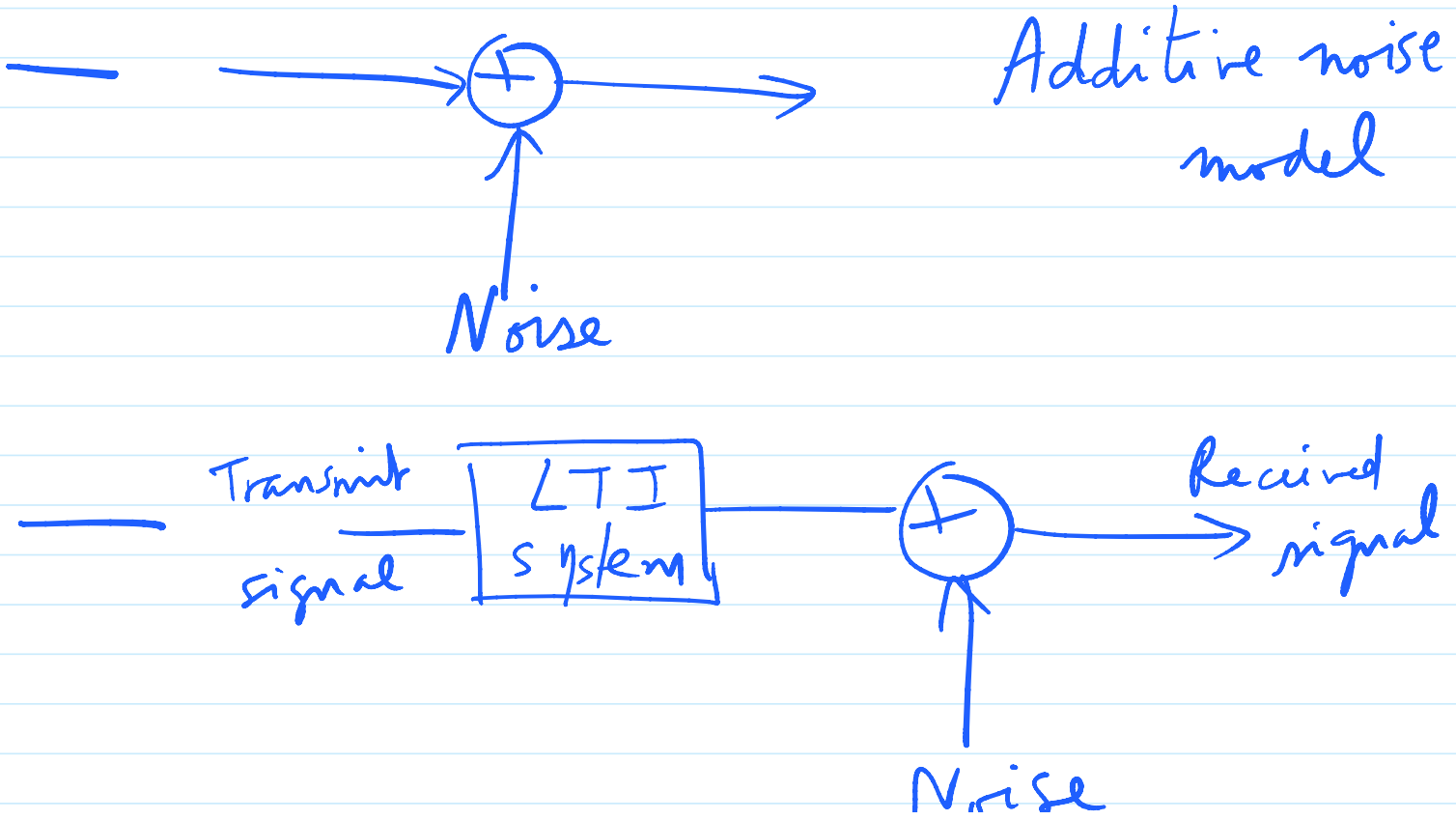
Questions:

- How should we model the channel?
- Given the model, how should we design

The communication system?

* Build from simple models

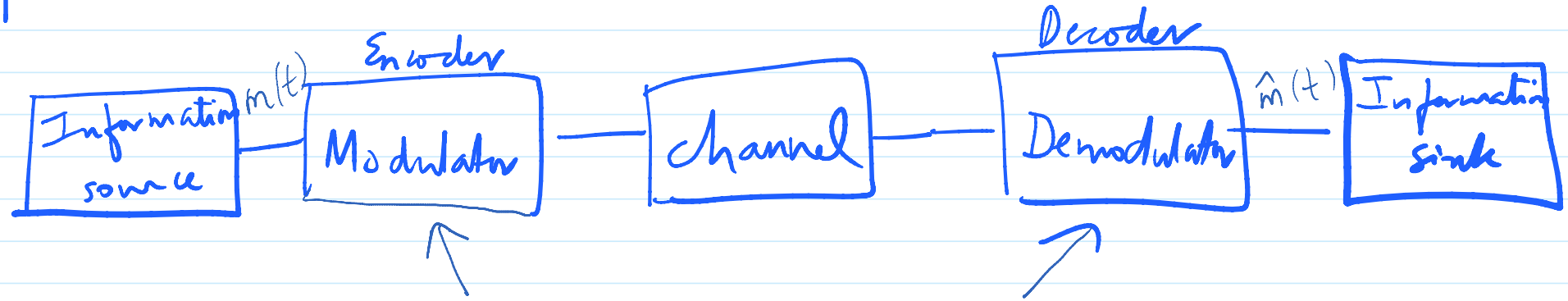
  — Ideal band-limited channel



Noise

* Designing for the simple model
  - gives us insights
  - generalize later for more realistic models

Typical communication system



Analog communication



Analog communication

$m(t)$ → $x(t)$ → channel

$x(t)$: Band-limited signals

Digital communication $x(t)$: One of a finite number of waveforms.

Lecture 2:

"Analog" signals: continuous time continuous valued

$m(t)$, $s(t)$, etc Eg Speech, music, etc

"Digital" signals: discrete time, discrete valued

Eg: files on a computer

Performance criteria:

Analog communication systems:

(1) How close is $\hat{m}(t)$ to $m(t)$?

eg. $\int_{-\infty}^{\infty} |m(t) - \hat{m}(t)|^2 dt$

(2) $\frac{\text{Bandwidth of } m(t)}{\text{Bandwidth of channel}}$

Digital communication systems:

(1) Probability of error

(2) Message transmission rate (bits/s) & spectral efficiency (bits/s/Hz).

Review: Signals and Systems

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- Focus on communication systems

  Classify signals/systems based on frequency band they occupy/operate

Complex numbers:

$$z = x + jy \qquad j = \sqrt{-1}$$

$$= re^{j\theta}$$

Addition: Use $x+jy$ form

Multiplication: Use $re^{j\theta}$ form

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \text{(Euler's formula)}$$

Signals: some examples & notation

Continuous time signals $x(t), s(t), \ldots$

Discrete time signals $x[n]$

(1) Sinusoid of frequency f_0

$$s(t) = A\cos(2\pi f_0 t + \theta)$$

$$A > 0, \quad \theta \in [0, 2\pi]$$

$$= (A\cos\theta)\cos 2\pi f_0 t - (A\sin\theta)\sin 2\pi f_0 t$$

(2) Complex exponential at frequency f_0

$$s(t) = A e^{j(2\pi f_0 t + \theta)} \qquad A > 0, \ \theta \in (0, 2\pi]$$

$$= (Ae^{j\theta}) e^{j 2\pi f_0 t} = \underbrace{\alpha}_{\text{complex}} e^{j 2\pi f_0 t}$$

(3) Delta function

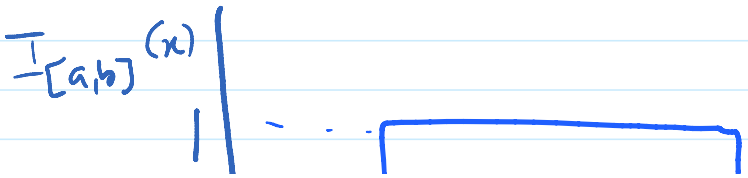
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ ? & t = 0 \\ \infty & \end{cases}$$

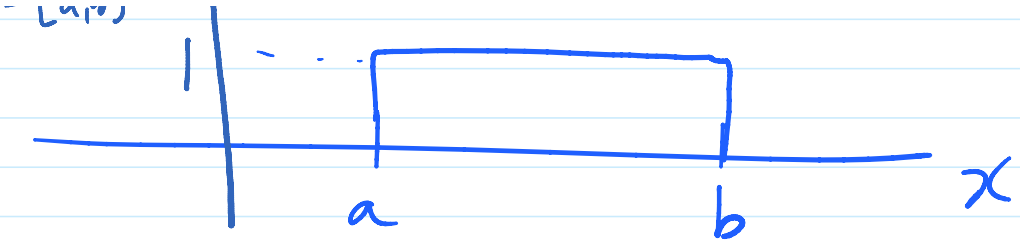
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad \int_{-\infty}^{\infty} s(t)\,\delta(t)\,dt = s(0)$$

$$\int_{-\infty}^{\infty} s(t)\,\delta(t-t_0)\,dt = s(t_0)$$

(4) Indicator function

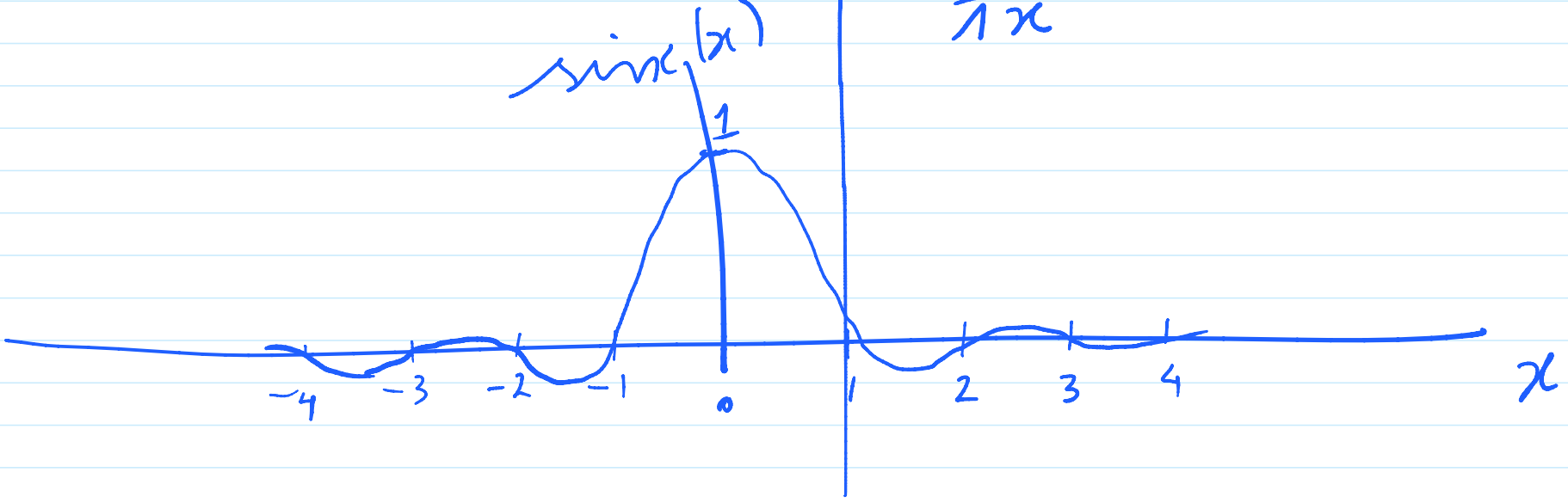
$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{else} \end{cases}$$





(5) Sinc function

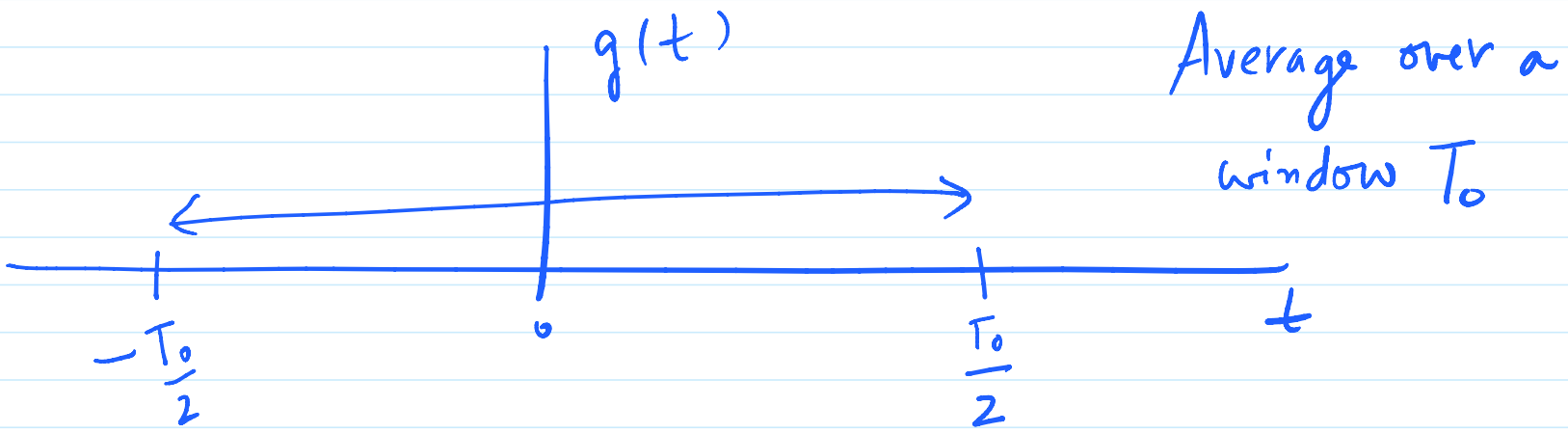
$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Lecture 3:

Time average of a signal $g(t)$

$$\overline{g} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t)\, dt$$



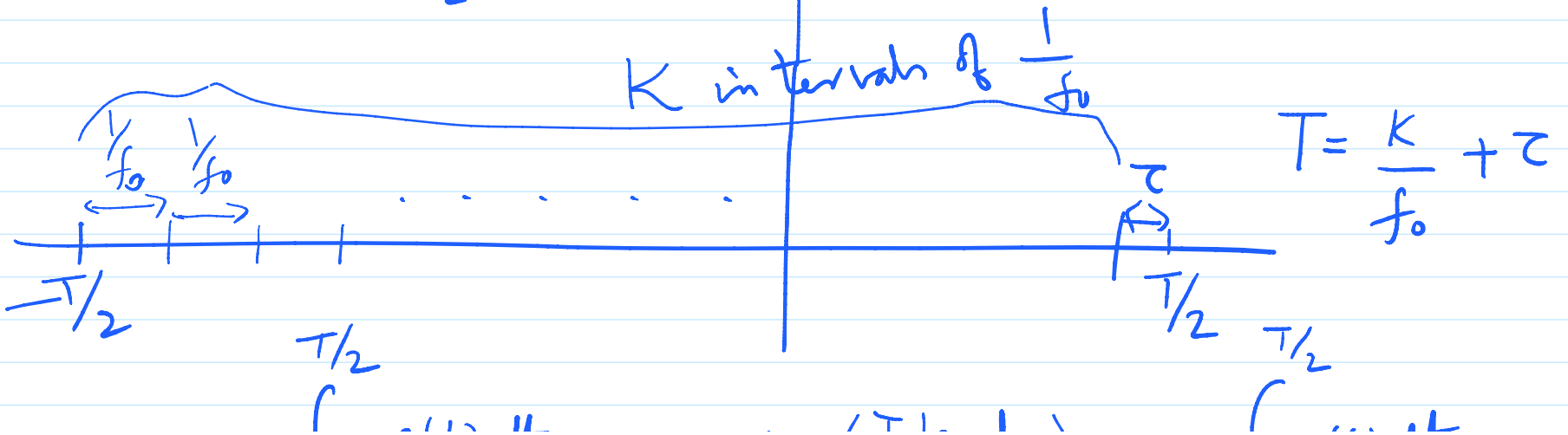
Eg: $s(t) = A e^{j(2\pi f_0 t + \theta)}$

$$\bar{s} = 0$$

* $s(t)$ is periodic with period $\frac{1}{f_0}$

Integral over a interval of length $\frac{1}{f_0}$ is zero.

$$\lim_{T \to \infty} \int_{-T/2}^{T/2} s(t)\,dt = ?$$



$$\int_{-T/2}^{T/2} s(t)\,dt = K\left(\begin{array}{l}\text{Integral}\\ \text{over 1 period}\end{array}\right) + \int_{\frac{T}{2}-\tau}^{T/2} s(t)\,dt$$

$$= \int_{T/2-\tau}^{T/2} s(t)\,dt$$

$$\bar{s} = \lim_{T\to\infty} \frac{1}{T} \underbrace{\int_{T/2-\tau}^{T/2} s(t)\,dt} = ?$$

$$-\int_{T/2}^{T/2} |s(t)|\,dt \leq \int_{T/2-\tau}^{T/2} s(t)\,dt \leq \int_{T/2-\tau}^{T/2} |s(t)|\,dt$$

$\int_{T/2-\tau}$ $\int_{T/2-\tau}$ $\int_{T/2-\tau}$

$\leq \tau A \leq \frac{A}{f_0}$

$$0 = -\lim_{T\to\infty} \frac{A}{Tf_0} \leq \bar{s} \leq \lim_{T\to\infty} \frac{A}{Tf_0} = 0$$

$$\boxed{\bar{s} = 0}$$

Definition: INNER PRODUCT.

$$\underline{s} = \begin{pmatrix} s[1] \\ s[2] \\ \vdots \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} r[1] \\ r[2] \\ \vdots \\ \vdots \end{pmatrix}$$

Two $m \times 1$ complex vectors

$$\underline{s} = \begin{pmatrix} s[2] \\ \vdots \\ s[m] \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} \vdots \\ r[m] \end{pmatrix}$$

vectors

Inner product $\langle \underline{s}, \underline{r} \rangle = \underline{r}^H \underline{s} = \sum_{i=1}^{m} s[i] r^*[i]$.

Two complex signals $s(t)$ and $r(t)$:

$$\langle s, r \rangle = \int_{-\infty}^{\infty} s(t) r^*(t) \, dt$$

Linearity $s = a_1 s_1 + a_2 s_2$

$$\langle s, r \rangle = a_1 \langle s_1, r \rangle + a_2 \langle s_2, r \rangle$$

$$r = a_1 r_1 + a_2 r_2$$

$$\langle s, r \rangle = a_1^* \langle s, r_1 \rangle + a_2^* \langle s, r_2 \rangle$$

Energy and norm of a signal:

Energy E_s of a signal $s = \langle s, s \rangle = \int_{-\infty}^{\infty} |s(t)|^2 dt$

$$\triangleq \|s\|^2$$

$\|s\|$: Norm of a signal $s(t)$

$E_s = 0 \Rightarrow s$ must be zero "almost everywhere"

"

(over any interval of non-zero length)

Eg (1) $s(t) = 2\, I_{[0,T)} + j\, I_{[\frac{T}{2}, 2T)}$

$$E_s = \|\tfrac{T}{2}$$

(2) $s(t) = e^{-3|t| + j2\pi t}$

$$E_s = \frac{1}{3}.$$

Finite signal energy examples

Power of a signal $s(t)$ is $\overline{|s(t)|^2}$

$$P_s = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt$$

$$P_s = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt$$

For Finite energy signals (E_s is finite), $P_s = 0$.

Eg (1) $s(t) = A e^{j(2\pi f_0 t + \theta)}$

$$P_s = A^2$$

(2) $s(t) = A \cos(2\pi f_0 t + \theta)$

$$\bar{s} = 0$$

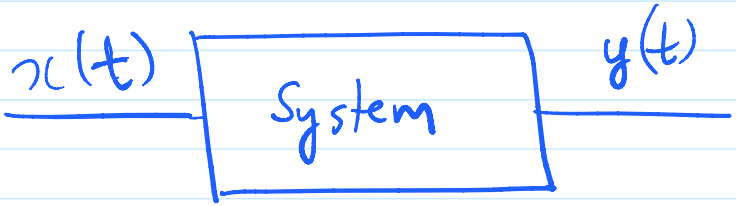
$$P_s = \overline{A^2 \cos^2(2\pi f_0 t + \theta)}$$

$$= \overline{\frac{A^2}{2}\left(1 + \cos(4\pi f_0 t + 2\theta)\right)}$$

$$= \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_0 t + 2\theta)$$

$$= \frac{A^2}{2} + 0 = \frac{A^2}{2}.$$

SYSTEMS



Input-output relationship

If we can determine output $y(t)$ for any possible input $x(t)$ in a given class of signals of interest, then we have characterized the system.

Lecture 4

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Linear systems

Let $x_1(t)$ and $x_2(t)$ be arbitrary input signals.
Let $y_1(t)$ and $y_2(t)$ be their corresponding outputs.

A system is linear if for arbitrary complex scalars a_1, a_2, the output corresponding to $a_1 x_1(t) + a_2 x_2(t)$ is $a_1 y_1(t) + a_2 y_2(t)$.

Time invariant systems

Let $x(t)$ be an arbitrary input signal.
Let $y(t)$ be the corresponding output.

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A system is time invariant if for each t_0 the output corresponding to $x(t-t_0)$ is $y(t-t_0)$.

Examples

(1) $y(t) = 2x(t-1) - jx(t-2)$ Linear? Time invariant?

(2) $y(t) = \int_{t-1}^{t+1} x(\tau)\,d\tau$

(3) $y(t) = e^{2x(t-1)}$ Check

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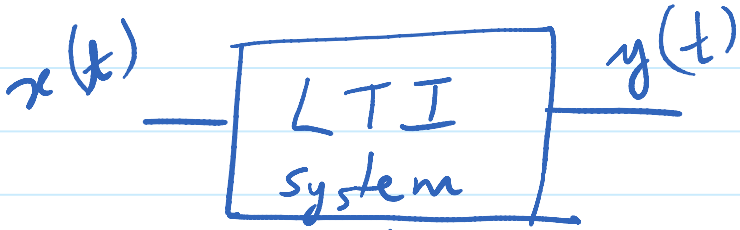

Both linear and time-invariant

For an LTI system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where $h(t)$ is the response (output) corresponding to input $\delta(t)$.

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Convolution of $x(t)$ & $h(t)$

$(x * h)(t)$

$(h * x)(t)$

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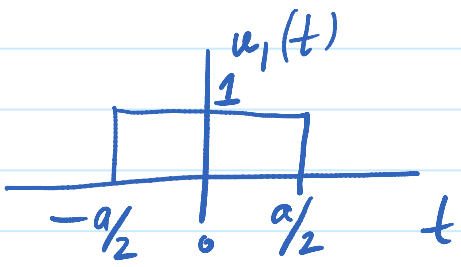
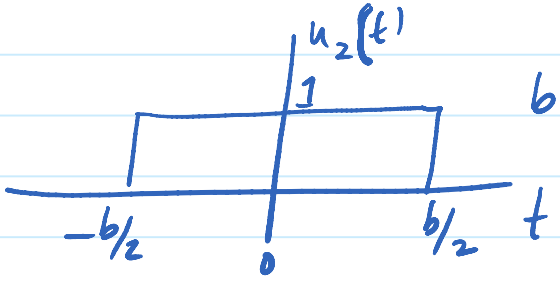
$$x * h = h * x$$

$$\int_{-\infty}^{\infty} x(\tau)\, h(\underbrace{t-\tau}_{\triangleq \nu})\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{+\infty}^{-\infty} x(t-\nu)\, h(\nu)\, (+d\nu)$$

Eg





$b > a$.

$u_2(\tau) \longrightarrow u_2(t-\tau)$

$= u_2(-(\tau - t))$

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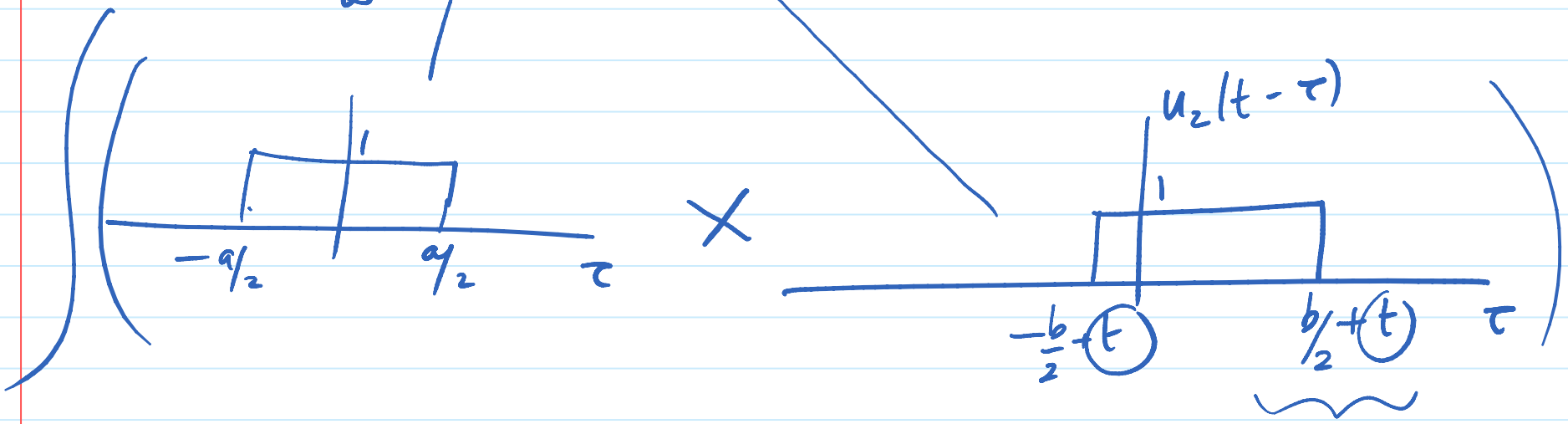
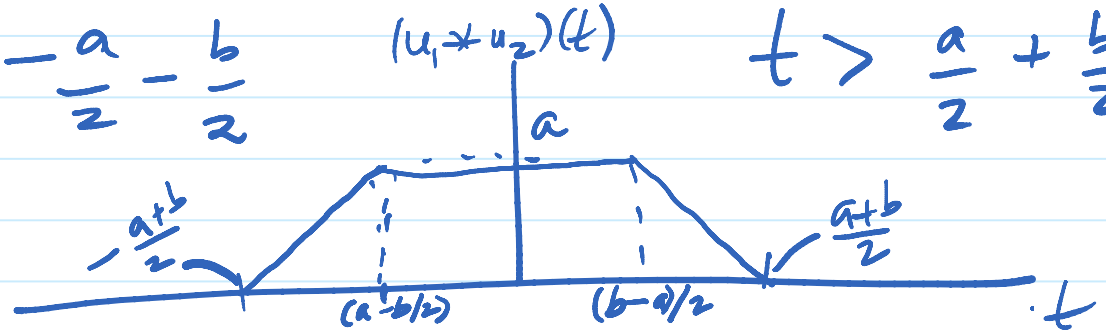
$$\int_{-\infty}^{\infty} u_1(\tau)\, u_2(t-\tau)\, d\tau$$



$\frac{b}{2}+t < -\frac{a}{2} \longrightarrow 0$ $\quad -\frac{b}{2}+t > \frac{a}{2} \longrightarrow 0$

$t < -\frac{a}{2}-\frac{b}{2}$ $\quad (u_1 * u_2)(t)$ $\quad t > \frac{a}{2}+\frac{b}{2}$



Lecture 4

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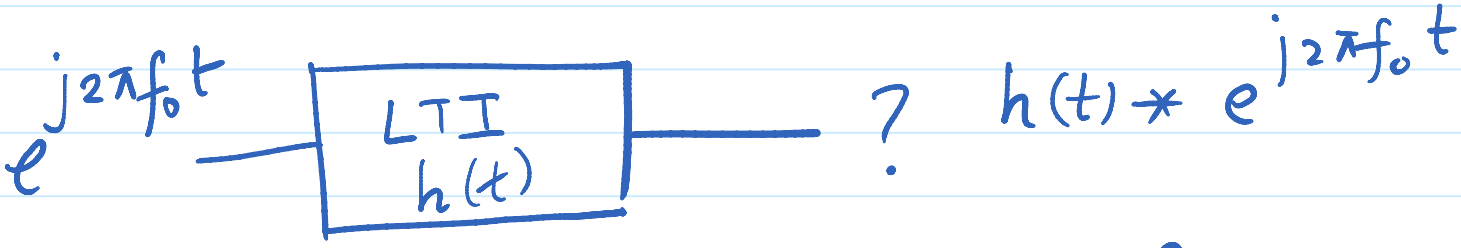
Linearity of convolution

$$(s_1(t) + s_2(t)) * r(t) = (s_1(t) * r(t)) + (s_2(t) * r(t))$$

Complex exponential through an LTI system



$$\text{output} = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{e^{j2\pi f_0 (t-\tau)}}_{e^{j2\pi f_0 t} \cdot e^{-j2\pi f_0 \tau}}\, d\tau = e^{j2\pi f_0 t} \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau}\, d\tau \right]}_{H(f_0)}$$

Lecture 5

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Fourier Series:

* Periodic signals

$u(t)$ is periodic with period T if

$\rightarrow u(t+T) = u(t) \quad \underline{\forall t}$

If $u(t)$ is periodic with period T, it is also periodic with period nT where n is any +ve integer. $(u(t+nT) = u(t) \quad \forall t)$

(T_0) Fundamental period: The smallest interval for which $u(t)$ is periodic

$1/T_0 \triangleq f_0$ = Fundamental frequency

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Any periodic signal $u(t)$ with period T_0 can be expressed as a linear combination of complex exponentials as (under some conditions)

$$u(t) = \underbrace{\sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}}$$

Dirichlet conditions (sufficient)

* $\int_{\text{over one period}} |u(t)|\,dt < \infty$
* Finite # of discontinuities in one period
* Finite # maxima & minima in one period

The series on the RHS converges to the function $u(t)$ at all pts (for all t) where the function $u(t)$ is continuous.

Let $\psi_m(t) = e^{j2\pi m f_0 t}$ $\quad m = 0, \pm 1, \pm 2, \ldots$

$$\left(u(t) = \sum_{n=-\infty}^{\infty} u_n \psi_n(t) \right)$$

$$\int_D^{D+T_0} \psi_0(t)\, dt = T_0$$

$$m \neq 0 \quad \int_D^{D+T_0} \psi_m(t)\, dt = \frac{e^{j2\pi m f_0 t}}{j2\pi m f_0} \Bigg|_D^{D+T_0}$$

$$= \frac{e^{j2\pi m f_0 (D+T_0)} - e^{j2\pi m f_0 D}}{e^{j2\pi m f_0}}$$

$$= \frac{e^{j2\pi m f_0 D}\left(e^{j2\pi m f_0 T_0} - 1\right)}{e^{j2\pi m f_0}} = 0$$

Consider $\int_D^{D+T_0} \psi_n(t)\psi_k^*(t)\,dt = \int_D^{D+T_0} e^{j2\pi(n-k)f_0 t}\,dt$

$\langle \psi_n, \psi_k \rangle_{T_0}$ (orthogonality) $= \begin{cases} 0 & n \neq k \\ T_0 & n = k \end{cases}$

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Therefore, we can now write a formula for u_n.

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}$$

$u_m = ?$

$$\int_D^{D+T_0} u(t)\,\psi_m^*(t)\,dt = \sum_{n=-\infty}^{\infty} u_n \int_D^{D+T_0} \psi_n(t)\,\psi_m^*(t)\,dt$$

$$= u_m T_0$$

$$u_m = \frac{1}{T_0}\int_D^{D+T_0} u(t)\,\psi_m^*(t)\,dt = \frac{1}{T_0}\langle u(t), \psi_m(t)\rangle_{T_0}$$

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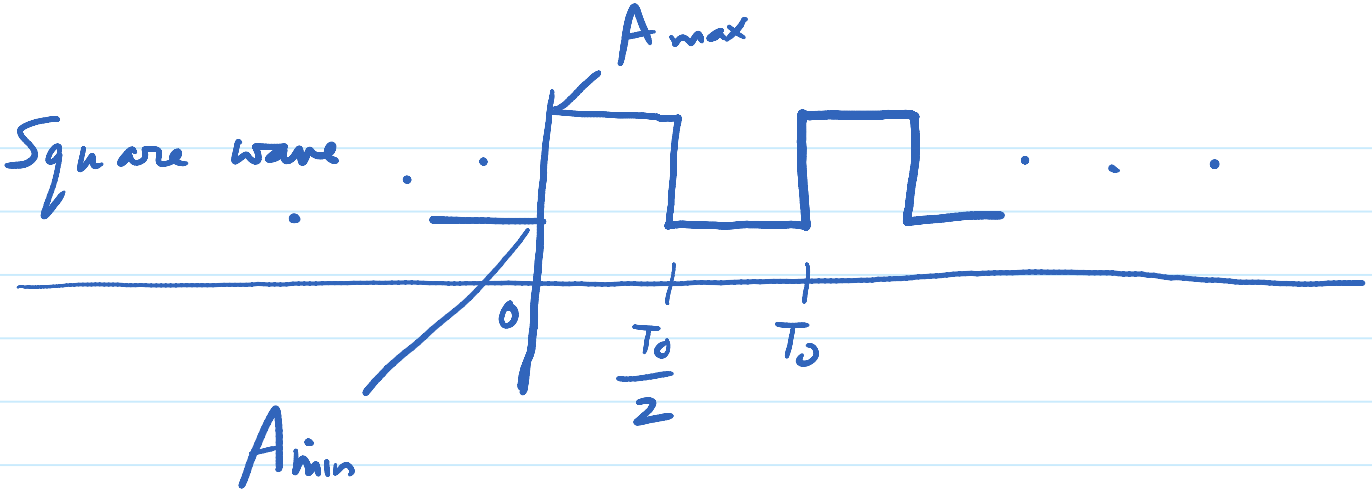
Examples (1) Square wave



$$u_0 = \frac{A_{max} + A_{min}}{2}$$

$$u(t) = \frac{A_{max} + A_{min}}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} u_k e^{j2\pi k f_0 t}$$

$$u_k = \begin{cases} 0 & k \text{ even} \\ \dfrac{A_{max} - A_{min}}{j\pi k} & k \text{ odd} \end{cases}$$

$$u_{-k} = -u_k = u_k^*$$

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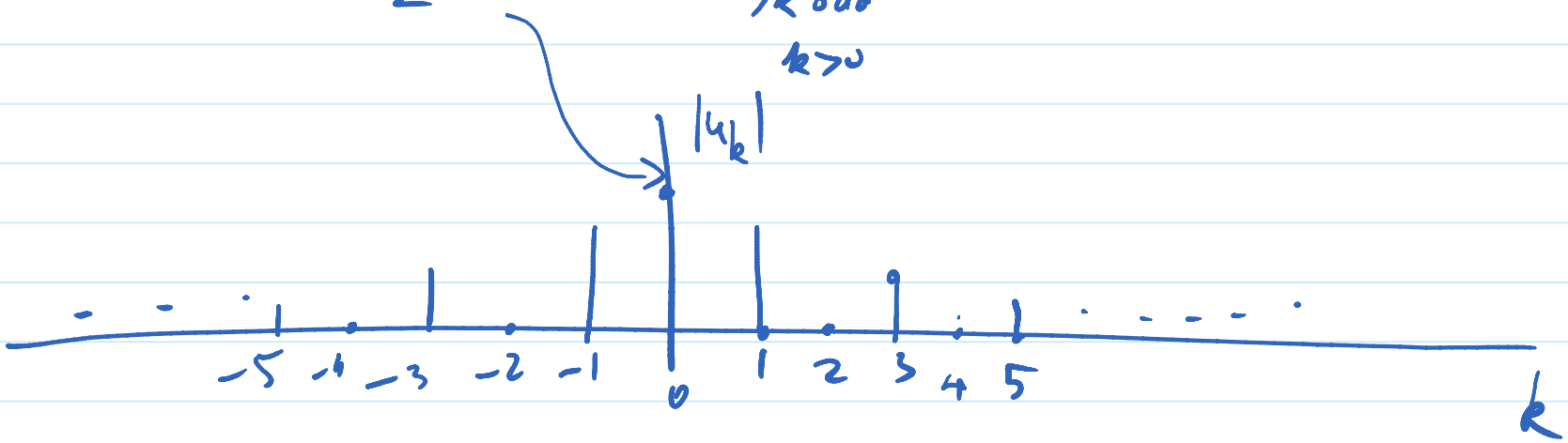
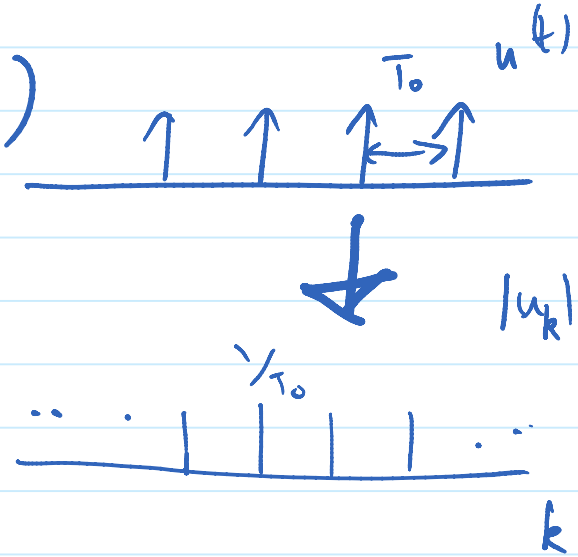
$$u(t) = \frac{A_{max} + A_{min}}{2} + \sum_{\substack{k\ odd \\ k>0}} \frac{2(A_{max} - A_{min})}{\pi k} \sin\left(\frac{2\pi k t}{T_0}\right)$$



(2) $u(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ (Impulse train)



$$u_k = \frac{1}{T_0}$$

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Properties: (without proof)

(1) **Linearity**: Let $u(t)$ & $v(t)$ be periodic with period T_0

Let $u(t) \longleftrightarrow \{u_k\}$

$v(t) \longleftrightarrow \{v_k\}$

Then $\alpha u(t) + \beta v(t) \longleftrightarrow \{\alpha u_k + \beta v_k\}$

Lecture 6

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2) If $u(t) \longleftrightarrow \{u_k\}$

then $u(t-d) \longleftrightarrow \{u_k e^{-j2\pi k f_0 d}\}$

$$u_k = \frac{1}{T_0} \int_{D}^{D+T_0} u(t) e^{-j2\pi k f_0 t} dt$$

3) For real $u(t)$, $\boxed{u_{-k} = u_k^*}$

$$u_{-k} = \frac{1}{T_0} \int_{D}^{D+T_0} u(t) e^{j2\pi k f_0 t} dt$$

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}$$

$$= u_0 + \sum_{n=1}^{\infty} \left(u_n e^{j2\pi n f_0 t} + u_n^* e^{-j2\pi n f_0 t} \right)$$

$$2|u_n| \cos\left(2\pi n f_0 t + \angle u_n\right)$$

4) Let $x(t) = \frac{d}{dt} u(t)$, $u(t) \leftrightarrow \{u_k\}$

$$\boxed{x_k = j2\pi k f_0 u_k}$$

Notes: $x_0 = 0$

Higher freq.'s amplified

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5) Parseval's identity

$$\langle u, v \rangle_{T_0} = \int_{T_0} u(t)\, v^*(t)\, dt = T_0 \sum_{k=-\infty}^{\infty} u_k v_k^*$$

$v = u$

$$\frac{1}{T_0} \int_{T_0} |u(t)|^2 dt = \sum_{k=-\infty}^{\infty} |u_k|^2$$

Fourier transform

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$u(t) \longleftrightarrow U(f)$

$U(f) = \mathcal{F}(u(t))$

* Consider signals with finite energy

$$\int_{-\infty}^{\infty} |u(t)|^2 dt < \infty$$

* Fourier transform $U(f)$ of $u(t)$ is

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-j2\pi f t} dt, \quad -\infty < f < \infty$$

Inverse Fourier transform

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df$$

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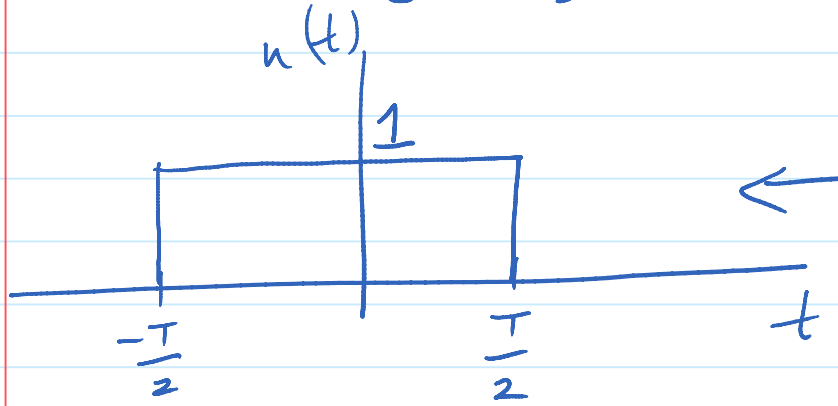
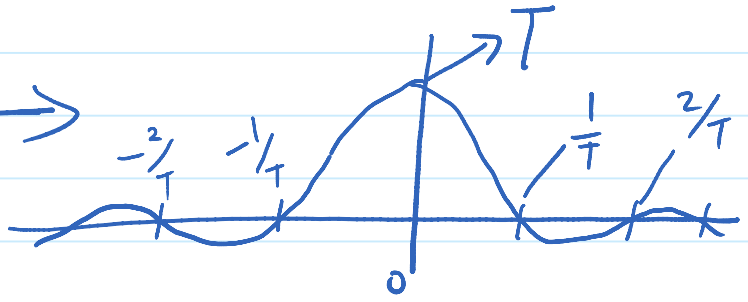
Example

$$u(t) = \Pi_{\left[-\frac{T}{2}, \frac{T}{2}\right]}(t)$$



$$\longleftrightarrow$$

$$U(f) = T\,\mathrm{sinc}(fT)$$



$$U(f) = \int_{-T/2}^{T/2} e^{-j2\pi f t}\,dt = \frac{e^{-j2\pi f t}}{-j2\pi f}\Bigg|_{-T/2}^{T/2}$$

$$= \frac{e^{-j\pi fT} - e^{j\pi fT}}{-j2\pi f} = T\frac{\sin(\pi fT)}{\pi fT} = T\,\mathrm{sinc}(fT)$$

08 August 2017 09:46

Other examples where signal is not a finite energy signal

(1) $u(t) = \delta(t)$

$$U(f) = \int_{-\infty}^{\infty} \underbrace{u(t)}_{\delta(t)} e^{-j2\pi ft}\, dt = 1$$

$$\boxed{\delta(t) \longleftrightarrow 1}$$

(2) $U(f) = \delta(f - f_0)$

$$u(t) = \int_{-\infty}^{\infty} U(f)\, e^{j2\pi ft}\, df = e^{j2\pi f_0 t}$$

$$\boxed{e^{j2\pi f_0 t} \longleftrightarrow \delta(f - f_0)}$$

Lecture 7

09 August 2017 07:46

Signals – frequency content

Fourier transform:

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-j2\pi f t} dt \quad \& \quad u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df$$

– Examples

$$I_{[-\frac{T}{2}, \frac{T}{2})}(t) \longleftrightarrow T \operatorname{sinc}(fT)$$

finite energy signal

$$\delta(t) \longleftrightarrow 1$$

$$e^{j2\pi f_0 t} \longleftrightarrow \delta(f - f_0)$$

Fourier transform properties:

Let $u(t) \longleftrightarrow U(f)$, $v(t) \longleftrightarrow V(f)$.

(1) Linearity $\quad \alpha u(t) + \beta v(t) \longleftrightarrow \alpha U(f) + \beta V(f)$ for any complex α, β

FT

09 August 2017 08:06

(2) Duality

If $u(t) \longleftrightarrow U(f)$

then $U(t) \longleftrightarrow u(-f)$

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df$$

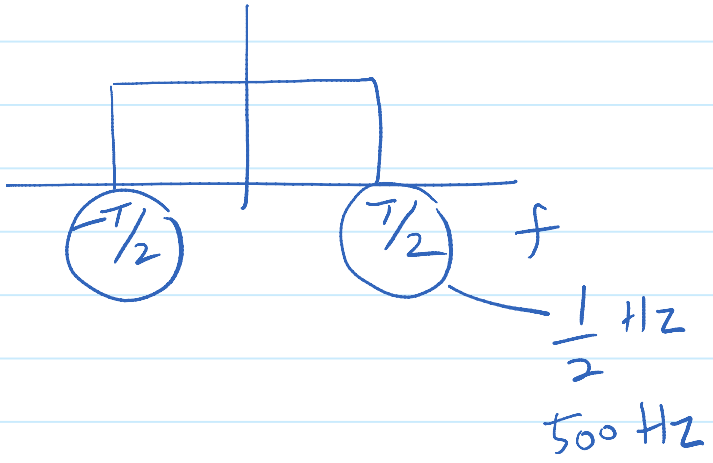
Proof:

$$\mathcal{F}[U(t)] = \int_{-\infty}^{\infty} U(t) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} U(t) e^{j2\pi(-f)t} dt = u(-f)$$

$T \operatorname{sinc}(tT) \longleftrightarrow$

first zero crossing at $\frac{1}{T}$ — 1 sec, 1 ms



FT

09 August 2017 08:13

(3) $u(t-t_0) \longleftrightarrow U(f)\, e^{-j2\pi f t_0}$

Time delay t_0

Linear phase term

phase $-2\pi f t_0$ is a linear fn of f!

(4) $u(t)\, e^{j2\pi f_0 t} \longleftrightarrow U(f-f_0)$

Frequency shift by f_0

multiplication by a complex exponential at freq f_0.
(modulation)

Proof: We know $u(t) = \int_{-\infty}^{\infty} U(f)\, e^{j2\pi f t}\, df$

FT

09 August 2017 08:19

Need to show $\int_{-\infty}^{\infty} U(f-f_0)\, e^{j2\pi f t}\, df = ?$

[Let $f - f_0 = \nu$]

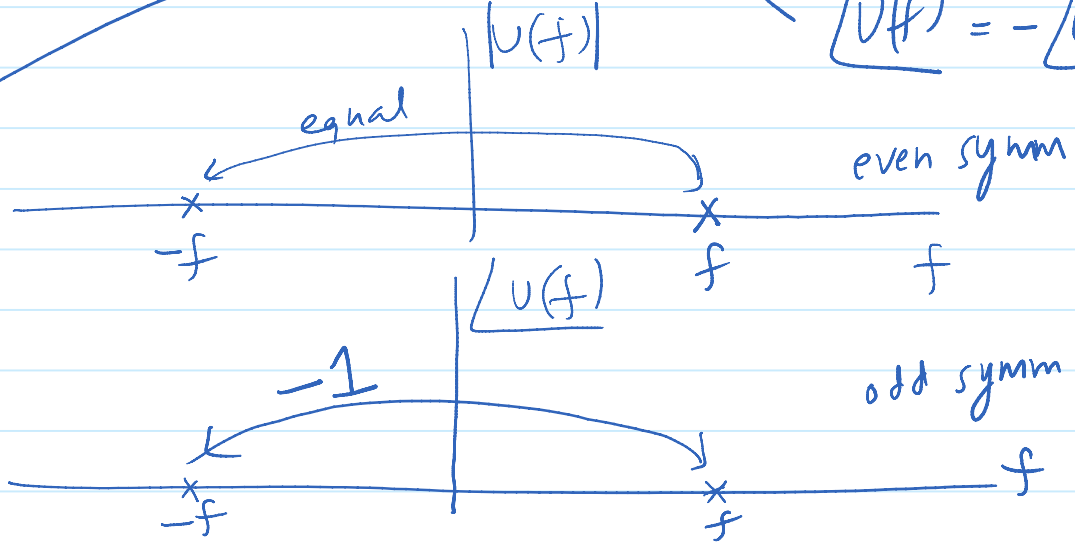
$$= \int_{-\infty}^{\infty} U(\nu)\, e^{j2\pi(\nu+f_0)t}\, d\nu$$

$$= e^{j2\pi f_0 t} \underbrace{\int_{-\infty}^{\infty} U(\nu)\, e^{j2\pi\nu t}\, d\nu}_{u(t)}$$

(5) If $u(t)$ is real, then $U(f) = U^*(-f)$ $\Rightarrow$ $|U(f)| = |U(-f)|$, $\angle U(f) = -\angle U(-f)$

$\mathrm{Re}(U(f)) = \mathrm{Re}(U(-f))$

$\mathrm{Im}(U(f)) = -\mathrm{Im}(U(-f))$



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(6) Differentiation $x(t) = \frac{d}{dt} u(t)$

$$x(t) \longleftrightarrow X(f) = \boxed{j2\pi f}\, U(f)$$

Note: $\boxed{X(0) = 0}$

$$U(0) = \int_{-\infty}^{\infty} (t)\, dt$$

Can we find $U(f)$ from $X(f)$?

$$U(f) = \frac{X(f)}{j2\pi f} \quad \text{for all } f \neq 0$$

If $U(0)$ is finite, it does not matter if we cannot find $U(0)$ exactly

F T

09 August 2017 08:33

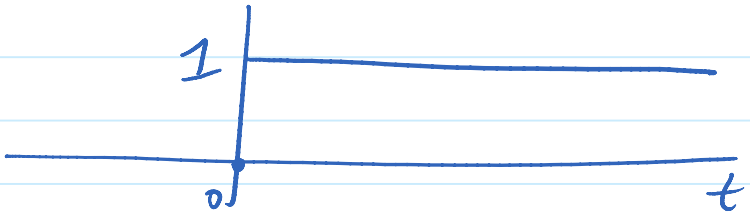
$\delta(t) \Leftrightarrow 1$

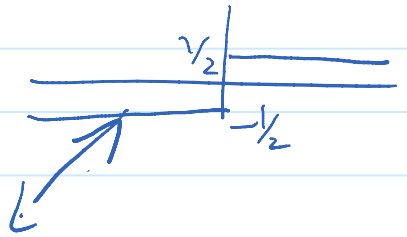
$1 \Leftrightarrow \delta(f)$

Let $U(0) = \int_{-\infty}^{\infty} u(t)\,dt$ is not finite

but $\bar{u}$ is finite $\quad \bar{u} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u(t)\,dt$

$\bar{u}(t) = \bar{u} \longleftrightarrow \bar{u}\,\delta(f)$





$\bar{u} = \frac{1}{2} \qquad U(f) = \underbrace{\frac{1}{j2\pi f}}_{} + \underbrace{\frac{1}{2}\,\delta(f)}_{\bar{u}}$

(7) Parseval's identity

$$\int_{-\infty}^{\infty} |u(t)|^2\,dt = \int_{-\infty}^{\infty} |U(f)|^2\,df \quad \xleftarrow{u=v} \quad \int_{-\infty}^{\infty} u(t)v^*(t)\,dt = \int_{-\infty}^{\infty} U(f)V^*(f)\,df$$

FT

09 August 2017 08:42

Proof: Use $\int_{-\infty}^{\infty} e^{j2\pi(f-f')t}\,dt = \delta(f-f')$

$$\int_{-\infty}^{\infty} u(t)v^*(t)\,dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} U(f)e^{j2\pi ft}\,df\right]\left[\int_{-\infty}^{\infty} V(f')e^{j2\pi f't}\,df'\right]^* dt$$

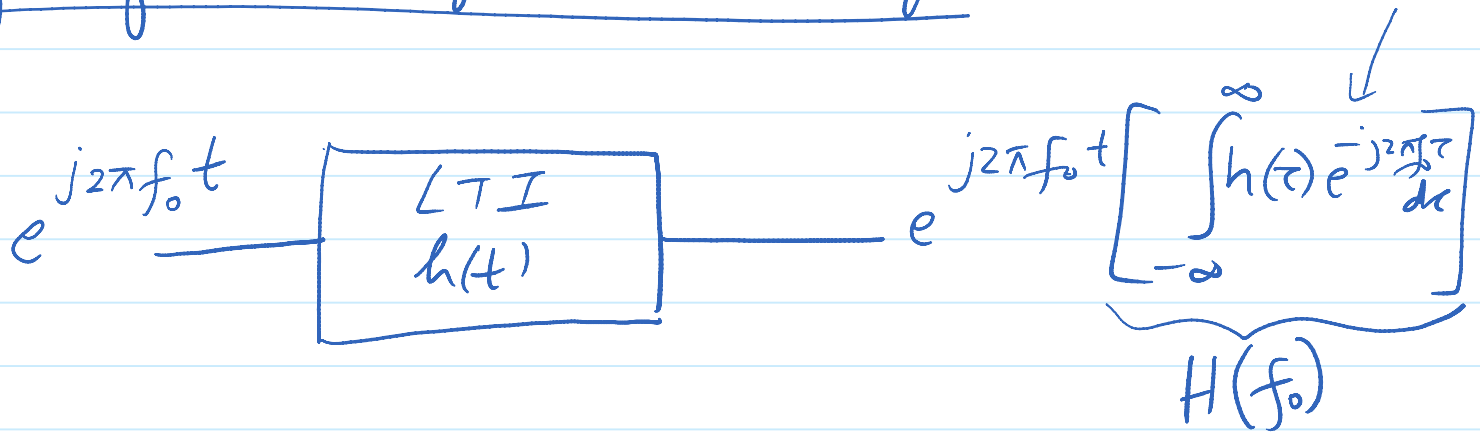
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(f)V^*(f')e^{j2\pi(f-f')t}\,df\,df'\right]dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(f)V^*(f')\,\delta(f-f')\,df\,df'$$

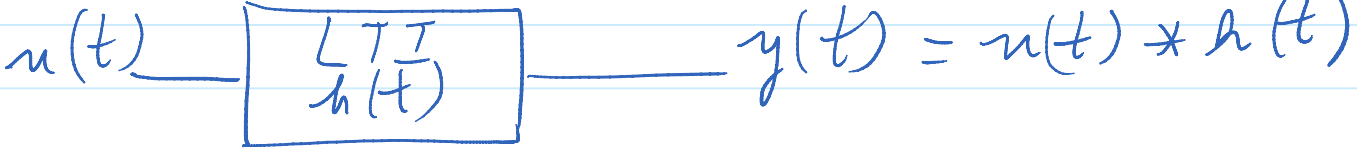
Lecture 8

14 August 2017 09:02

Transfer function of an LTI system



$\mathcal{F}(h(t)) \triangleq H(f)$ Transfer fn.



$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

LTI

14 August 2017 10:08

$$Y(f) = \int_{-\infty}^{\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(\tau) h(t-\tau)\, d\tau \right] e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} u(\tau) \underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi f t}\, dt \right]}_{H(f)\, e^{-j2\pi f \tau}} d\tau$$

$$= \underbrace{H(f) \int_{-\infty}^{\infty} u(\tau) e^{-j2\pi f \tau}\, d\tau}_{U(f)} = H(f)\, U(f)$$

$$y(t) = (u * h)(t) \longleftrightarrow U(f)\, H(f) = Y(f)$$

Conv in time domain — Mult in freq domain

LTI

14 August 2017 10:13

By duality

$u(t)\,v(t) \longleftrightarrow Y(f) = (U * V)(f)$

Multiplication in time domain

Convolution in freq domain

LTI system response to real-valued sinusoids

$u(t) = \cos(2\pi f_0 t + \theta)$ → [LTI, real $h(t)$] → $y(t) = ?$

$y(t) = (u * h)(t)$

$u(t) = \frac{1}{2} e^{j(2\pi f_0 t + \theta)} + \frac{1}{2} e^{-j(2\pi f_0 t + \theta)}$

$y(t) = \frac{H(f_0)}{2} e^{j(2\pi f_0 t + \theta)} + \frac{H(-f_0)}{2} e^{-j(2\pi f_0 t + \theta)}$

$h(t)$ real $\Rightarrow H(-f_0) = H^*(f_0)$

LTI

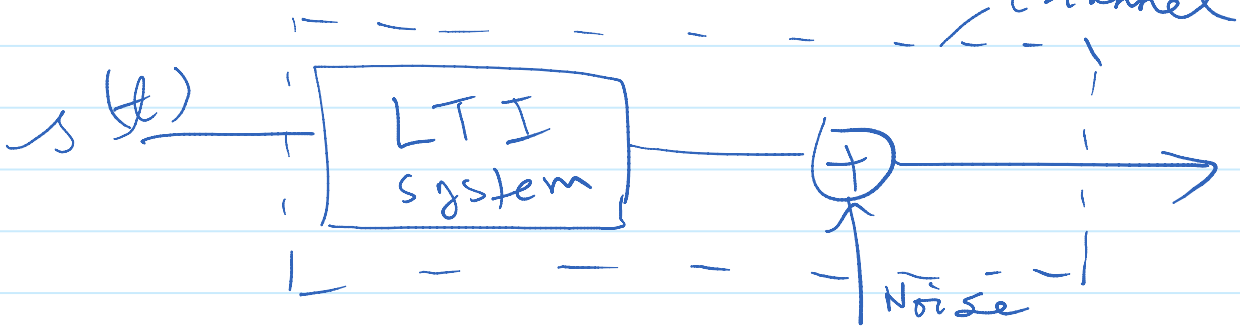
14 August 2017 10:20

$$= \frac{H(f_0)}{2} e^{j(2\pi f_0 t + \theta)} + \frac{H^*(f_0)}{2} e^{-j(2\pi f_0 t + \theta)}$$

$$= \frac{|H(f_0)|}{2} \left[e^{j(2\pi f_0 t + \theta + \angle H(f_0))} + e^{-j(2\pi f_0 t + \theta + \angle H(f_0))} \right]$$

$$= |H(f_0)| \cos(2\pi f_0 t + \theta + \angle H(f_0))$$

Example: (LTI system model for a wireless communication channel)



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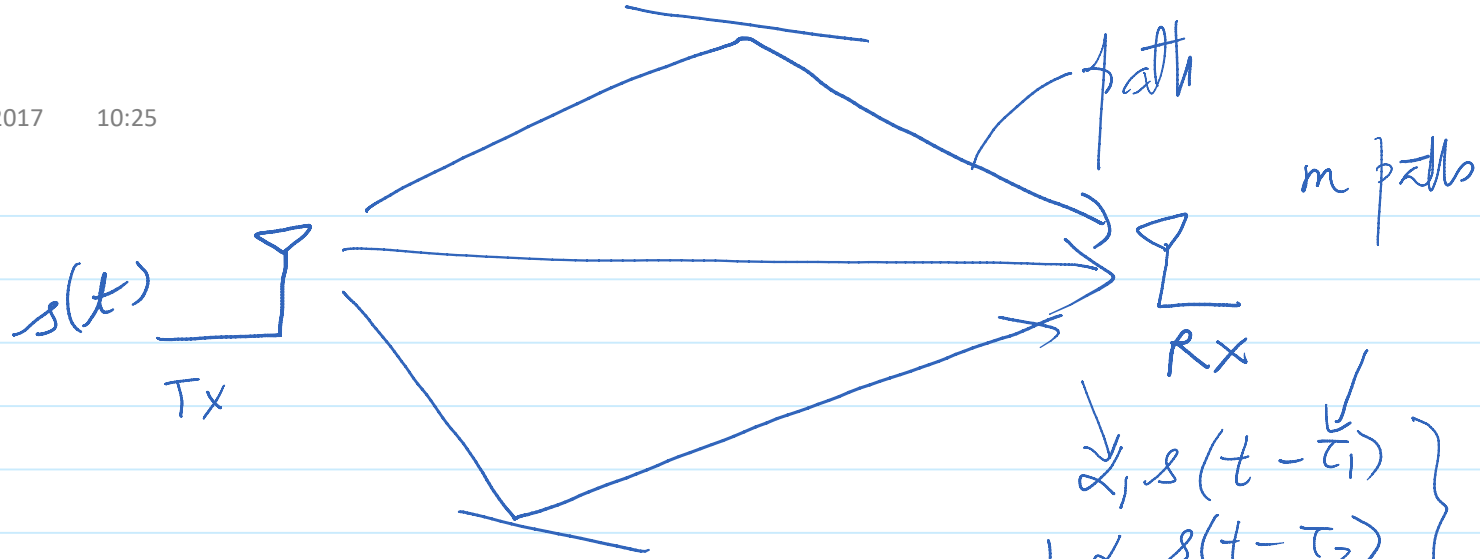


$$\alpha_1 s(t-\tau_1) + \alpha_2 s(t-\tau_2) + \cdots + \alpha_m s(t-\tau_m)$$

+ Noise

Multipath channel

path: delay τ, channel coeff α

$s(t)$ → $h(t)$? → $\sum_{i=1}^{m} \alpha_i s(t-\tau_i)$

$$h(t) = \sum_{i=1}^{m} \alpha_i \delta(t-\tau_i)$$

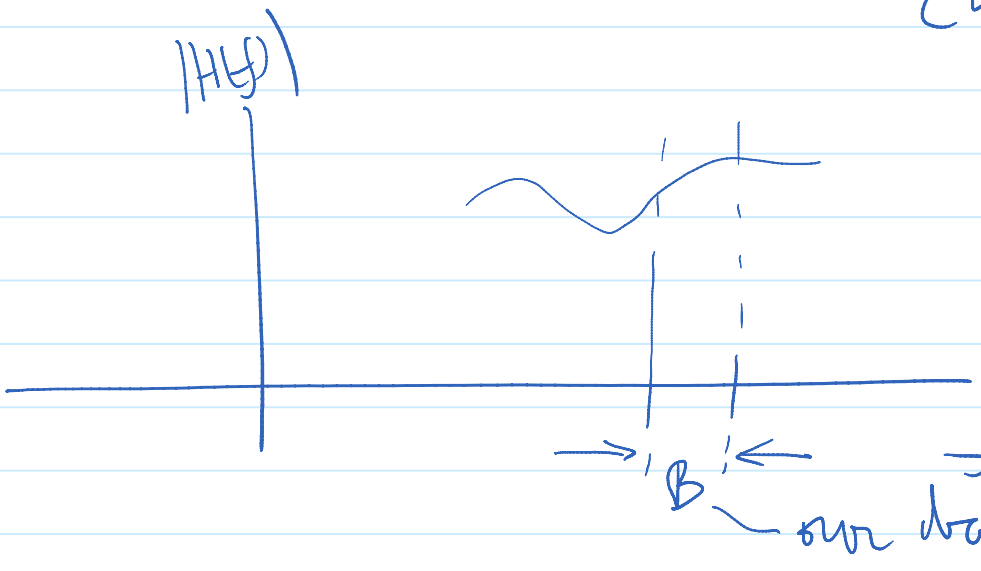
channel model

Channel

14 August 2017 10:31

$$H(f) = \alpha_1 e^{-j2\pi f \tau_1} + \alpha_2 e^{-j2\pi f \tau_2} + \dots + \alpha_m e^{-j2\pi f \tau_m}$$

→ Linear combination of complex exponentials (in freq. domain)



$$H(f) = \sum_{k=1}^{m} \alpha_k e^{-j2\pi f \tau_k} = e^{-j2\pi f \tau_1} \left[\sum_{k=1}^{m} \alpha_k e^{-j2\pi f (\tau_k - \tau_1)} \right]$$

channel

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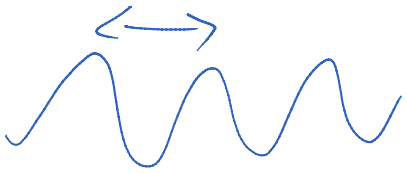


(Assume w.l.o.g. $\tau_1 < \tau_2 < \cdots < \tau_m$)

$$H(f) = e^{-j2\pi f \tau_1}\left[\alpha_1 + \sum_{k=2}^{m} \alpha_k e^{-j2\pi f(\tau_k - \tau_1)}\right]$$

Period $\frac{1}{\tau_k - \tau_1}$

$$\min_k \frac{1}{\tau_k - \tau_1} \quad (\text{or}) \quad \max_k (\tau_k - \tau_1) = \tau_m - \tau_1$$

$\triangleq$ Delay spread τ_d

If $B \ll \frac{1}{\text{Delay spread}}$, $H(f) \approx$ flat

Coherence Bandwidth $= \frac{1}{\tau_d}$

Channel

14 August 2017 10:45

$3 \times 10^8 \text{ m/s} \times 10^{-6} \text{ s}$

300 m

Example

$$h(t) = \delta(t-1) + 0.5\delta(t-3.5) - 0.5\delta(t-1.5)$$

Delays in μs

Delay spread = 2.5 μs

$$\frac{1}{2.5\mu s} = 400 \text{ kHz}$$

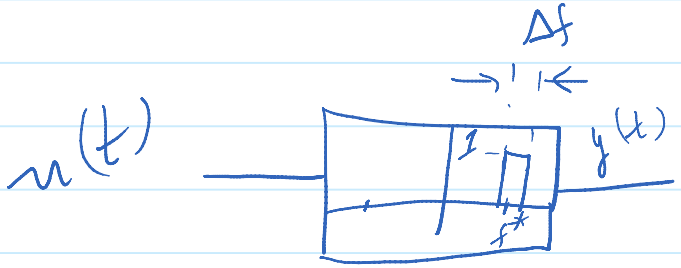
Lecture 9

16 August 2017 07:47

Energy spectral density & Bandwidth

$u(t) \longleftrightarrow U(f)$

Energy spectral density: A measure of how energy is spread out in frequency domain



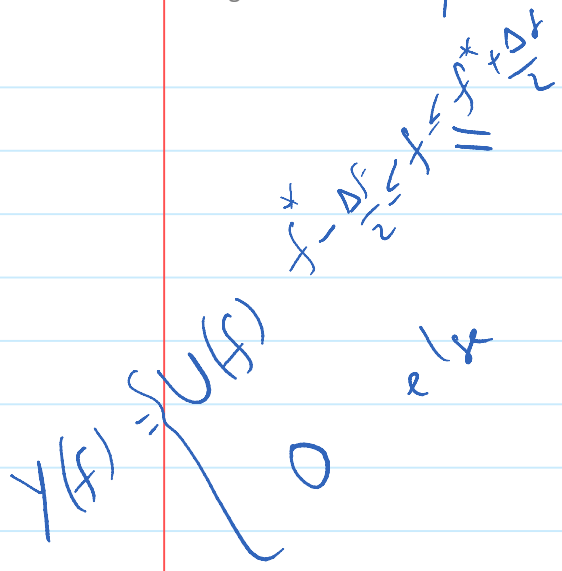
energy $\int_{-\infty}^{\infty} |y(t)|^2 dt$

We want to know energy content of $u(t)$ in a small band of frequencies around frequency f^*.

Energy spectral density at $f^* = \lim_{\Delta f \to 0} \frac{\text{Energy of } y(t)}{\Delta f}$

Energy Spectral Density

16 August 2017 08:07



$$\lim_{\Delta f \to 0} \frac{\int_{-\infty}^{\infty} |y(t)|^2 dt}{\Delta f} = \lim_{\Delta f \to 0} \frac{\int_{-\infty}^{\infty} |Y(f)|^2 df}{\Delta f}$$

$$= \lim_{\Delta f \to 0} \frac{\int_{f^* - \frac{\Delta f}{2}}^{f^* + \frac{\Delta f}{2}} |U(f)|^2 df}{\Delta f}$$

$$= |U(f^*)|^2$$

$$E_u(f) = |U(f)|^2$$

Energy spectral density of $u(t)$ at f

ESD

16 August 2017 08:11

What is the inverse Fourier transform of $E_u(f)$?

$$E_u(f) = |U(f)|^2 \longleftrightarrow ?$$

$$U(f)\,U^*(f) \longleftrightarrow u(\tau) * u^*(-\tau)$$

$$\int_{-\infty}^{\infty} u(t)u^*(t-\tau)\,dt \triangleq R_u(\tau)$$

$$\left[f(\tau) * g(\tau) = \int_{-\infty}^{\infty} f(t)\,g(\tau - t)\,dt \right]$$

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-j2\pi ft}\,dt$$

$$U^*(f) = \int_{-\infty}^{\infty} u^*(t)e^{j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} u^*(-\tau)e^{-j2\pi f\tau}\,d\tau$$

16 August 2017 08:21

Auto correlation function : $R_u(\tau)$ of $u(t)$

τ : time variable (convention for autocorrelation)

$$= \int_{-\infty}^{\infty} u(t)\, u^*(t-\tau)\, dt$$

Inner product of $u(t)$ with $u(t-\tau)$

Bandwidth :

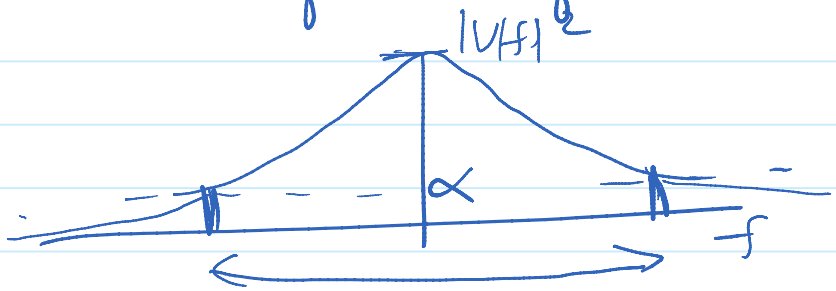
A measure of the band of frequencies "occupied" by a signal

(1) "Size" of band over which $U(f)$ is non-zero

Bandwidth

16 August 2017 08:27

(2) "Size" of the band over which $|U(f)|^2$ is within some fraction of its peak value.



(eg. $\alpha = \frac{1}{2}$)
3-dB bandwidth

(3) "Size" of smallest band over which a specified fraction of signal energy is contained
(Energy-containment bandwidth)

Examples (1) $u(t) = \text{sinc}(2t) \longleftrightarrow \frac{1}{2} I_{[-1,1]}(f)$

Bandwidth

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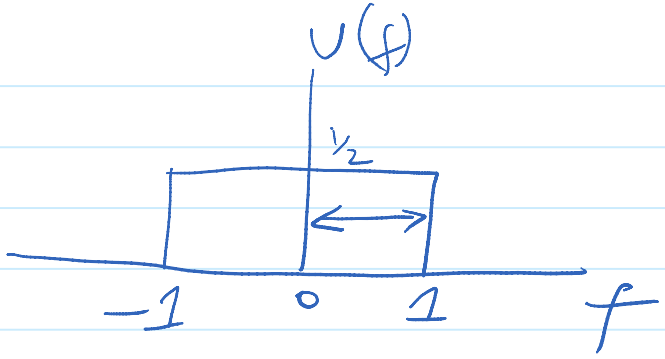
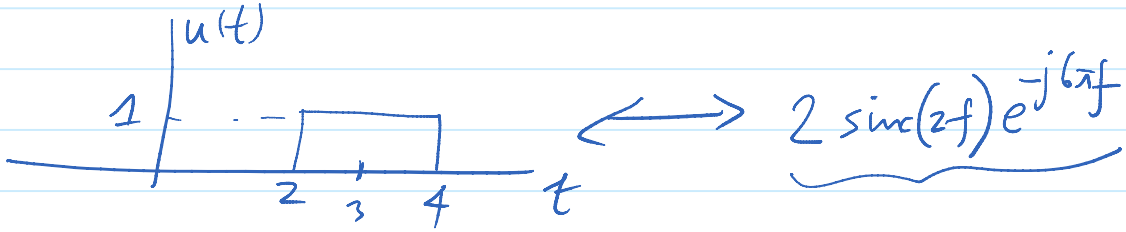


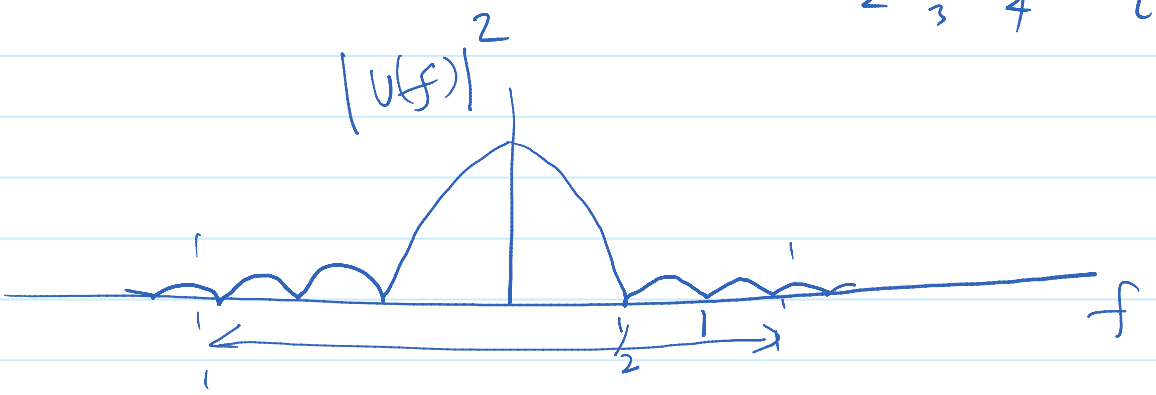
(Using defn (1)) Bandwidth = 1 Hz (Measuring only positive frequencies)

good if $u(t)$ is real

(2) $u(t) = I_{[2,4]}(t)$



$\longleftrightarrow$ $2\,\mathrm{sinc}(2f)\,e^{-j6\pi f}$



Defn (1): BW = ∞

(2):

(3): 99% Energy containment 5.1 Hz

Baseband and passband signals

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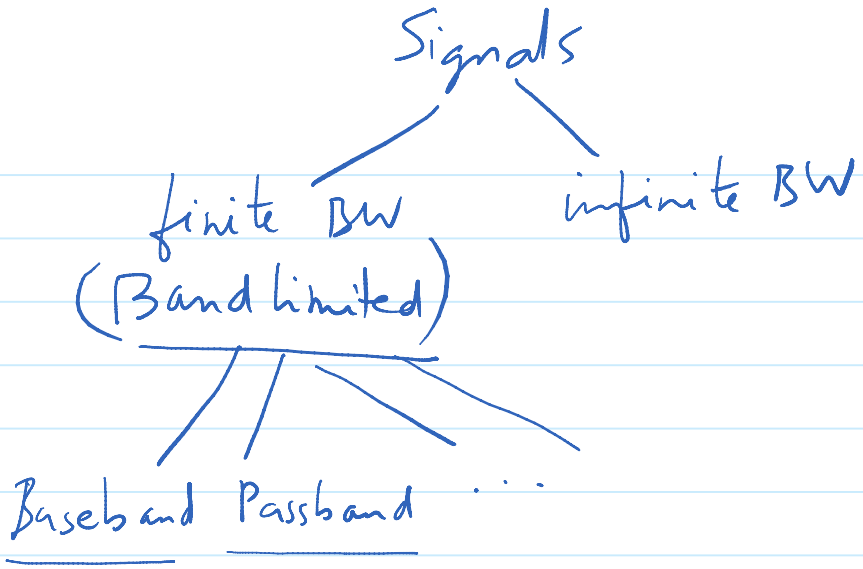
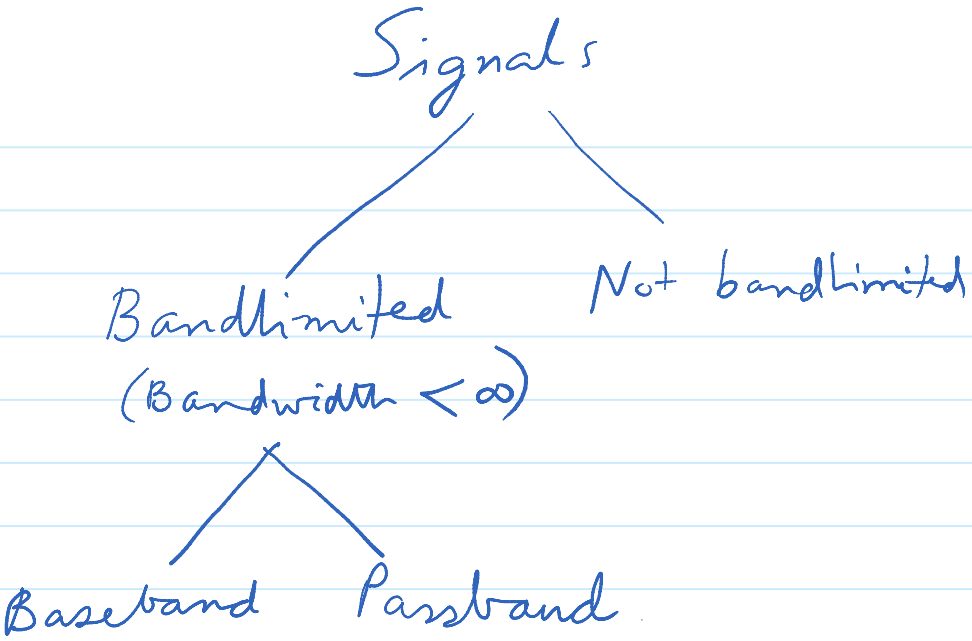


Lecture 10

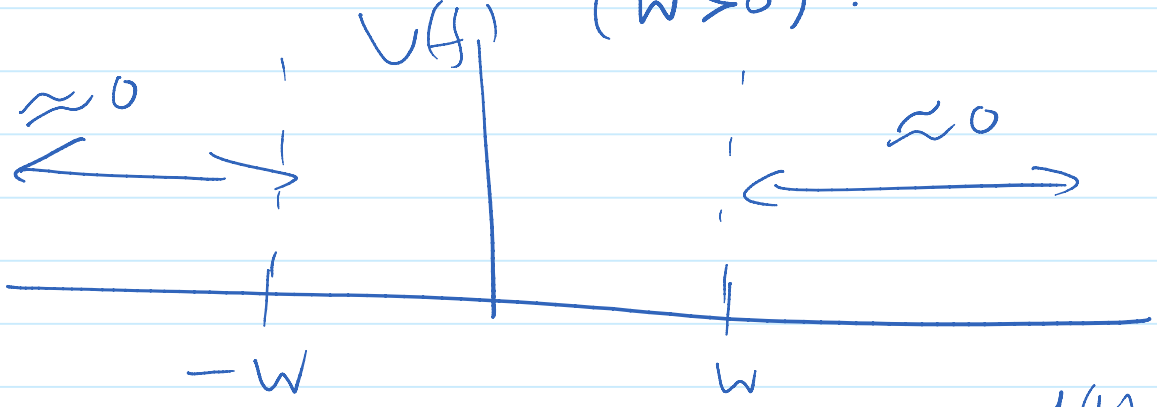
17 August 2017 16:32



Baseband signals

A signal $u(t)$ is said to be a baseband signal if the signal energy is concentrated in a band around DC ($f = 0$), and

$$U(f) \approx 0 \text{ for } |f| > W \quad (W > 0).$$



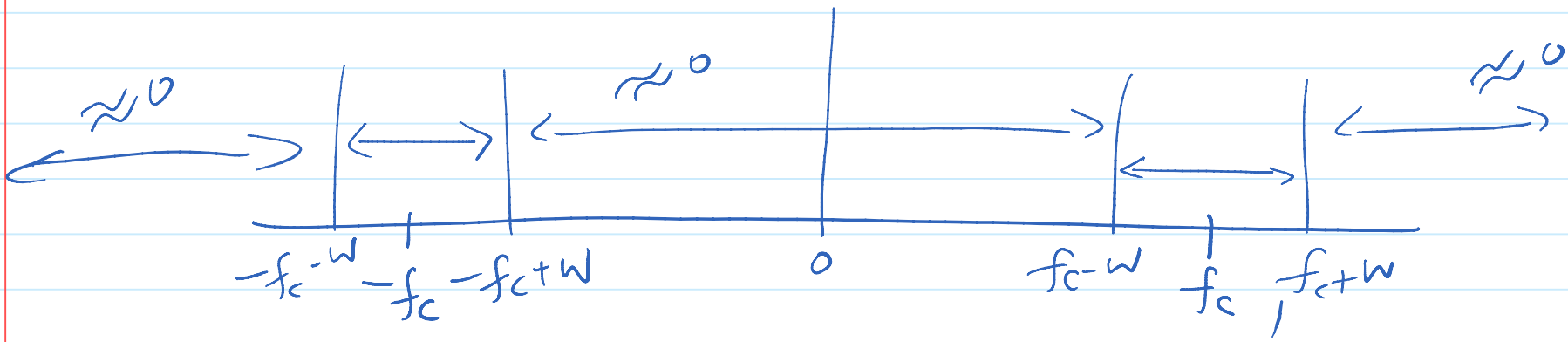
An LTI system is baseband if $h(t)$ / $H(f)$ is baseband.

Passband signals

18 August 2017 13:06

A signal $u(t)$ is said to be passband if its energy is concentrated in a band away from DC, with

$$U(f) \approx 0 \qquad \underbrace{|f \pm f_c| > W} \qquad \text{where } f_c > W > 0.$$



$f > f_c + W$, $f < f_c - W$ $\qquad$ $f + f_c > W$, $\qquad$ $(f + f_c) < -W$

$f > -f_c + W$, $f < -f_c - W$ $\qquad$ $\underline{f - f_c > W}$, $\qquad$ $\underline{(f - f_c) < -W}$

<u>Passband systems</u>: (LTI) if $h(t)$ is passband

BB, PB

18 August 2017 13:17

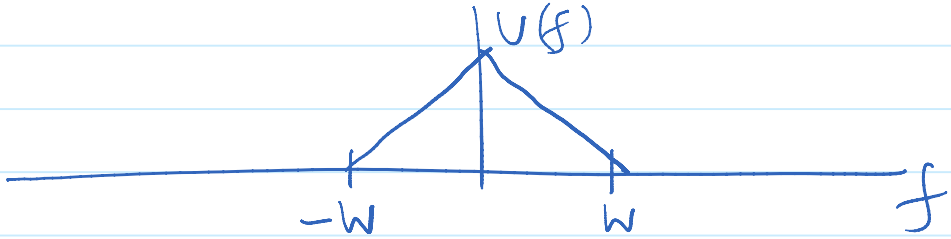
In communication, we often have:

(1) baseband message signals (eg. voice, music, etc.)
(2) passband channels (eg. wireless, AM/FM radio, satellite, ...)

Baseband channels (wired telephone)

* How do we translate baseband signals to passband and vice versa?

— Let $u(t)$ be a real baseband signal of bandwidth W



$U(-f) = U^*(f)$

PB

18 August 2017 13:23

$$u(t)\cos 2\pi f_c t = u(t)\left(\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right) \longleftrightarrow \frac{U(f-f_c)+U(f+f_c)}{2}$$

Passband signal

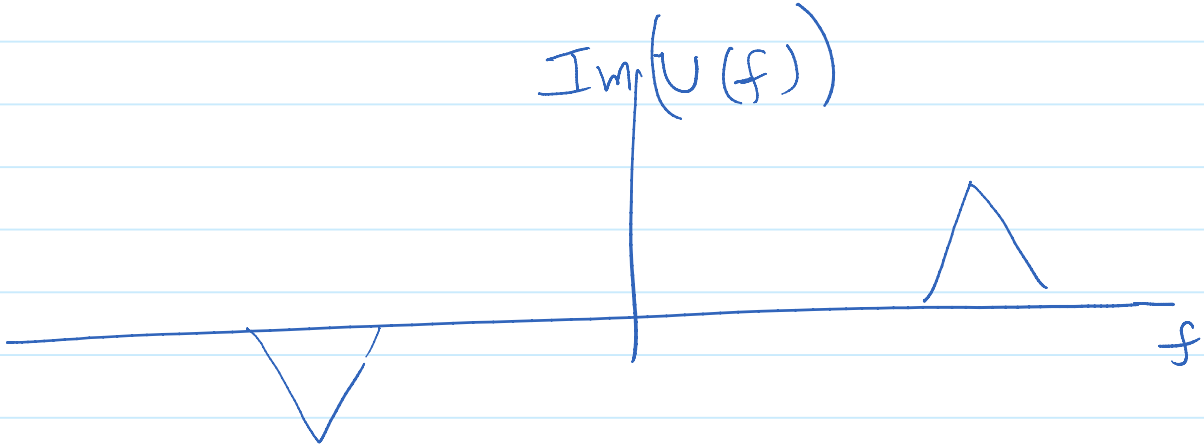
bandwidth $2W$



$$u(t)\sin 2\pi f_c t$$

passband signal

bandwidth $2W$



PB

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$u_c(t)$, $u_s(t)$: Two baseband signals (BW W)

$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$ Passband signal (BW 2W)

Questions

1) Can we get $u_c(t)$ and $u_s(t)$ back from $u_p(t)$?

2) Can any real passband signal be represented like this?

Q1) $u_p(t)$

Consider $u_p(t)\cos 2\pi f_c t$

PB

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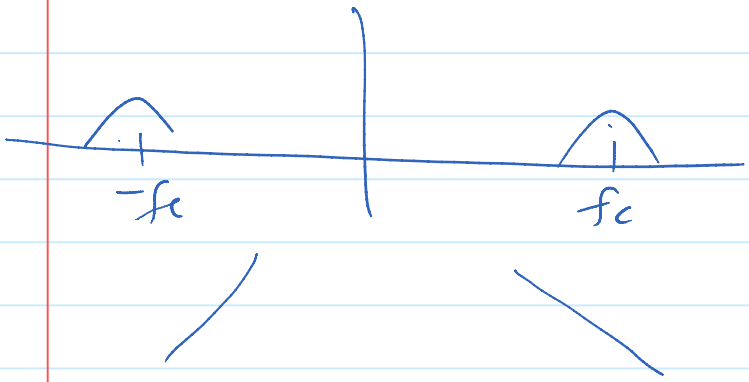
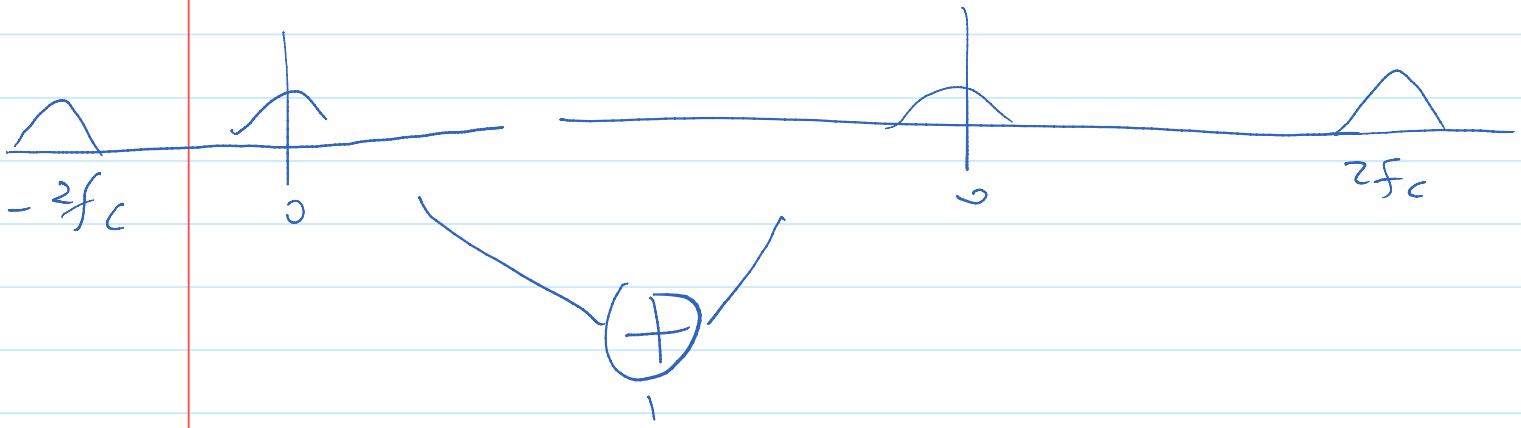
Freq domain

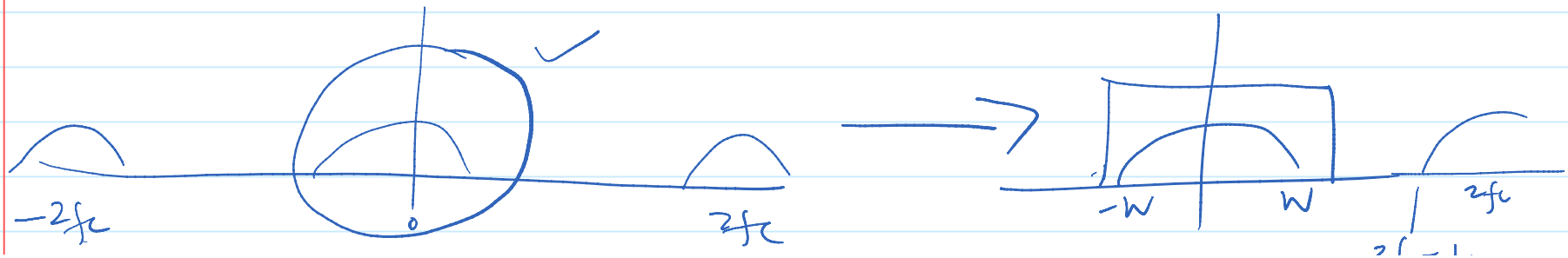


Need

$2f_c - W > W$

$\Rightarrow f_c > W$





Time domain

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

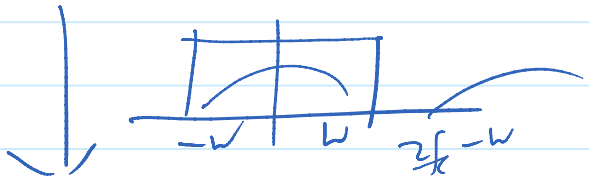
$$u_p(t)\cos 2\pi f_c t = u_c(t)\left[\frac{1+\cos 4\pi f_c t}{2}\right] - \frac{u_s(t)\sin 4\pi f_c t}{2}$$

$$= \frac{u_c(t)}{2} + \frac{u_c(t)\cos 4\pi f_c t}{2} - \frac{u_s(t)\sin 4\pi f_c t}{2}$$

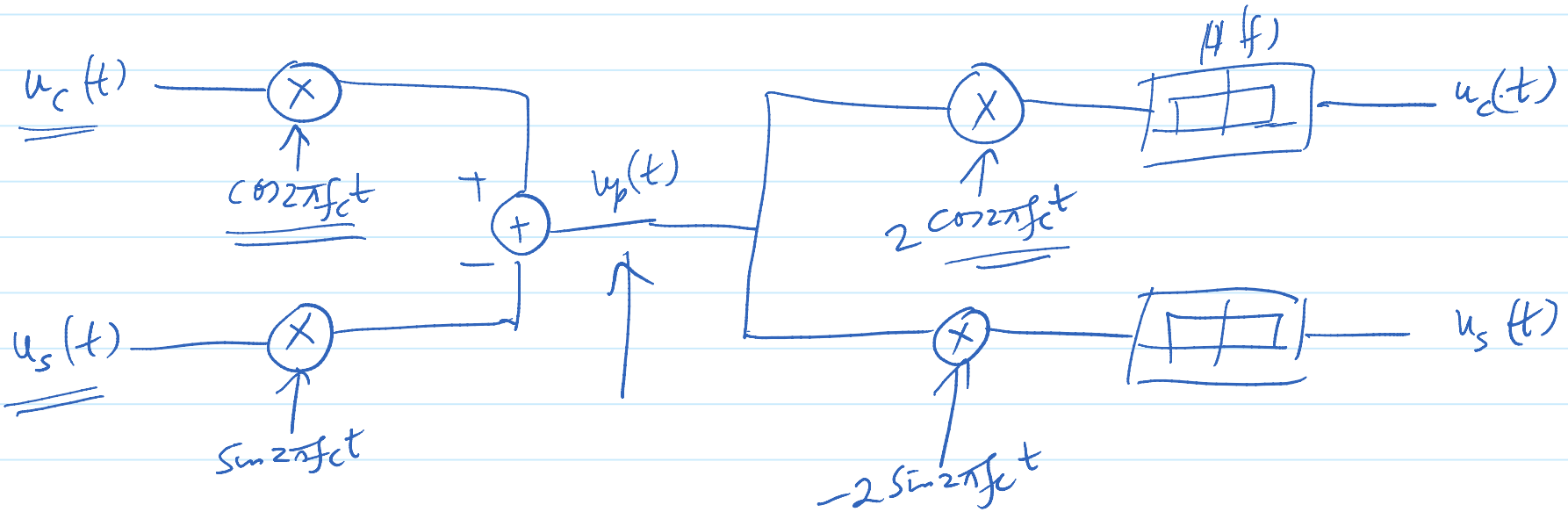
PB

18 August 2017 13:42

$$u_p(t)\sin 2\pi f_c t = \frac{u_c(t)\sin 4\pi f_c t}{2} - u_s(t)\left(\frac{1-\cos 4\pi f_c t}{2}\right)$$



$$-\frac{u_s(t)}{2}$$



Lecture 11

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$f_c > W$

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t \quad \text{(A)}$$

$u_p(t)$: real Pass band signal, BW $2W$

$u_c(t)$, $u_s(t)$: Two real baseband signals, BW W

Qn: 2) Can any real passband signal be represented like this? **Yes**

(A) can be written as

$$u_p(t) = \underbrace{e(t)}_{\text{envelope}} \cos\left(2\pi f_c t + \underbrace{\theta(t)}_{\text{phase}}\right)$$

L 11

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$$e(t) = \sqrt{u_c^2(t) + u_s^2(t)} \qquad \theta(t) = \tan^{-1}\left(\frac{-u_s(t)}{u_c(t)}\right)$$

$$\left[u_c(t) = e(t)\cos\theta(t) \;;\; u_s(t) = e(t)\sin\theta(t)\right]$$

(A) can also be written as

$$\boxed{u_p(t) = \mathrm{Re}\left\{\underset{\sim}{u}(t)\, e^{j2\pi f_c t}\right\}}$$ where

$$u(t) = u_c(t) + j\,u_s(t) = e(t)\,e^{j\theta(t)}$$

[Define complex baseband signal $\underset{\sim}{u}(t) = u_c(t) + j\,u_s(t) = e(t)e^{j\theta(t)}$

$$u(t)\,e^{j2\pi f_c t} = e(t)\,e^{j(2\pi f_c t + \theta(t))}$$]

L 11

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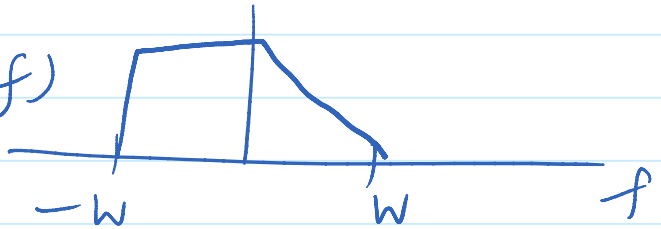
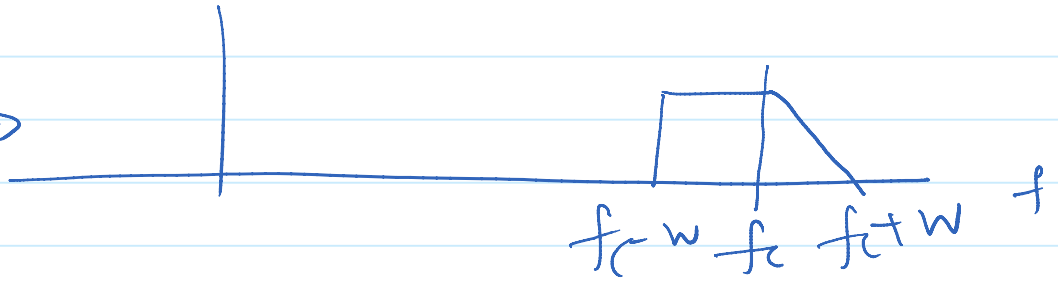
We have three equivalent representations.

Consider the 3rd. $u_p(t) = \mathrm{Re}\left\{ \underbrace{u(t)}_{\text{complex baseband signal.}} e^{j2\pi f_c t} \right\}$.

✓ $u(t) \longleftrightarrow U(f)$



$u(t) e^{j2\pi f_c t} \longleftrightarrow$



$u_p(t) = \dfrac{u(t) + u^*(t)}{2}$

$u_p(t) \longleftrightarrow \dfrac{U(f) + U^*(-f)}{2}$



L11

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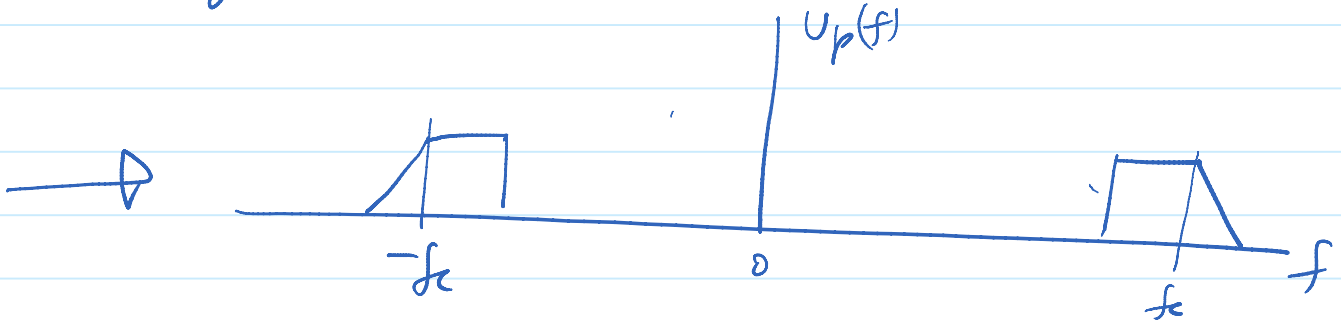
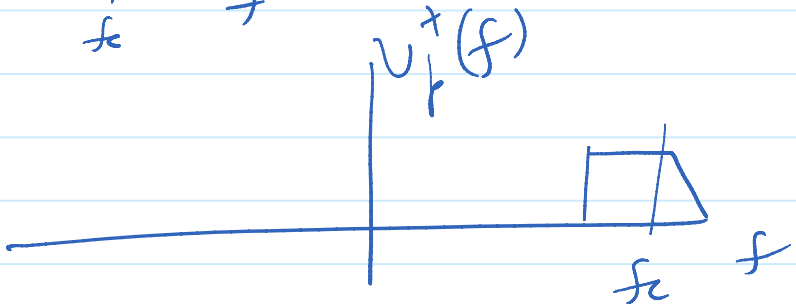
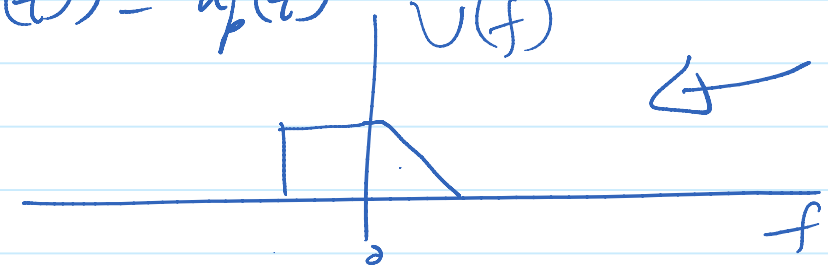
Start from an arbitrary real passband signal $u_p(t)$



Define $U_p^+(f) = \begin{cases} U_p(f) & f \geq 0 \\ 0 & f < 0 \end{cases}$



Define $C(f) = 2U_p^+(f) \rightarrow \mathrm{Re}(c(t)) = u_p(t)$



$$\underline{\underline{u(t)}} = c(t)e^{-j2\pi f_c t}$$

$$u_p(t) = \mathrm{Re}\left\{\underline{\underline{u(t)}} e^{j2\pi f_c t}\right\}$$

$\underline{\underline{u(t)}} = u_c(t) + j u_s(t)$

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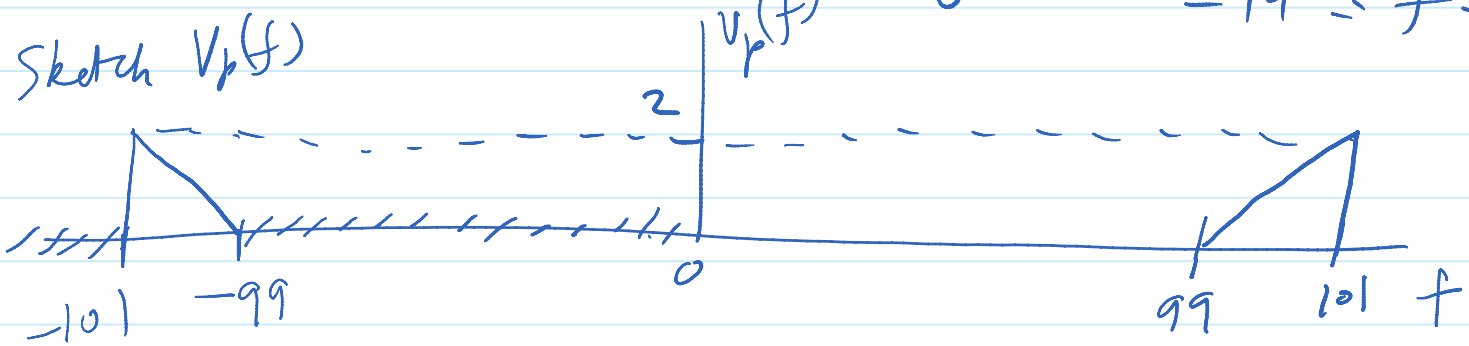
Example: (Madhow's book 2.8.4)

$v_p(t)$ is a real passband signal with $V_p(f)$ defined (for $f<0$) as

$$V_p(f) = \begin{cases} -(f+99) & -101 \le f \le -99 \\ 0 & f < -101 \\ 0 & -99 \le f \le 0. \end{cases}$$

(a) Sketch $V_p(f)$



(b) Is $v_p(t) = v_p(-t)$?

$$v_p(-t) \longleftrightarrow \int_{-\infty}^{\infty} v_p(-t)\, e^{-j2\pi f t}\, dt \qquad \text{Let } \tau = -t$$

$$= \int_{-\infty}^{\infty} v_p(\tau)\, e^{j2\pi f \tau}\, d\tau$$

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$$= \left[\int_{-\infty}^{\infty} v_p(\tau)\, e^{-j2\pi f \tau}\, d\tau \right]^* \qquad (v_p(t) \text{ real})$$

$$= V_p^*(f)$$

$$v_p(-t) \longleftrightarrow V_p^*(f)$$

Is $v_p(t) = v_p(-t)$? $\equiv$ Is $V_p(f) = V_p^*(f)$? TRUE

Yes.

(c) Sketch $V_c(f), V_s(f)$ for $f_c = 99$.

$V_p(f) \longrightarrow U(f)$



$$v_c(t) = \mathrm{Re}(u(t))$$

$$V_c(f) = \frac{U(f) + U^*(-f)}{2}$$

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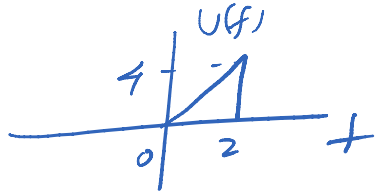
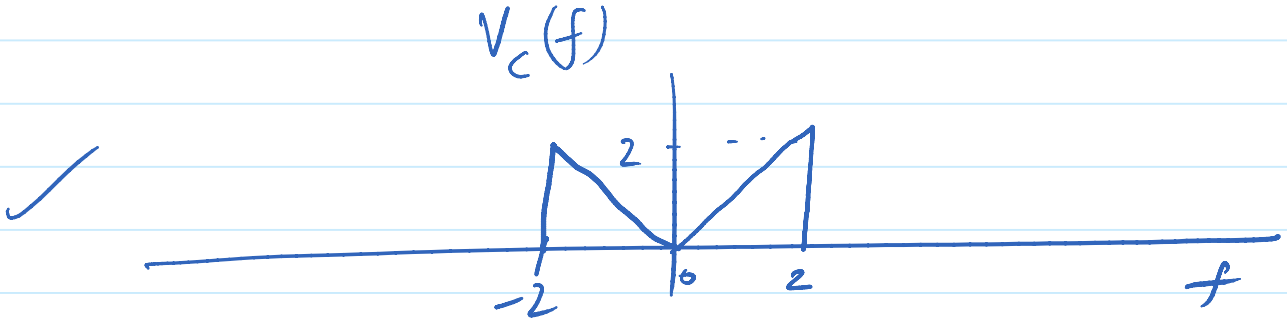


$v_c(t) + j v_s(t)$

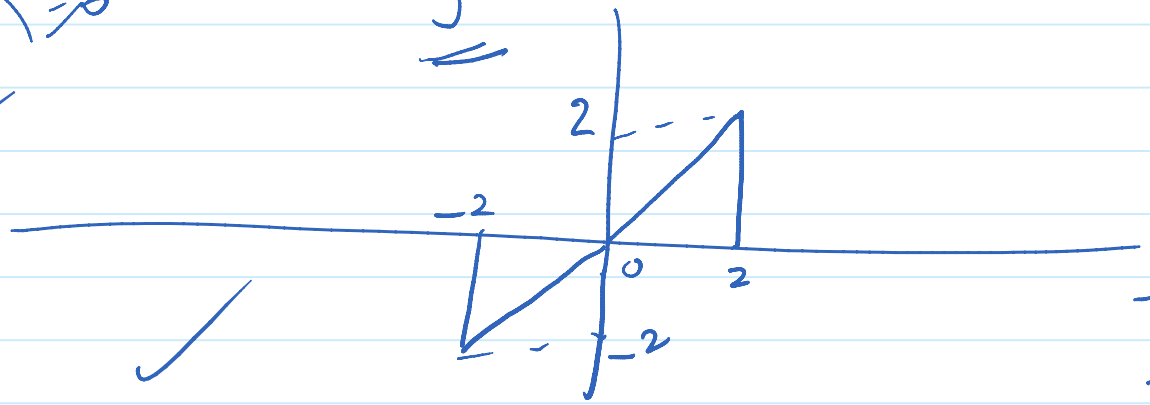
$$V_c(f) = \frac{U(f) + U^*(-f)}{2}$$

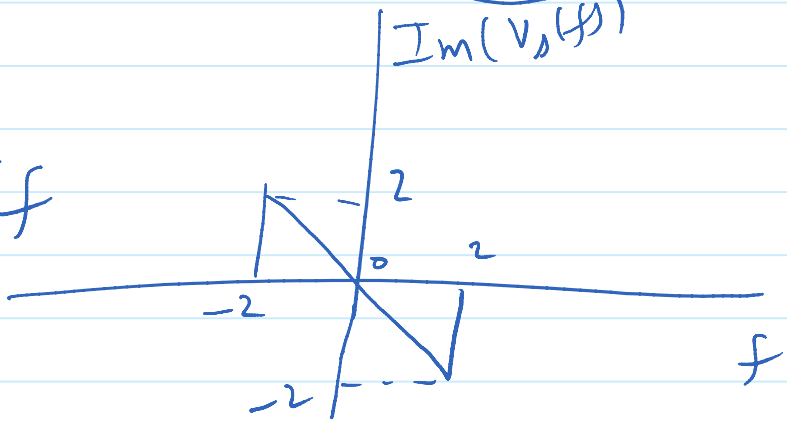


$$V_s(f) = \frac{U(f) - U^*(-f)}{2j}$$

$j V_s(f)$

$Re(V_s(f)) = 0$

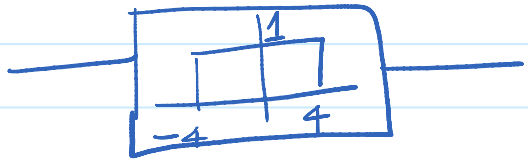




Lecture 12

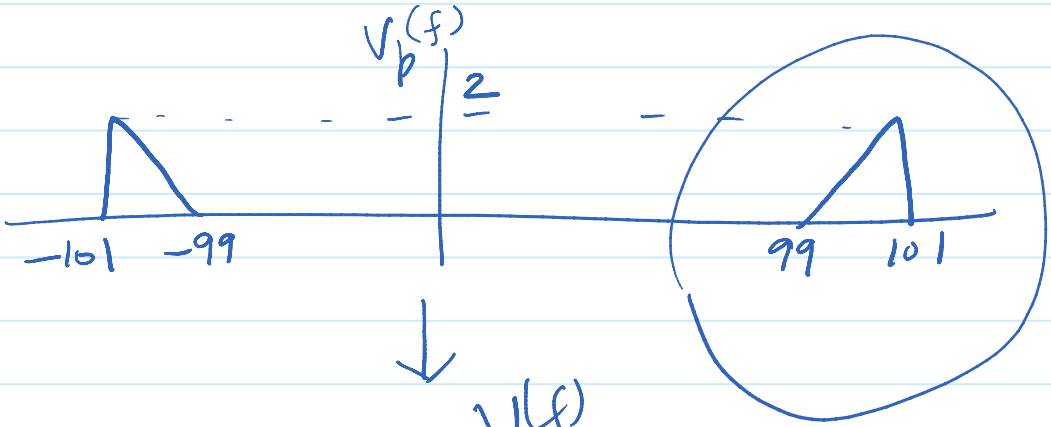
23 August 2017 07:46

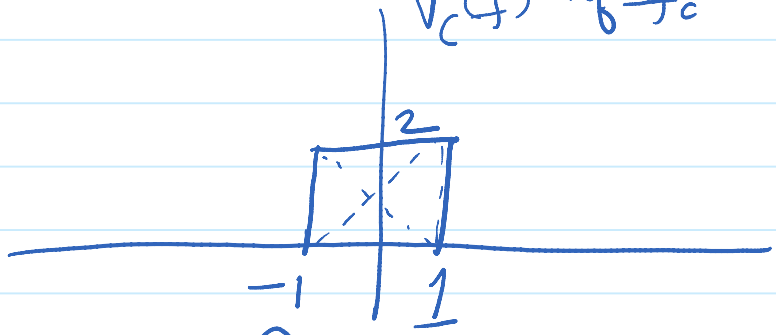
(d) $v_p(t)\cos(200\pi t)$

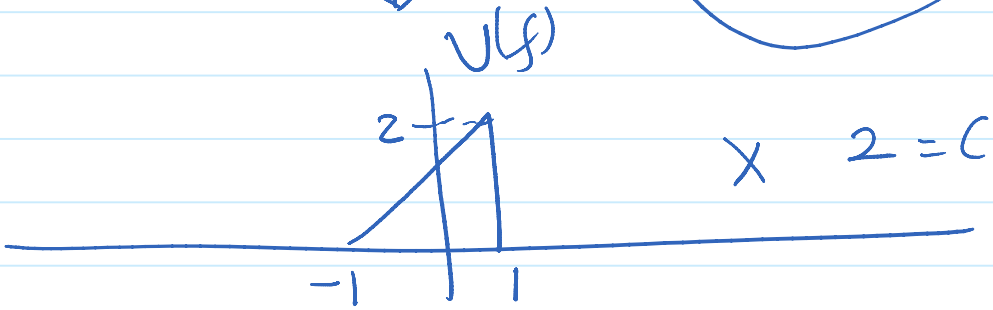


? $\frac{v_c(t)}{2} = 2\,\mathrm{sinc}(2t)$

$V_c(f)$ if $f_c = 100$







$\times\ 2 = C(f)$

$\frac{C(f) + C^*(-f)}{2}$

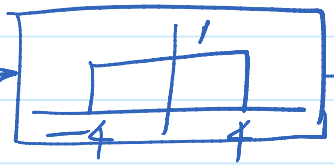
$2\,I_{[-1,1]}(f)$

$\longleftrightarrow 2(2\,\mathrm{sinc}(2f))$

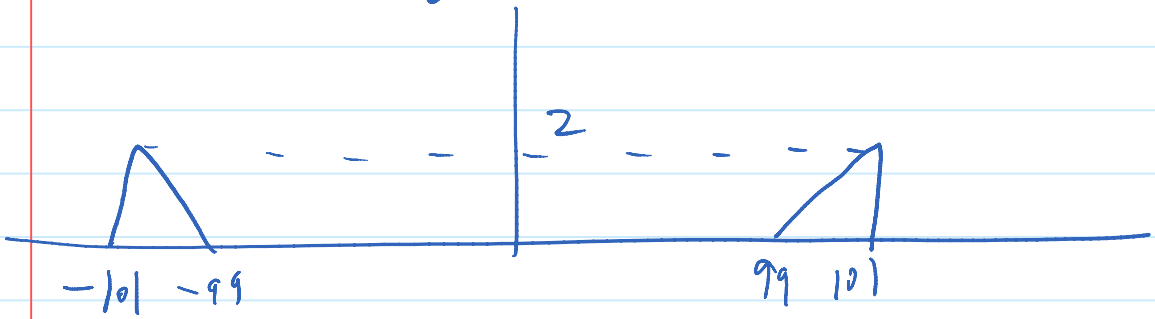
$v_c(t) = 4\,\mathrm{sinc}(2t)$

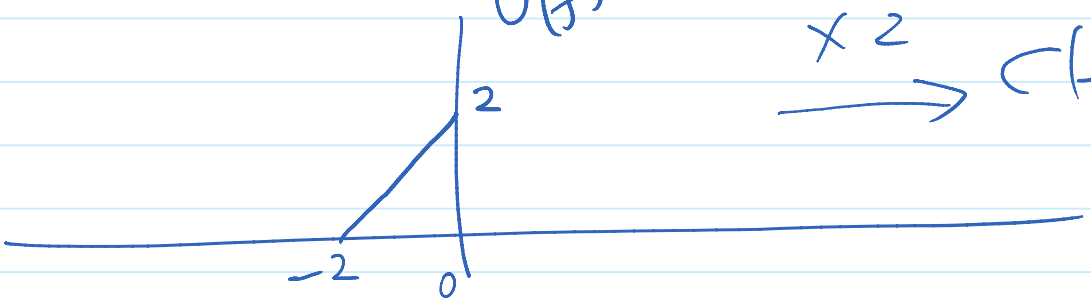
L12

23 August 2017 08:11

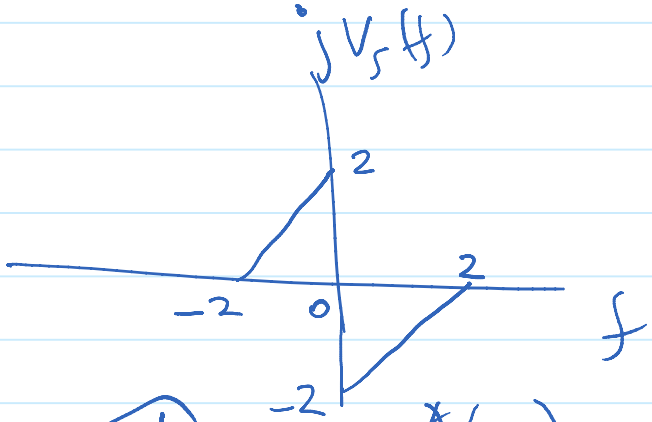
(e) $v_p(t)\,\underbrace{\sin(202\pi t)}_{\frac{e^{j202\pi t}-e^{-j202\pi t}}{2j}}$ $\longrightarrow$  $\longrightarrow$? $\dfrac{-v_o(t)}{2} = \dfrac{1-\mathrm{sinc}(4t)}{\pi t}$

taking $f_c = 101$





$\xrightarrow{\times 2} C(f)$



$\dfrac{C(f)-C^*(-f)}{2j}$

L12

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① We can define the complex baseband representation of a real passband signal using many different f_c.

② We can also introduce a phase reference θ.

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

① → Multiple representations possible with different f_c's.

② → $u_p(t) = u_c(t)\cos(2\pi f_c t + \theta) - u_s(t)\sin(2\pi f_c t + \theta)$

$u_p(t)$ real passband signal

$u_c(t) + j u_s(t)$ complex baseband equivalent signal

$u_c(t)$ in-phase component of the complex baseband signal

$u_s(t)$ quadrature component

L 12

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Take message signals ⟶ In-phase & Quadrature components

choose $f_{c,0}$ f_1, θ_1

$u_p(t)$ (Passband signal for transmission)

Use $f_{c,0}$ f_2, θ_2

Message ⟵ In-phase & quadrature components

$$u_p(t) = u_{c_1}(t)\cos(2\pi f_1 t + \theta_1) - u_{s_1}(t)\sin(2\pi f_1 t + \theta_1)$$

$$= u_{c_2}(t)\cos(2\pi f_2 t + \theta_2) - u_{s_2}(t)\sin(2\pi f_2 t + \theta_2)$$

L12

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Easy to relate $\underbrace{u_{c_1}(t) + j u_{s_1}(t)}_{\triangleq u_1(t)}$ to $\underbrace{u_{c_2}(t) + j u_{s_2}(t)}_{\triangleq u_2(t)}$

$$u_p(t) = Re\left[u_1(t) e^{j(2\pi f_1 t + \theta_1)}\right] \leftarrow$$
$$= Re\left[u_2(t) e^{j(2\pi f_2 t + \theta_2)}\right]$$

$$= Re\left[u_1(t) e^{j(2\pi f_1 - f_2)t + \theta_1 - \theta_2)} \cdot e^{j(2\pi f_2 t + \theta_2)}\right]$$

$$u_2(t) = u_1(t)\, e^{j(2\pi f_1 - f_2)t + \theta_1 - \theta_2)}$$

Let $\Delta f = f_1 - f_2$, $\Delta\theta = \theta_1 - \theta_2$ $\quad u_2(t) = u_1(t)\, e^{j(2\pi \Delta f t + \Delta\theta)}$

L12

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$$u_2(t) = u_1(t)\, e^{j(2\pi \Delta f t + \Delta\theta)}$$

$$u_2(t) = u_{c_2}(t) + j\, u_{s_2}(t) \qquad u_1(t) = u_{c_1}(t) + j\, u_{s_1}(t)$$

$$u_{c_2}(t) + j\, u_{s_2}(t) = \left(u_{c_1}(t) + j\, u_{s_1}(t)\right) e^{j(2\pi \Delta f t + \Delta\theta)}$$

$$\begin{cases} u_{c_2}(t) = u_{c_1}(t)\cos(2\pi \Delta f t + \Delta\theta) - u_{s_1}(t)\sin(2\pi \Delta f t + \Delta\theta) \\ u_{s_2}(t) = u_{c_1}(t)\sin(2\pi \Delta f t + \Delta\theta) + u_{s_1}(t)\cos(2\pi \Delta f t + \Delta\theta) \end{cases}$$

Example

$$u_p(t) = I_{[-1,1]}(t)\cos(400\pi t)$$

→ Not exactly bandlimited
But Approximately bandlimited.

L12

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(a) Find the output when $u_p(t)\cos(401\pi t)$ is passed thro' a lowpass filter $\longrightarrow \frac{1}{2}u_{c_2}(t)$.

$$u_{c_1}(t) = I_{[-1,1]}(t)$$

$$u_{c_2}(t) = \underbrace{I_{[-1,1]}(t)}\cos(\pi t) \quad (\approx \text{approx BW})$$

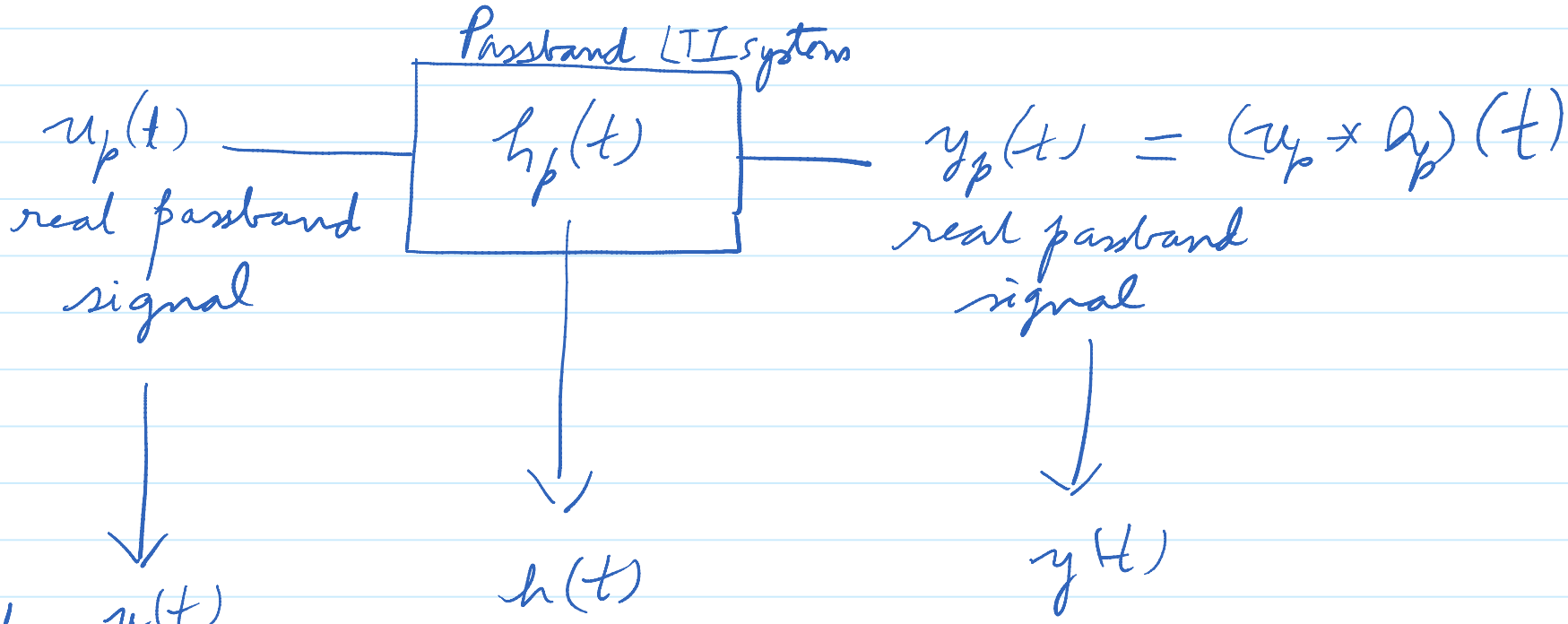
(b) Find the output when $u_p(t)\sin(400\pi t - \pi/4)$ is passed thro' a lowpass filter

$$u_p(t) = \mathrm{Re}\left[\underbrace{I_{[-1,1]}(t)e^{j\pi/4}}\ \underbrace{e^{j(400\pi t - \pi/4)}}\right]$$

$$\text{output } u_{s_2}(t) = -\frac{1}{2}I_{[-1,1]}(t)\sin\frac{\pi}{4} = -\frac{1}{2\sqrt{2}}I_{[-1,1]}(t)$$

Lecture 13

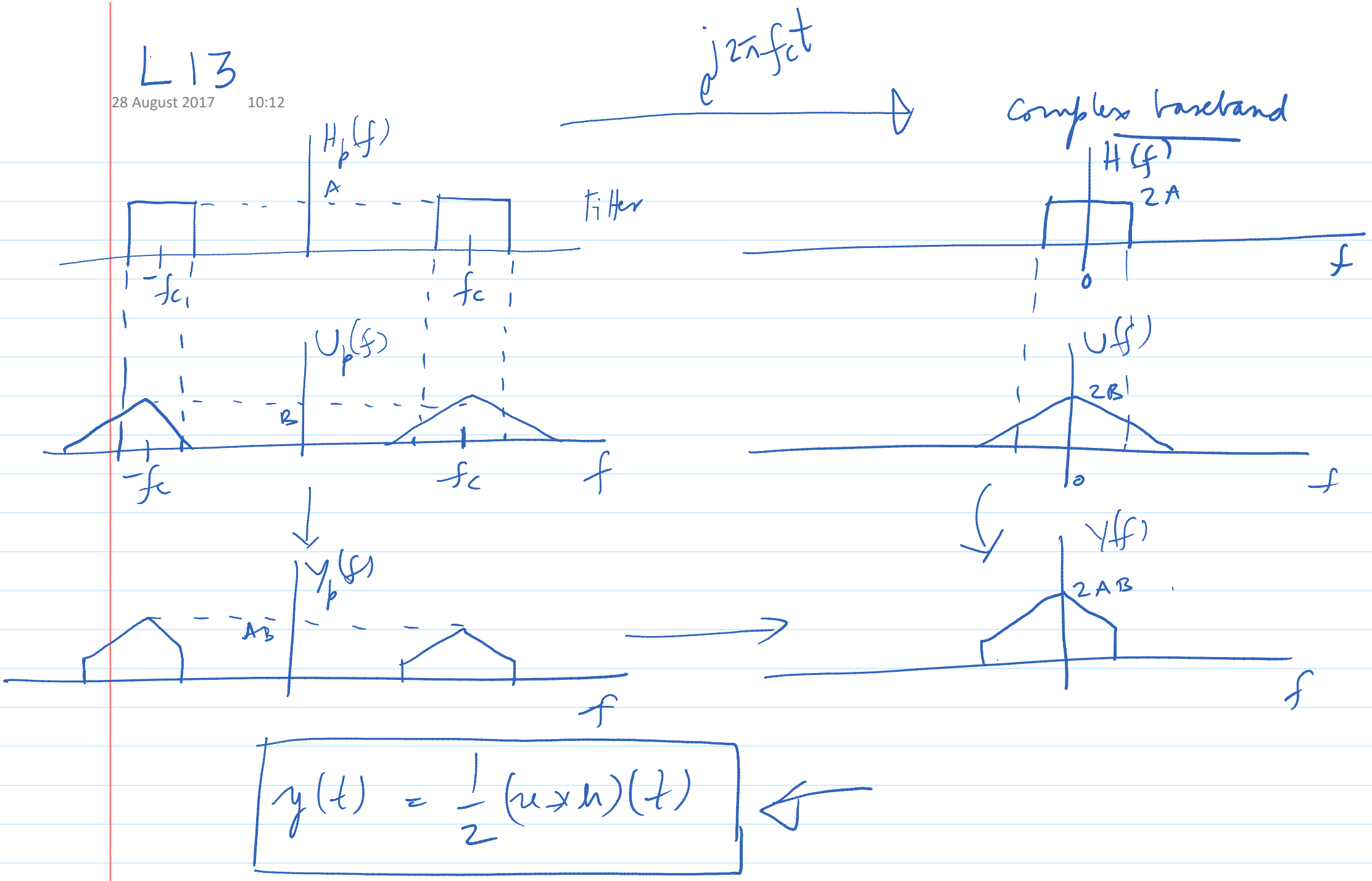
28 August 2017 09:09



How are $y(t)$, $u(t)$ and $h(t)$ related?

L 13

28 August 2017 10:12



$$y(t) = \frac{1}{2}(u * h)(t)$$

L13

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$y_c(t) + j y_s(t) = y(t) = \frac{1}{2}(u * h)(t)$

$u = u_c + j u_s$

$h = h_c + j h_s$

$$y_c(t) = \frac{1}{2}\left((u_c * h_c)(t) - (u_s * h_s)(t)\right)$$

$$y_s(t) = \frac{1}{2}\left((u_s * h_c)(t) + (u_c * h_s)(t)\right)$$

Any Real passband signal $\Rightarrow$ Equivalent complex baseband signal.

$$u_p(t) = \mathrm{Re}\left[\underbrace{u(t)}_{u_c(t) + j u_s(t)} e^{j 2\pi f_c t}\right]$$

What we did not discuss?

$$\mathrm{Im}\left[u(t) e^{j 2\pi f_c t}\right] = \underbrace{u_c(t) \sin 2\pi f_c t + u_s(t) \cos 2\pi f_c t}_{\text{Hilbert transform of } u(t)}$$

L13

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Energy and power of passband & equivalent complex baseband signals

$$\text{Energy} = \int_{-\infty}^{\infty} u_p^2(t)\,dt = \|u_p\|^2$$

$$u_c(t) + j\,u_s(t) = u(t)$$

$$\text{Energy} = \int_{-\infty}^{\infty} |u(t)|^2\,dt = \|u\|^2$$

$$= \|u_c\|^2 + \|u_s\|^2$$

$$\boxed{\|u_p\|^2 = \frac{1}{2}\|u\|^2}$$

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

L13

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$$\int_{-\infty}^{\infty} u_p^2(t)\,dt = \int_{-\infty}^{\infty} \left(u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t\right)^2 dt$$

$$= \int_{-\infty}^{\infty} u_c^2(t)\cos^2 2\pi f_c t\,dt + \int_{-\infty}^{\infty} u_s^2(t)\sin^2 2\pi f_c t\,dt \;\rightarrow\; \frac{\|u_s\|^2}{2} + (\,)$$

$$- 2\int_{-\infty}^{\infty} u_c(t)u_s(t)\cos 2\pi f_c t \sin 2\pi f_c t\,dt$$

$$\int_{-\infty}^{\infty} u_c^2(t)\left(\frac{1+\cos 4\pi f_c t}{2}\right)dt = \frac{\|u_c\|^2}{2} + \left(\quad\right)$$

$U_c(f)$ Bandlimited BW W_1 $U_s(f)$ Bandlimited BW W_2.

L13

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$$u_c^2(t) \longleftrightarrow (V_c * V_c)(f) \quad BW \le 2W_1$$

$$u_s^2(t) \longleftrightarrow (V_s \times V_s)(f) \quad BW \le 2W_2$$

$$u_c(t)\, u_s(t) \longleftrightarrow (V_c * V_s)(f) \quad BW \le W_1 + W_2$$

Assume $2f_c > W_1 + W_2$, $2f_c > 2W_1$, $2f_c > 2W_2$.

$$\int_{-\infty}^{\infty} (u_c^2(t)) \cos 4\pi f_c t \, dt = 0$$

Correlation between two signals

$$\langle u_p, v_p \rangle = \frac{1}{2} \mathrm{Re}\left[\langle u, v \rangle\right]$$

2 real passband signals

2 complex baseband equivalent signals.

L13

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$$\langle u_p, v_p \rangle = \int_{-\infty}^{\infty} u_p(t) v_p(t)\, dt$$

$$= \int_{-\infty}^{\infty} (u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t)\,(v_c(t)\cos 2\pi f_c t - v_s(t)\sin 2\pi f_c t)\, dt$$

$$= \frac{1}{2}\langle u_c, v_c \rangle + \frac{1}{2}\langle u_s, v_s \rangle = \frac{1}{2}\mathrm{Re}\left[\langle u, v \rangle\right]$$

Lecture 14

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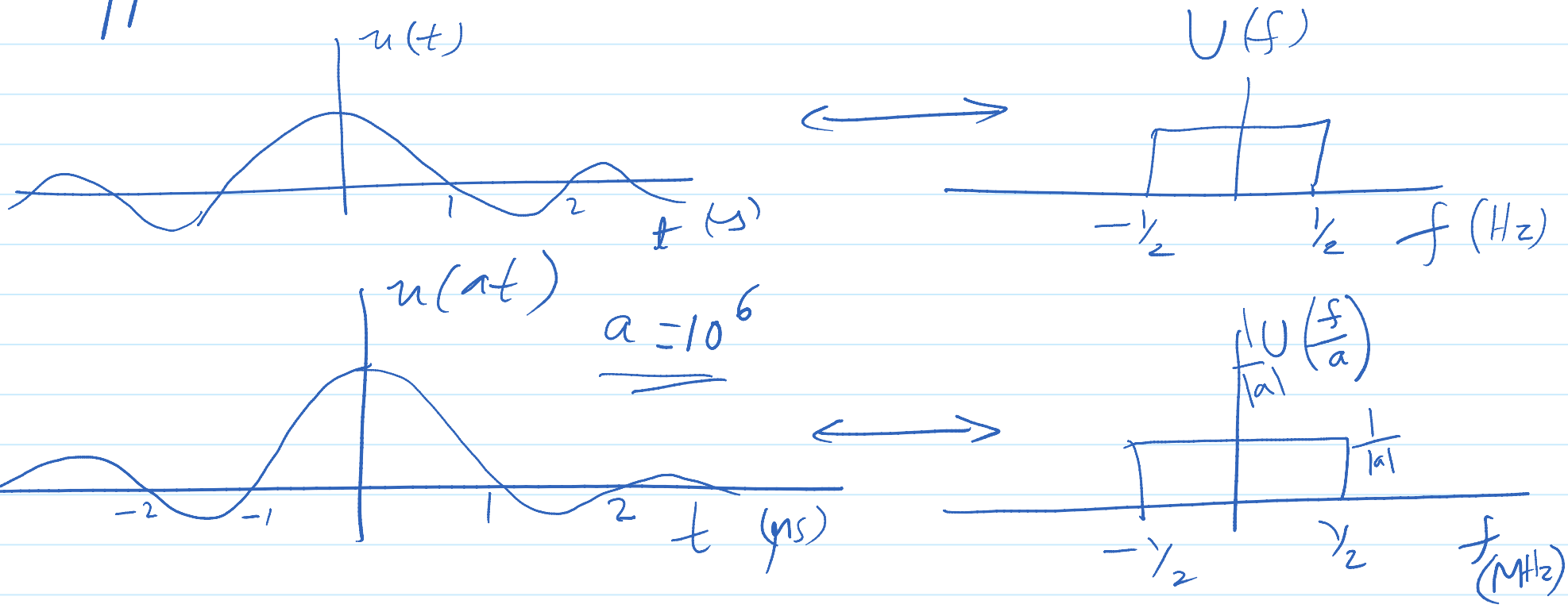
Time-scaling

If $u(t) \longleftrightarrow U(f)$,

then $u(at) \longleftrightarrow \frac{1}{|a|} U\left(\frac{f}{a}\right)$

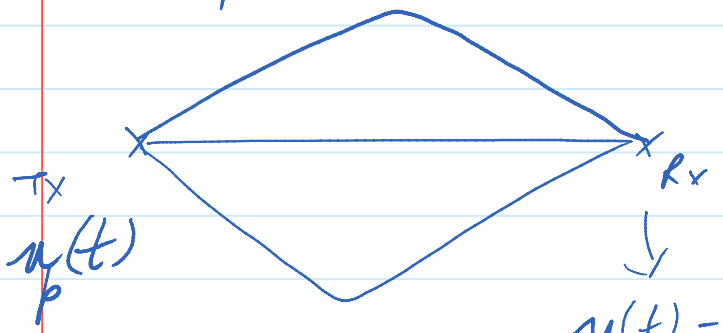
Suppose we scale the time axis



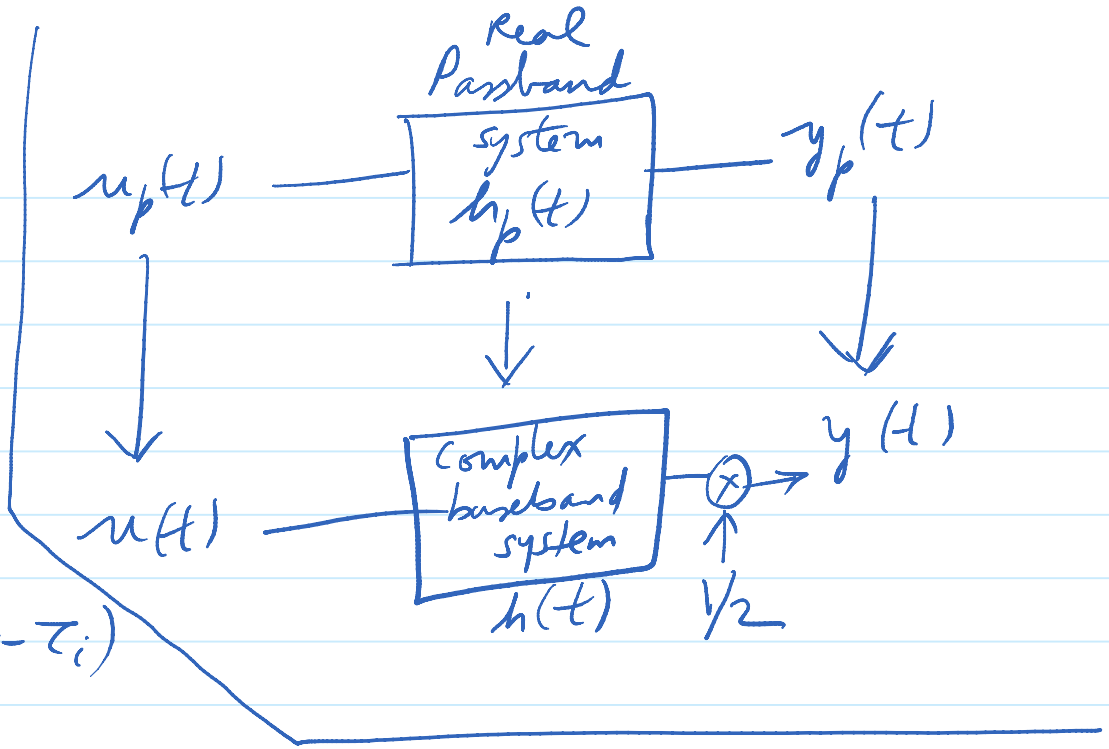
L14

29 August 2017 09:11

Multipath channel.



$$y(t) = \sum_{i=1}^{m} \alpha_i \, u_p(t - \tau_i)$$



Wireless channel modeling in complex baseband

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

$$= e(t)\cos(2\pi f_c t + \theta(t))$$

$$u(t) = e(t)\, e^{j\theta(t)}$$

L14

29 August 2017 09:19

(For each path) Attenuation factor $\propto \frac{1}{r}$ ← distance covered

Phase shift (reflections)

$$\text{Delay} = \frac{r}{c}$$

Passband received signal for path i

$$\underbrace{\underbrace{\frac{A_i}{r_i}}_{\text{Attenuation}} e(t-\tau_i) \cos\left(2\pi f_c (t-\tau_i) + \theta(t-\tau_i) + \phi_i\right)}_{\text{delay}}.$$

Phase shift

$$\sum_i \left(\quad \right)$$ overall passband received signal.

L14

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$$\frac{A_i}{r_i}\, e(t-\tau_i)\cos\left(2\pi f_c(t-\tau_i)+\theta(t-\tau_i)+\phi_i\right)$$

Complex baseband equivalent received signal:

$$\sum_i \frac{A_i}{r_i}\, u(t-\tau_i)\, e^{-j(2\pi f_c \tau_i + \phi_i)}$$

Complex baseband channel model $h(t)$

$$h(t) = \sum_i \frac{A_i}{r_i} e^{-j(2\pi f_c \tau_i + \phi_i)} \delta(t-\tau_i)$$

complex

$$H(f) = \sum_i \frac{A_i}{r_i} e^{-j(2\pi f_c \tau_i + \phi_i)} e^{-j2\pi f \tau_i}$$

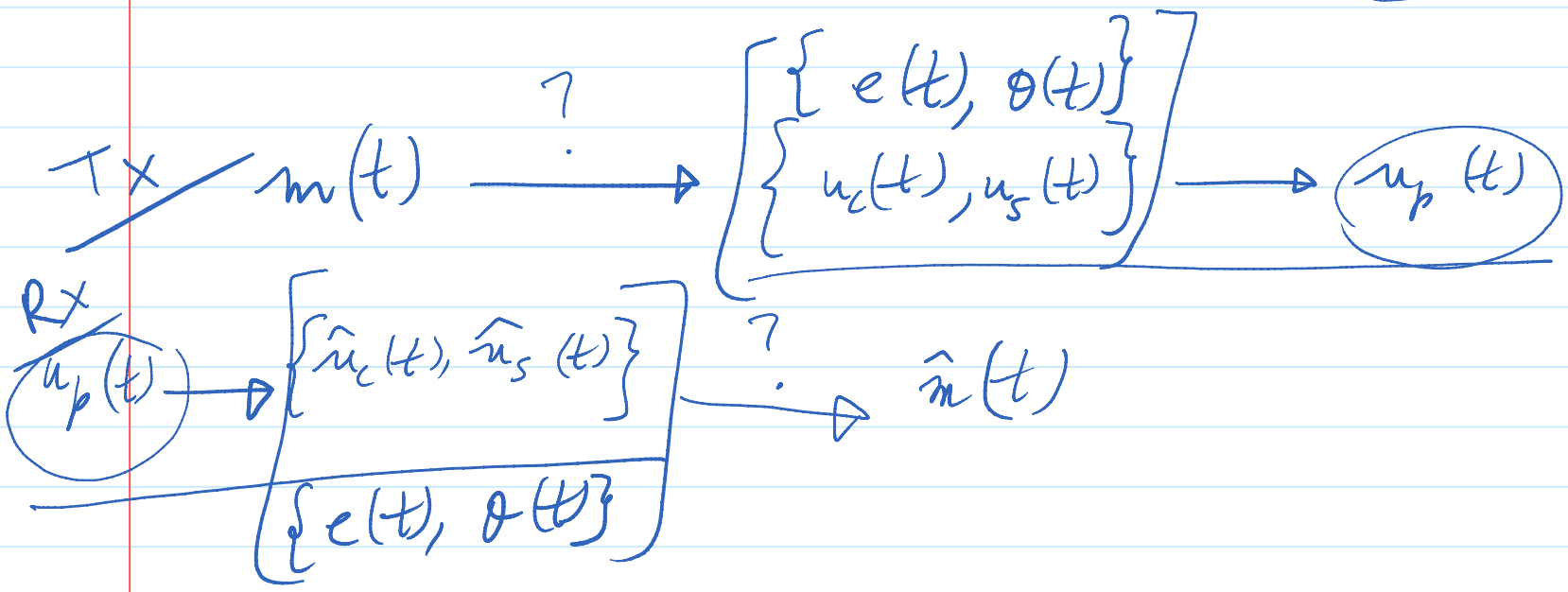
L14 Analog communication techniques

29 August 2017 09:29

"Message" signal $m(t)$: bandlimited / approx. bandlimited

Baseband.

Channel : (AM/FM radio, TV, ...) Passband, bandlimited



L14

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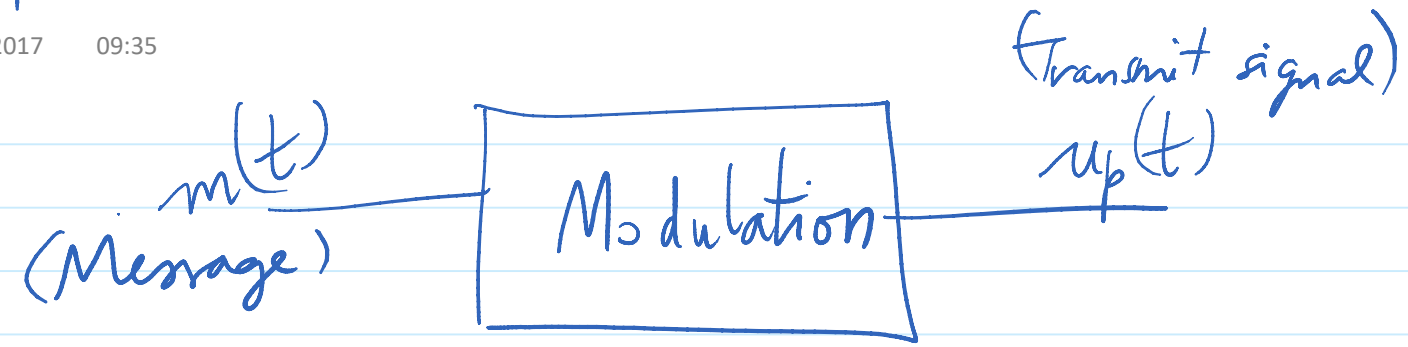
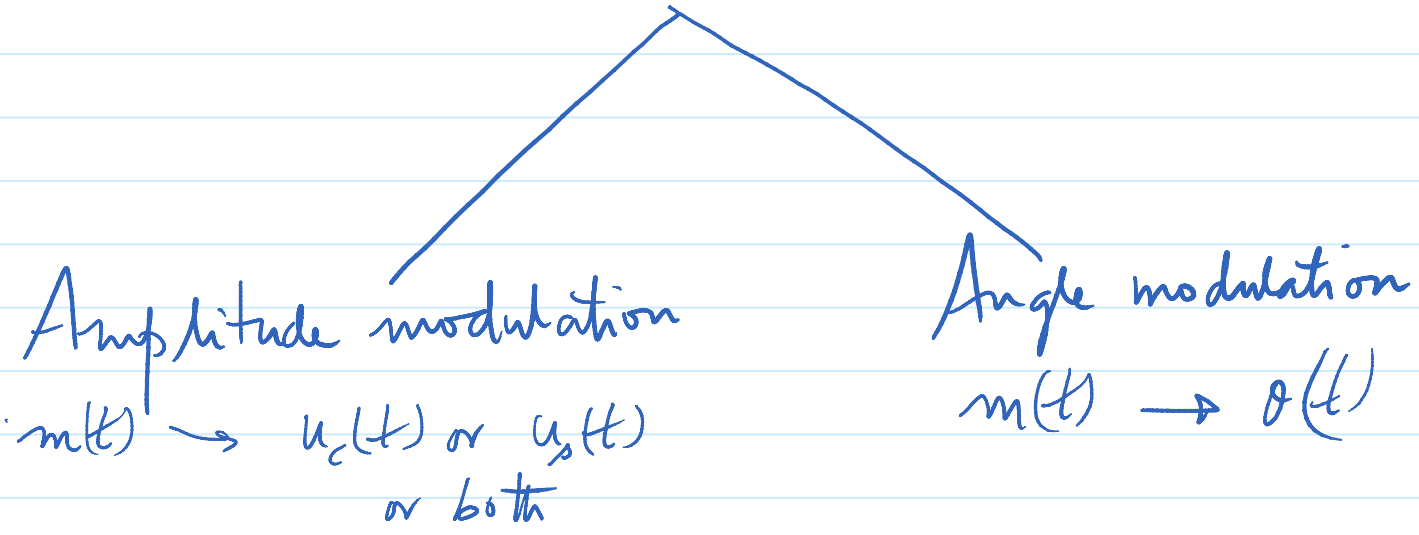


Modulation: Process of encoding the information signal into the transmit signal

Two broad classes



L 14

29 August 2017 09:39

Amplitude modulation (AM)

* Number of variants

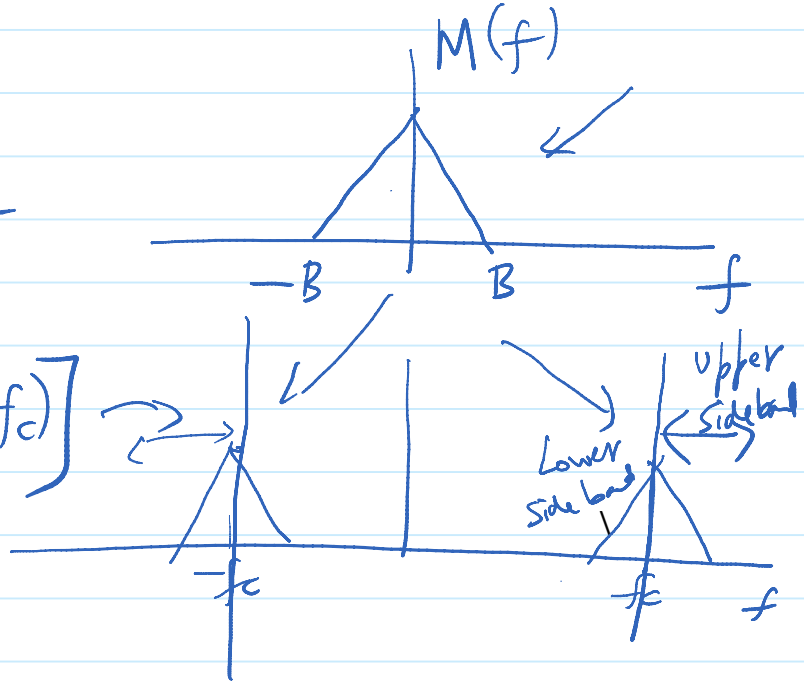
1) Double side band suppressed carrier (DSB-SC) AM

$u_c(t) = Am(t)$, $u_s(t) = 0$.

$$u_p(t) = Am(t)\cos 2\pi f_c t$$

$$\updownarrow$$

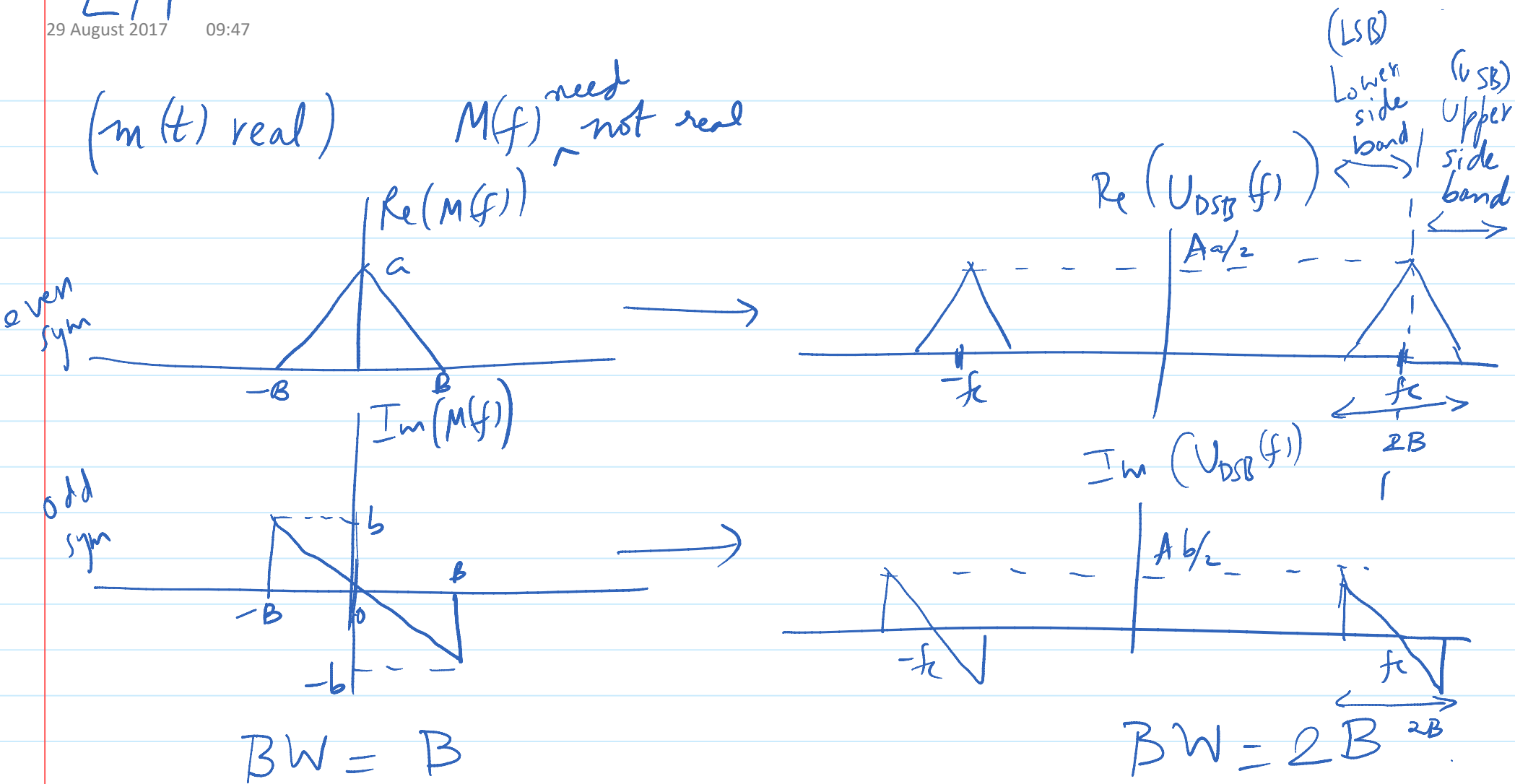
$$\frac{A}{2}\left[M(f-f_c) + M(f+f_c)\right]$$



L 14

29 August 2017 09:47

(m(t) real) M(f) need not real



BW = B

BW = 2B

Lecture 15

31 August 2017 09:41

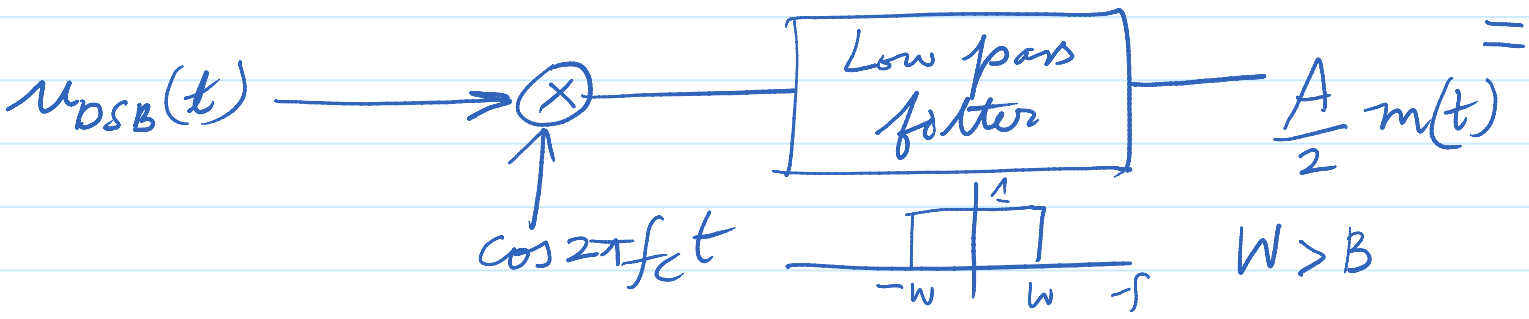
(1) If $m(t)$ has bandwidth B Hz, Then $u_{DSB}(t)$ has bandwidth $2B$ Hz

(2) If $m(t)$ is real, the upper side band of $u_{DSB}(t)$ can be determined from the lower side band (LSB) of $u_{DSB}(t)$. (and viceversa).

Demodulation of DSB-SC AM:

→ Need to recover the in-phase (I) component $u_c(t)$ of $u_{DSB}(t)$.

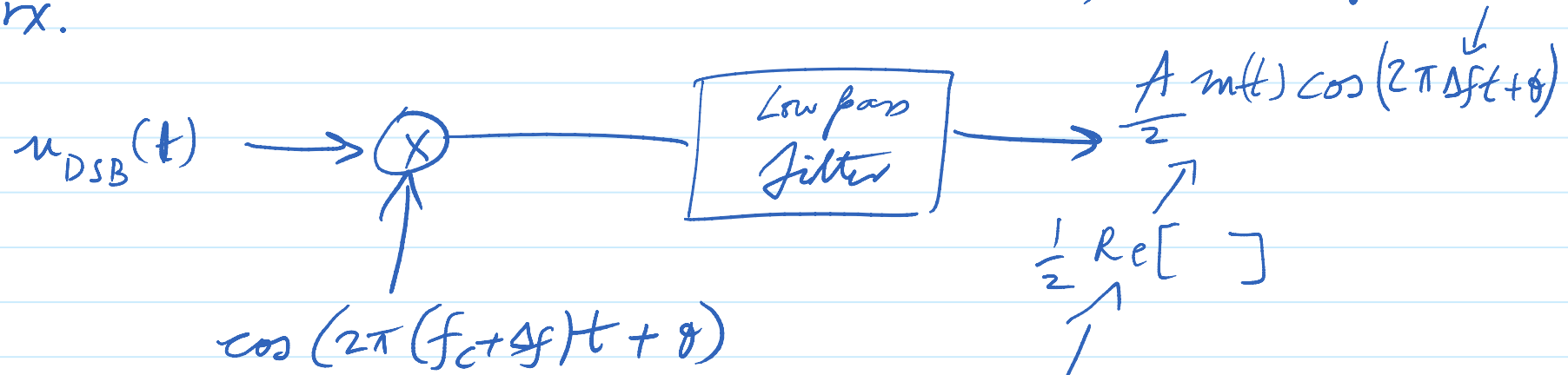
$u_c(t) = Am(t)$



L15

01 September 2017 13:10

This demodulation scheme requires $\cos 2\pi f_c t$ exactly at the rx.



$$u_{DSB}(t) = Re\left[A\, m(t)\, e^{j2\pi f_c t}\right]$$

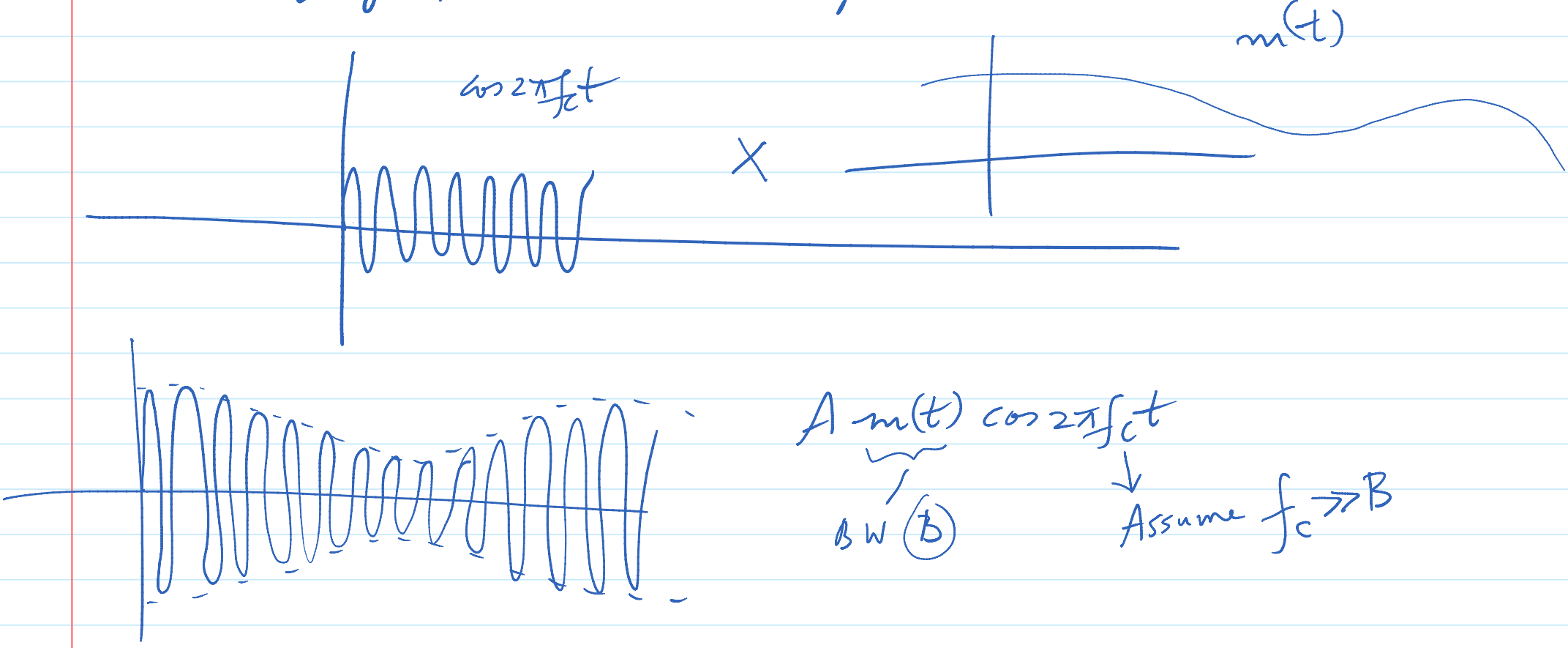
$$= Re\left[\underbrace{A\, m(t)\, e^{-j2\pi \Delta f t}\, e^{-j\theta}}\cdot \underbrace{e^{j(2\pi(f_c+\Delta f)t+\theta)}}\right]$$

Coherent demodulation : Needs the oscillator at the rx to be synchronized with the oscillator at the tx

L15

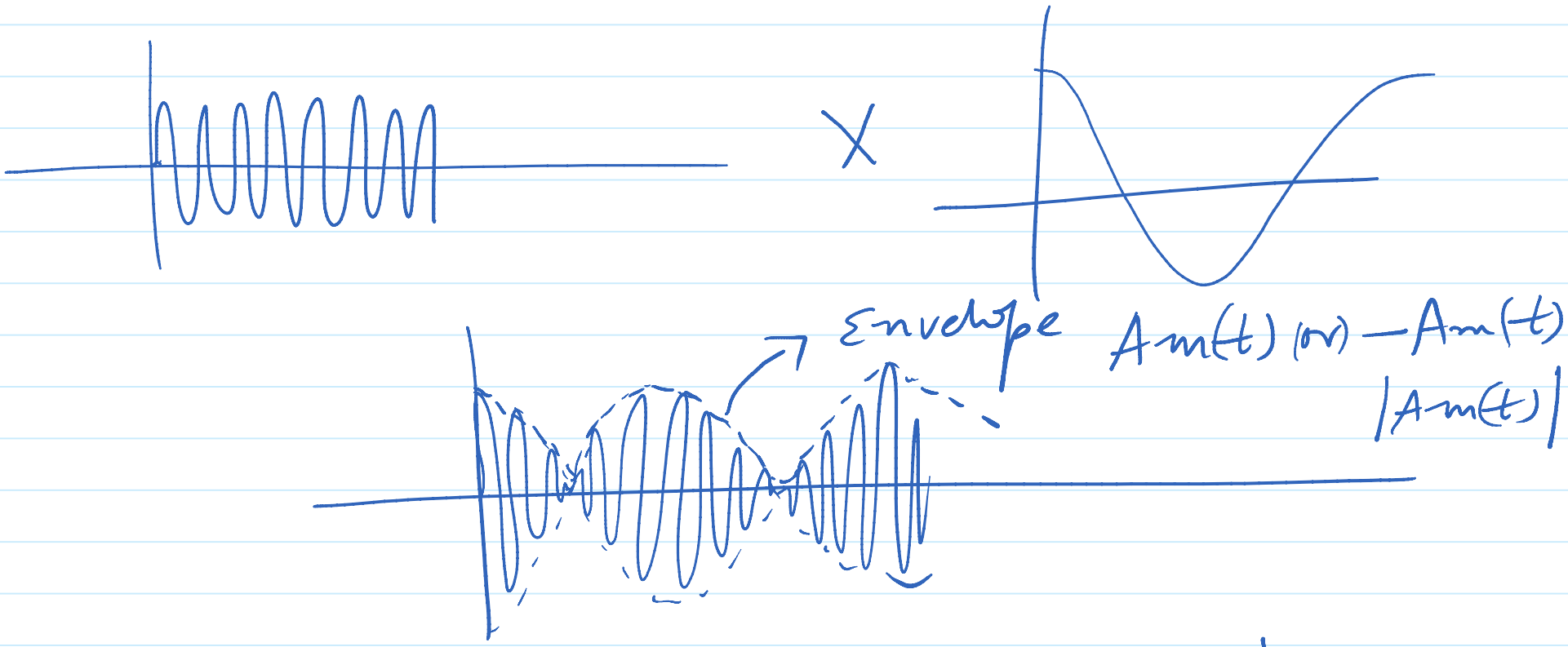
01 September 2017 13:15

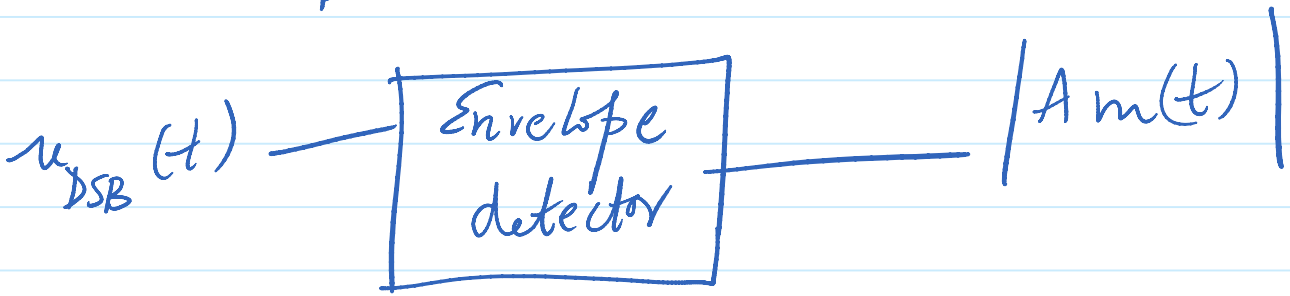
Visualizing the DSB-SC signal $u_{DSB}(t)$.



L15

01 September 2017 13:20





L15

01 September 2017 13:25

(2) Conventional AM

$$u_{AM}(t) = u_c(t)\cos 2\pi f_c t = A\,m(t)\cos 2\pi f_c t + A_c \cos 2\pi f_c t$$

$u_c(t) = A\,m(t) + A_c$

$A\,m(t)\cos 2\pi f_c t$: DSB-SC signal

$A_c \cos 2\pi f_c t$: Carrier

$$U_{AM}(f) = U_{DSB}(f) + \frac{A_c}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right)$$

Envelope $e(t) = |A\,m(t) + A_c|$

L15

01 September 2017 13:31

* How should we choose A_c to enable the use of envelope detection?

Need $A\,m(t) + A_c \geq 0 \quad \forall t$

Let $\min_t m(t) = -M_0$ where $M_0 > 0$.

$$-M_0 A + A_c \geq 0 \Rightarrow \boxed{\frac{A M_0}{A_c} \leq 1}$$

$\frac{A M_0}{A_c} \triangleq a_{mod}$ — Modulation index

$$u_{AM}(t) = (A\,m(t) + A_c)\cos 2\pi f_c t$$

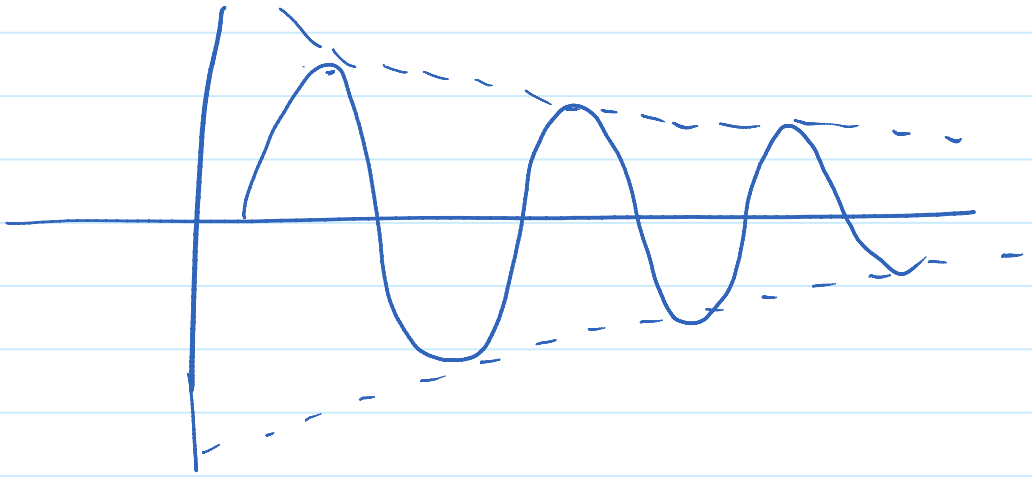
$$= A_c\left(\frac{A}{A_c} m(t) + 1\right)\cos 2\pi f_c t = A_c\left(1 + a_{mod}\frac{m(t)}{M_0}\right)\cos 2\pi f_c t.$$

L15

01 September 2017 13:39

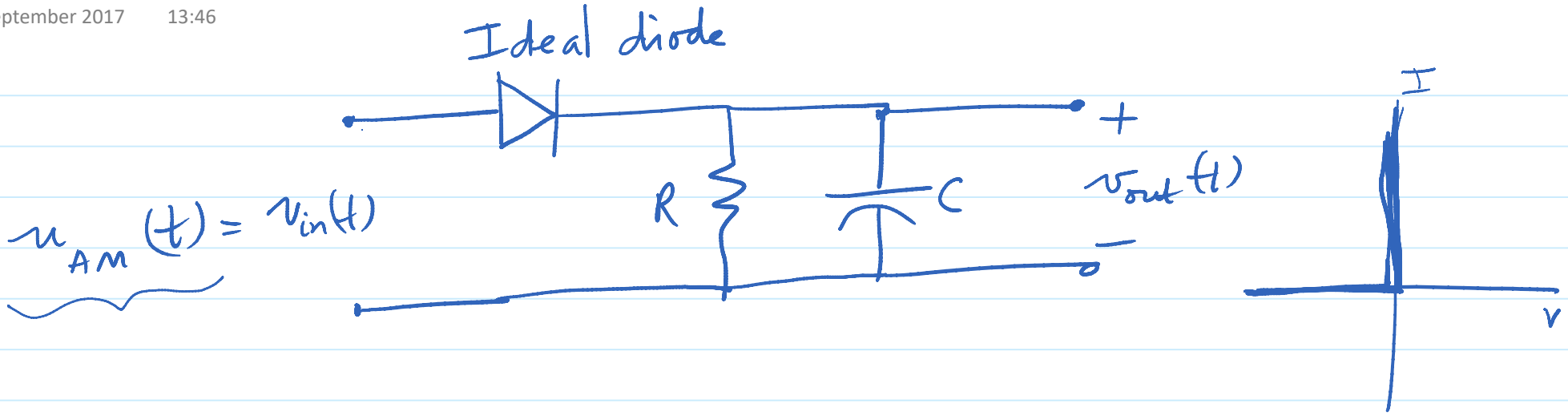
* Envelope detection is easier to implement than coherent demodulation
* But, a significant fraction of the transmit power is used to transmit $A_c \cos 2\pi f_c t$ which does not depend on the message. May be acceptable for broadcast radio where the # receivers is large for each transmitter.

Envelope detection



L15

01 September 2017 13:46

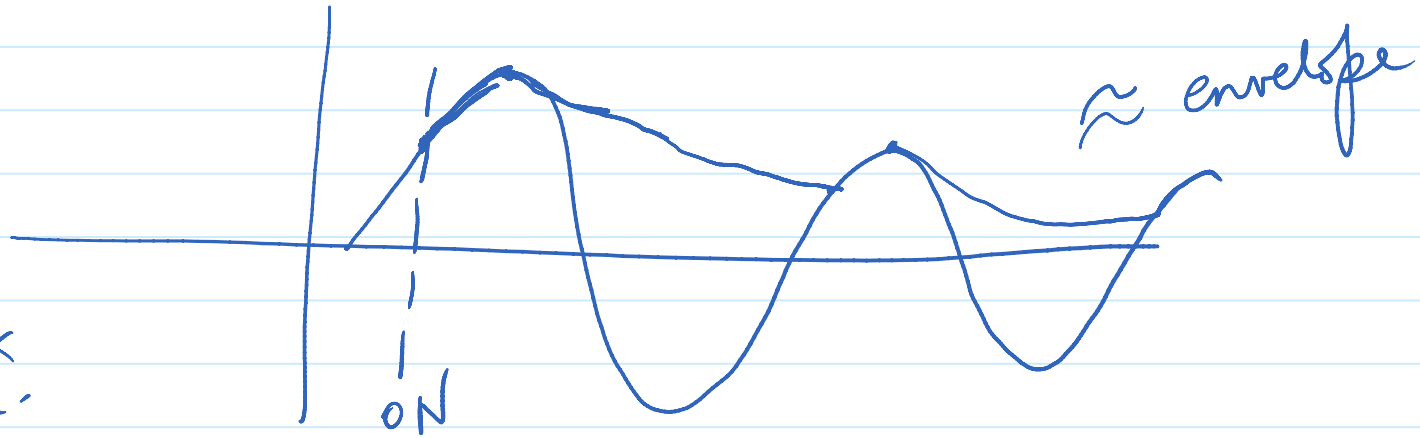


$u_{AM}(t) = v_{in}(t)$

$\frac{1}{B} \gg$ Time constant $(RC) \gg \frac{1}{f_c}$

to track $m(t)$

to not track the carrier.



Lecture 16

04 September 2017 10:00

DSB-SC AM $u_{DSB}(t) = A\, m(t) \cos 2\pi f_c t$

Conventional AM $u_{AM}(t) = (A\, m(t) + A_c) \cos 2\pi f_c t$

Power efficiency of conventional AM:

$$\text{Power efficiency} = \frac{\text{Power of the part of the signal that carries the message}}{\text{Power of } u_{AM}(t)}$$

$$u_{AM}(t) = (A\, m(t) + A_c) \cos 2\pi f_c t = A_c \left(1 + \underbrace{\frac{A M_0}{A_c}}_{a_{mod}} \underbrace{\frac{m(t)}{M_0}}_{m_n(t)}\right) \cos 2\pi f_c t$$

$$\left[-M_0 = \min_t m(t) \;;\; \frac{m(t)}{M_0} = m_n(t) \;\; \text{Normalized message}\right] \quad \min_t m_n(t) = -1$$

L16

04 September 2017 10:09

$$u_{AM}(t) = A_c\left(1 + a_{mod}\, m_n(t)\right)\cos 2\pi f_c t$$

$$\text{Power efficiency} = \frac{A_c^2 a_{mod}^2 \overline{m_n^2(t)\cos^2 2\pi f_c t}}{A_c^2 \overline{\left(1 + a_{mod}\, m_n(t)\right)^2 \cos^2 2\pi f_c t}}$$

$$= \frac{a_{mod}^2 \overline{m_n^2(t)\cos^2 2\pi f_c t}}{\overline{\left(1 + a_{mod}\, m_n(t)\right)^2 \cos^2 2\pi f_c t}}$$

$$= \frac{\frac{1}{2} a_{mod}^2 \overline{m_n^2(t)}}{\frac{1}{2}\overline{\left(1 + a_{mod}\, m_n(t)\right)^2}} = \frac{a_{mod}^2 \overline{m_n^2(t)}}{1 + a_{mod}^2 \overline{m_n^2(t)}}$$

(Assume $\overline{m_n(t)} = 0$)

L16

04 September 2017 10:17

$$a_{mod} = \frac{AM_0}{A_c}$$

$$A_c\left(1 + a_{mod}\, m_n(t)\right)\cos 2\pi f_c t$$

($m_n(t)$: -1)

For envelope detection, we need $a_{mod} \le 1$

$$\text{Power efficiency} \le \frac{\overline{m_n^2}}{1 + \overline{m_n^2}}$$

If $\overline{m_n^2} < 1$, Power efficiency $< \frac{1}{2}$

$1 + a_{mod}\, m_n(t) \ge 0 \Rightarrow a_{mod} \le 1$

$$\frac{a_{mod}^2\,\overline{m_n^2}}{1 + a_{mod}^2\,\overline{m_n^2}}$$

$$\frac{xc}{1+xc}$$

$$\frac{(1+xc)c - xc(c)}{(1+xc)^2} = \frac{c}{(1+xc)^2} > 0$$

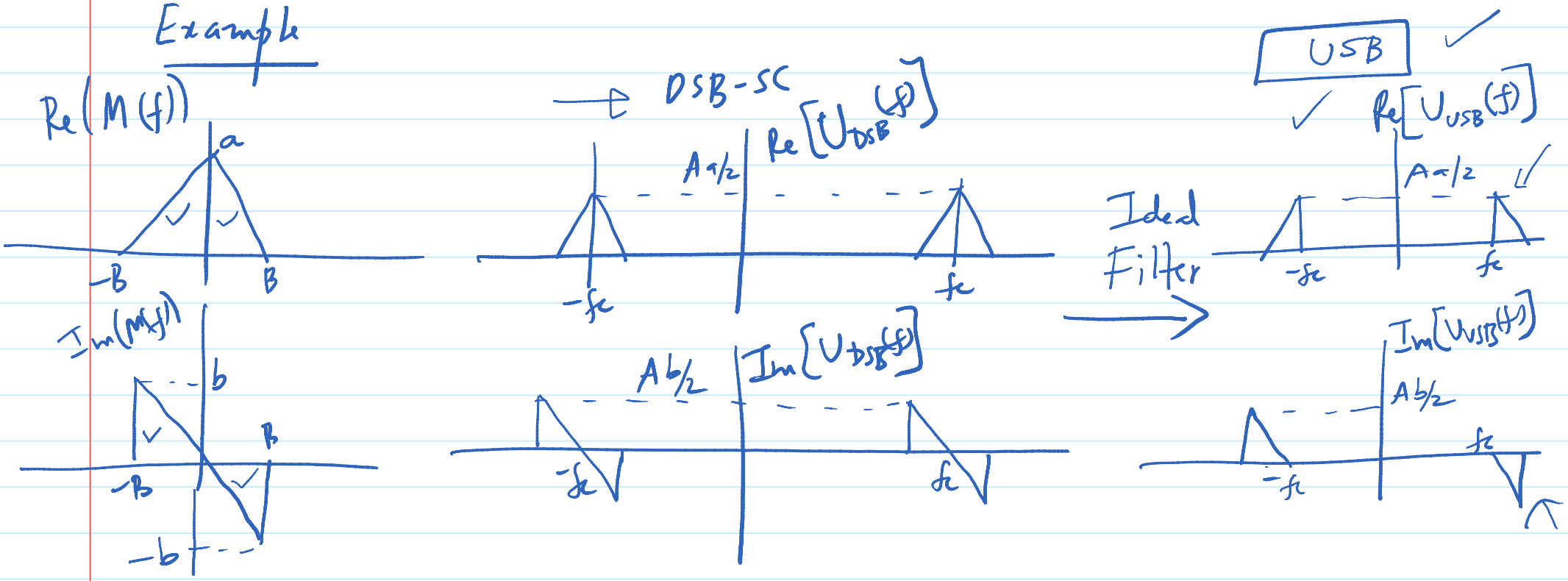
L16

04 September 2017 10:25

3) Single-sideband modulation (SSB AM)

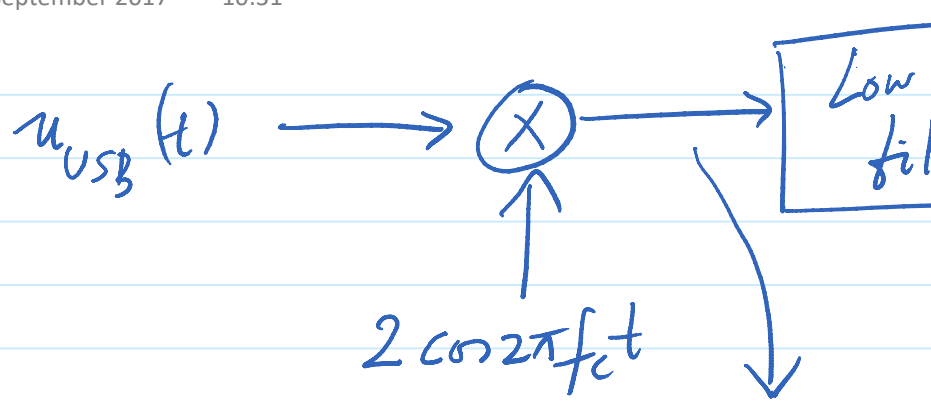
- We know that USB or LSB should be sufficient to recover $m(t)$.
- How can we recover $m(t)$?

Example

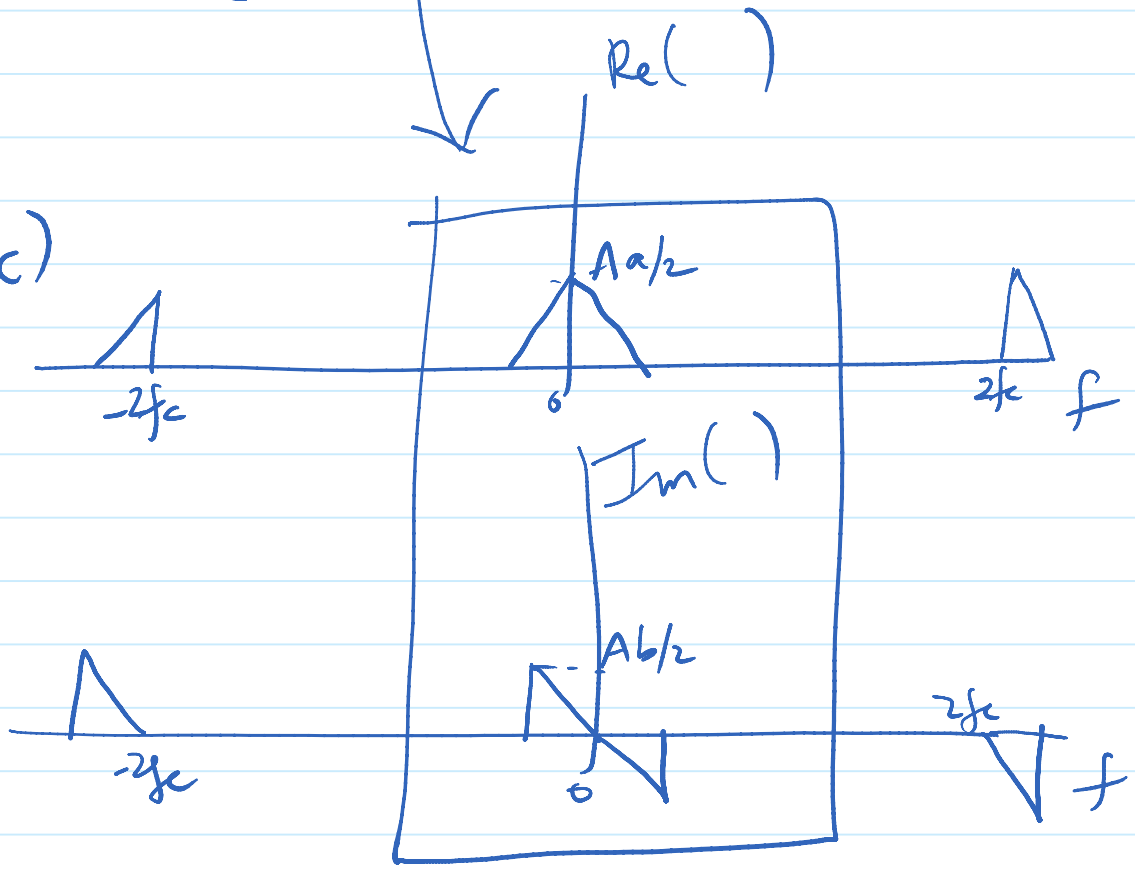


L16

04 September 2017 10:31



$$U_{USB}(f-f_c) + U_{USB}(f+f_c)$$



The same demodulation method works for both DSB-SC and USB. (also for LSB).

L16

04 September 2017 10:38

SSB AM uses only bandwidth B for message BW B.

DSB —— h —— 2B —— h ——

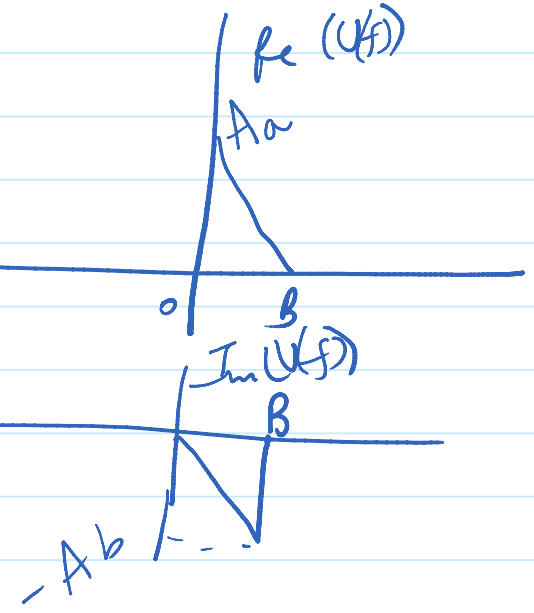
$u_{USB}(t)$ —(in-phase component)→ $\frac{A}{2} m(t)$

—(Quadrature component)→ ?

$u_s(t) \longleftrightarrow U_s(f) = \dfrac{U(f) - U^*(-f)}{2j}$

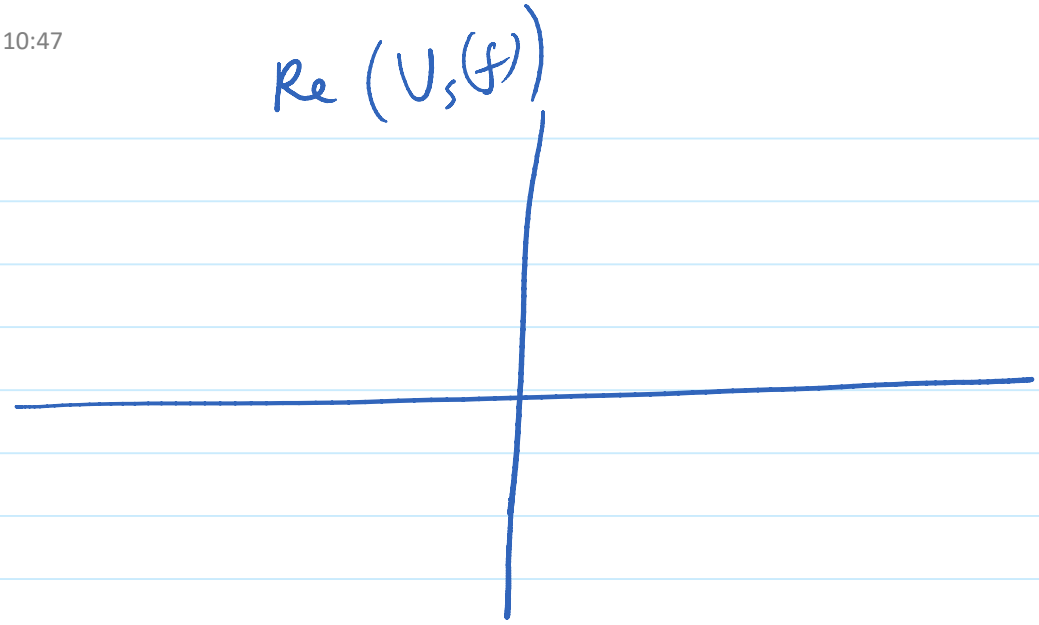
$\hookrightarrow \dfrac{u(t) - u^*(t)}{2j}$

$\mathcal{F}$[Complex envelope of $u_{USB}(t)$]



L16

04 September 2017 10:47



$$\frac{U(f) - U^*(-f)}{2j}$$

$$U(f) = Re(U(f)) + j\,Im(U(f))$$

$$U^*(-f) = Re(U(-f)) - j\,Im(U(-f))$$

Lecture 17

04 September 2017 10:47

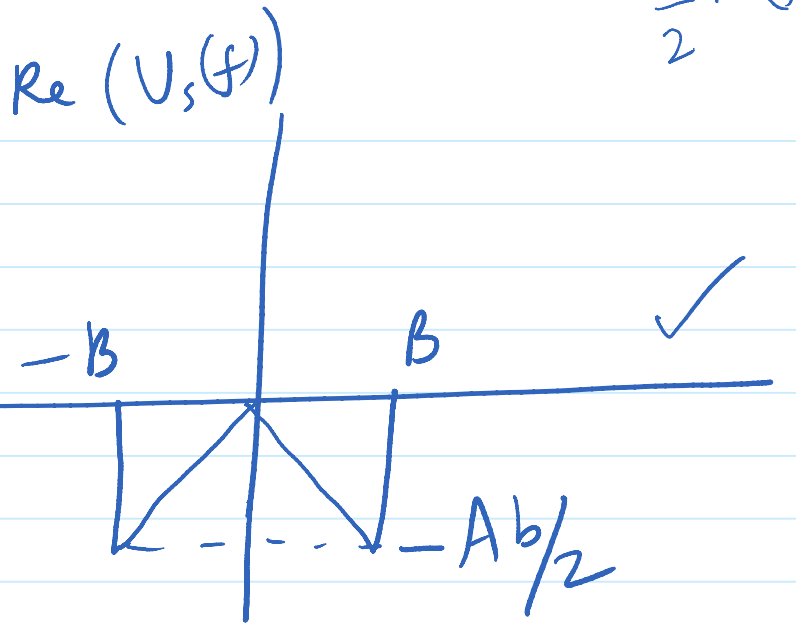
SSB

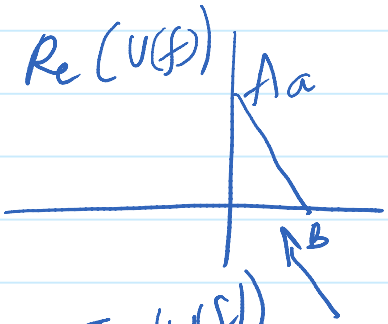
$$\frac{A}{2} m(f) = \frac{U(f) + U^*(-f)}{2} = U_c(f) \checkmark$$

$$? = \frac{U(f) - U^*(-f)}{2j} = U_s(f)$$

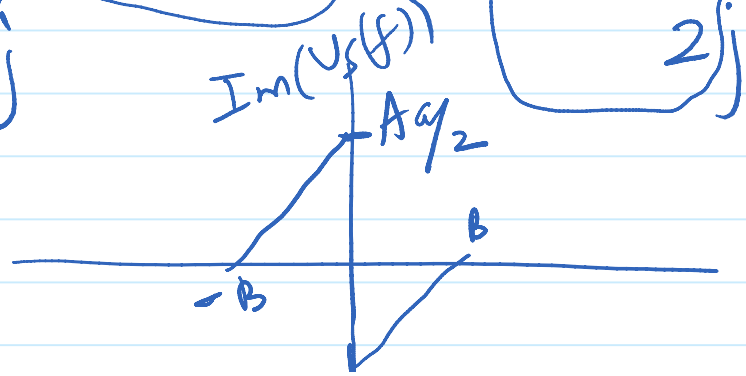
$$U(f) = Re(U(f)) + j\,Im(U(f))$$

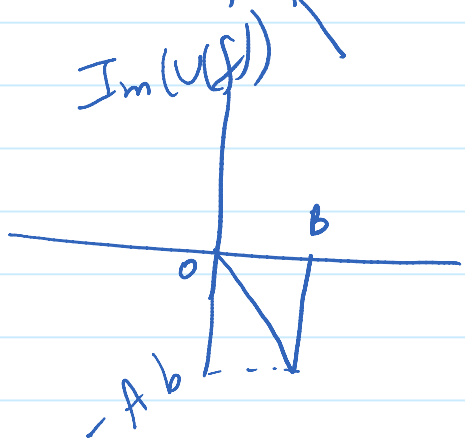
$$U^*(-f) = Re(U(-f)) - j\,Im(U(-f))$$





$$\frac{Re(U(f)) - Re(U(-f))}{2j} + j\,\frac{Im(U(f)) + Im(U(-f))}{2j}$$





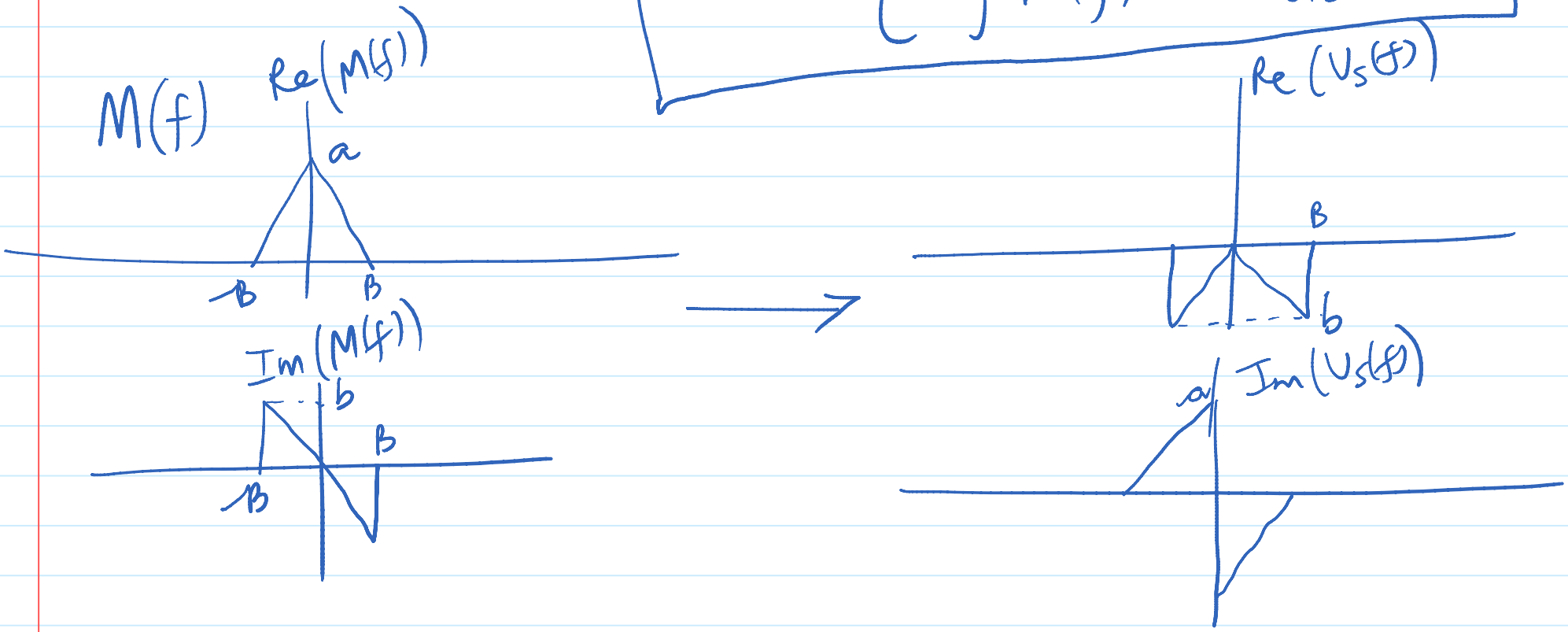
L17

05 September 2017 09:17

$f \gtrless 0$ $\quad \mathrm{Re}(U_s(f)) + j\,\mathrm{Im}(U_s(f)) = \mathrm{Im}(M(f)) - j\,\mathrm{Re}(M(f))$

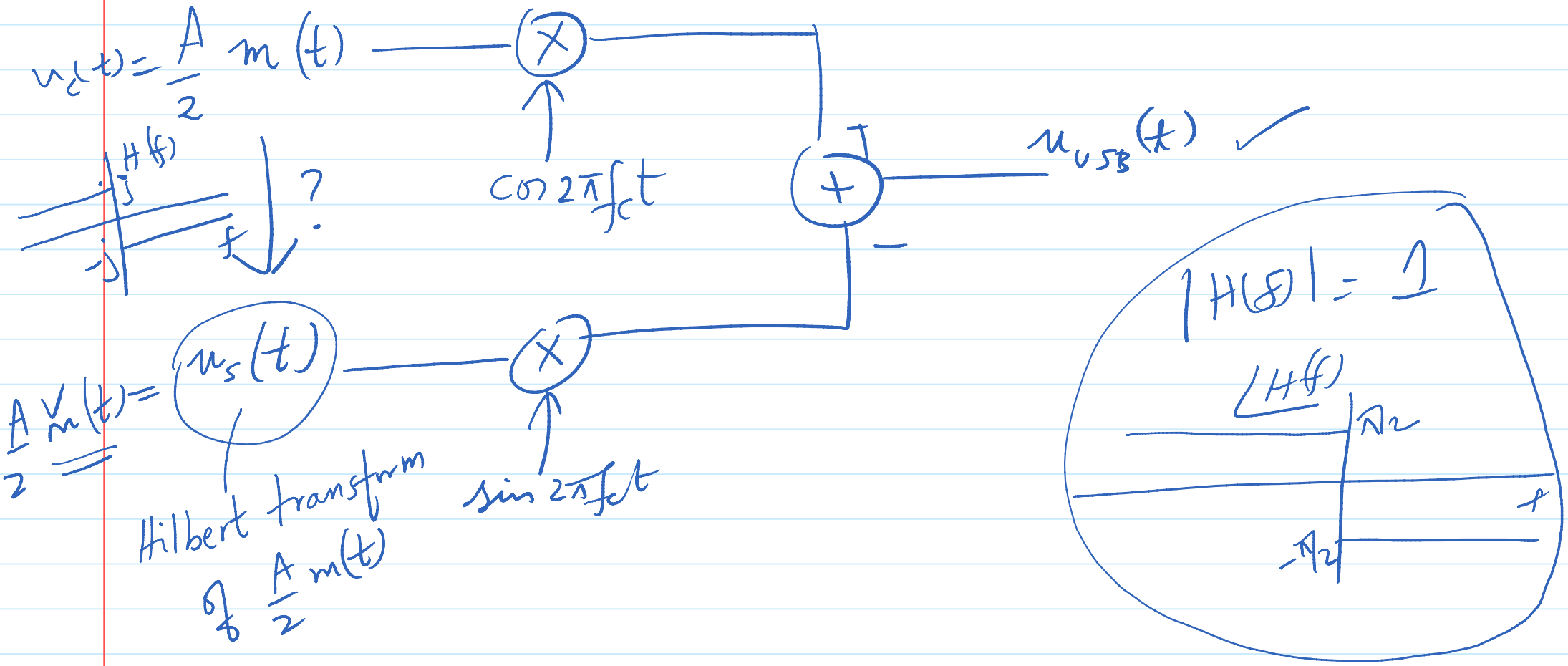
We can show that

$$U_s(f) = \frac{A_x}{2} \begin{cases} -j\,M(f) & f > 0 \\ j\,M(f) & \text{else} \end{cases}$$



L17

05 September 2017 09:23



L17

05 September 2017 09:30

$$u_{USB}(t) = \frac{A}{2}\left[m(t)\cos 2\pi f_c t - \hat{m}(t)\sin 2\pi f_c t\right]$$

Hilbert transform of a sinusoid

Let $m(t) = \cos(2\pi f_0 t + \phi) \longrightarrow \hat{m}(t) = ?$

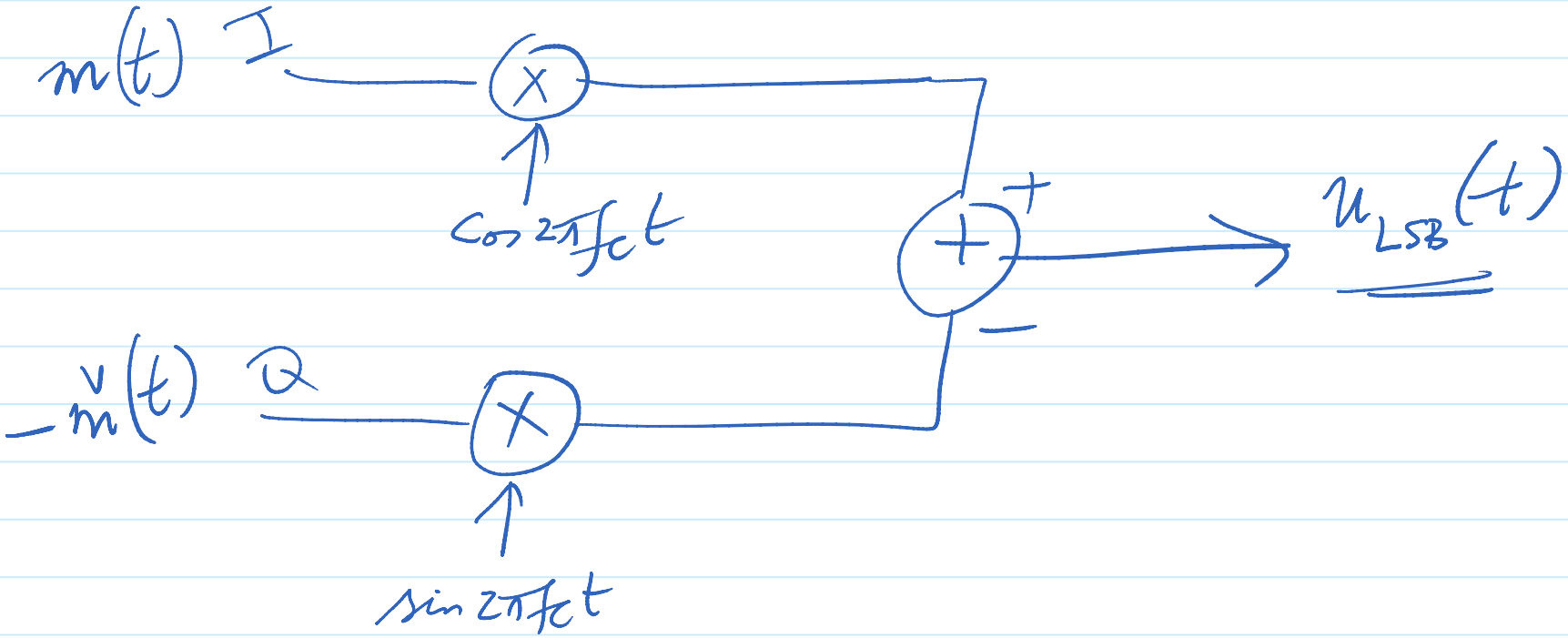
$$\frac{e^{j(2\pi f_0 t+\phi)} + e^{-j(2\pi f_0 t+\phi)}}{2} \longrightarrow \frac{-je^{j(2\pi f_0 t+\phi)} + je^{-j(2\pi f_0 t+\phi)}}{2}$$

$$= \sin(2\pi f_0 t + \phi).$$

L17

05 September 2017 09:34

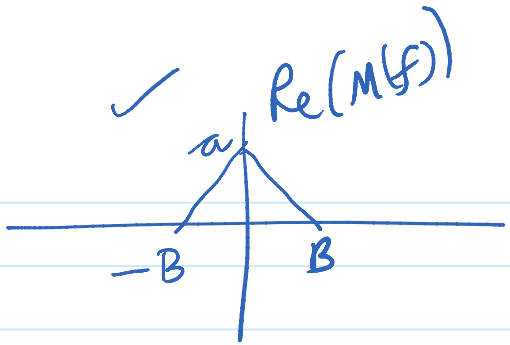
How can we obtain the LSB signal?

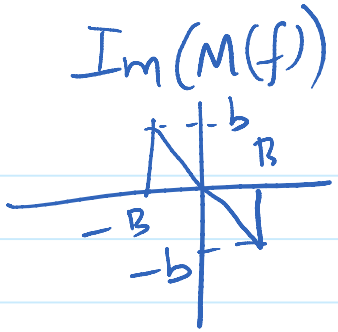


L17

05 September 2017 09:38

Example

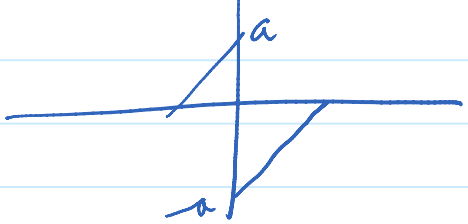


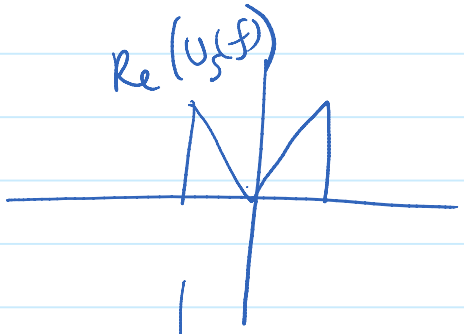


In-phase.

H.T.







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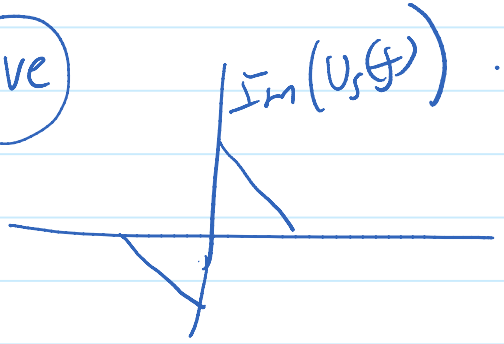


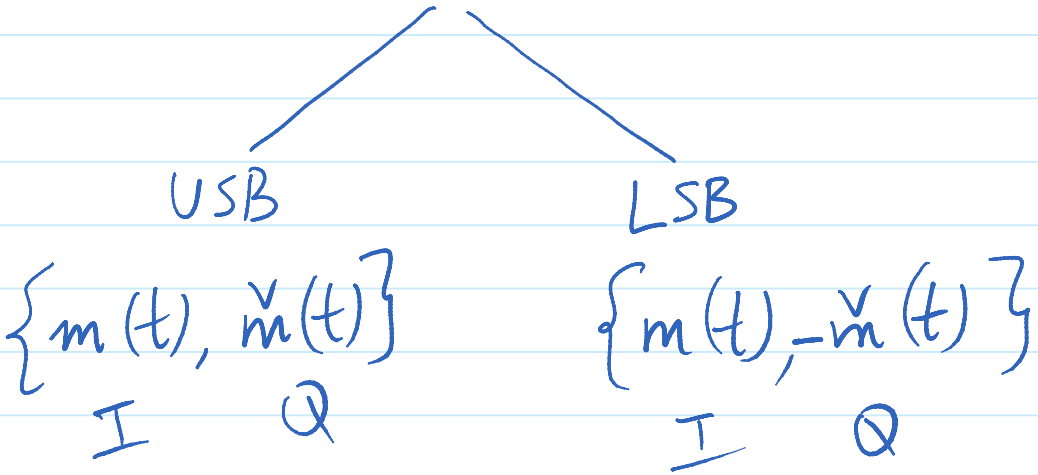
Quadrature



L17

05 September 2017 09:47

SSB modulation



Uses B Hz bandwidth for a message with BW B Hz.

Lecture 18

08 September 2017 11:57

So far:

	Message BW	Transmit signal BW
DSB-SC	B	2B
Conventional AM	B	2B
SSB	B	B

For SSB AM, we need an ideal filter (to filter out one sideband) or an ideal Hilbert transformer. Ideal filters are generally not realizable.

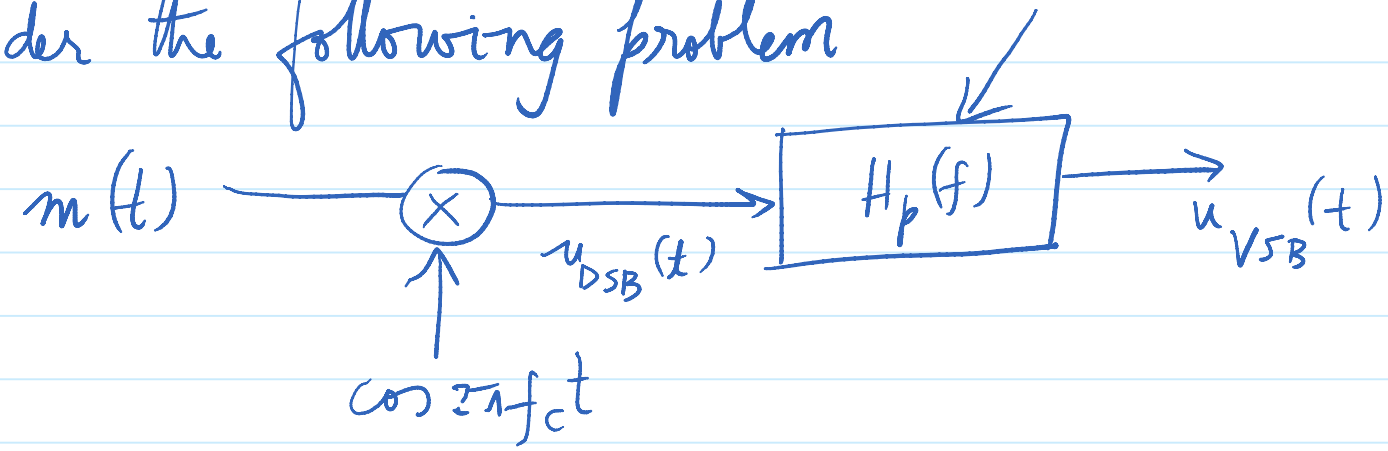
Vestigial sideband (VSB) modulation:

- Non-ideal filters used — Which filters will work?
- BW used $(B+\Delta)$

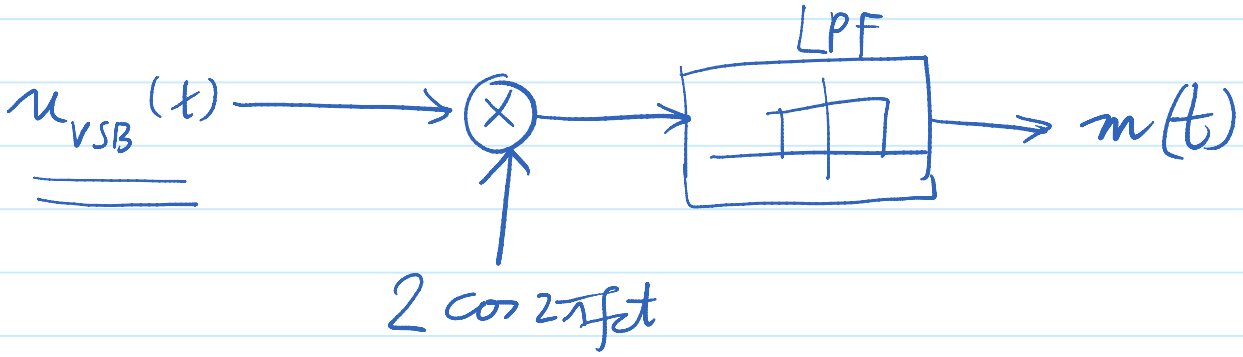
L18

08 September 2017 13:06

* Consider the following problem



Receiver:



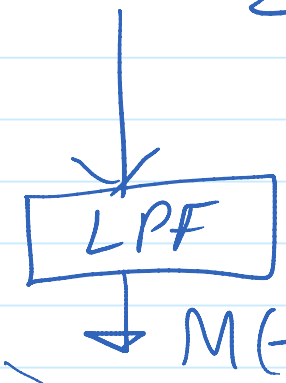
$$U_{VSB}(f) = U_{DSB}(f) H_p(f)$$

L18

08 September 2017 13:11

$$U_{VSB}(f) = \left[\frac{M(f-f_c) + M(f+f_c)}{2} \right] H_p(f) = \frac{M(f-f_c)H_p(f) + M(f+f_c)H_p(f)}{2}$$

$$u_{VSB}(t) \cdot 2\cos 2\pi f_c t \longleftrightarrow \frac{\left[M(f-2f_c)H_p(f-f_c) + M(f)H_p(f-f_c) + M(f)H_p(f+f_c) + M(f+2f_c)H_p(f+f_c)\right]}{2}$$



$$M(f)\left[H_p(f-f_c) + H_p(f+f_c)\right] = M(f) \cdot \text{constant} \quad \text{for all } |f| \le B$$

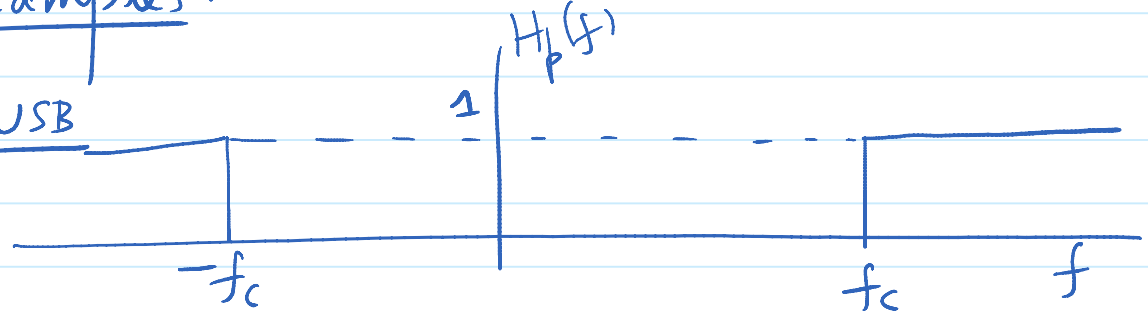
L18

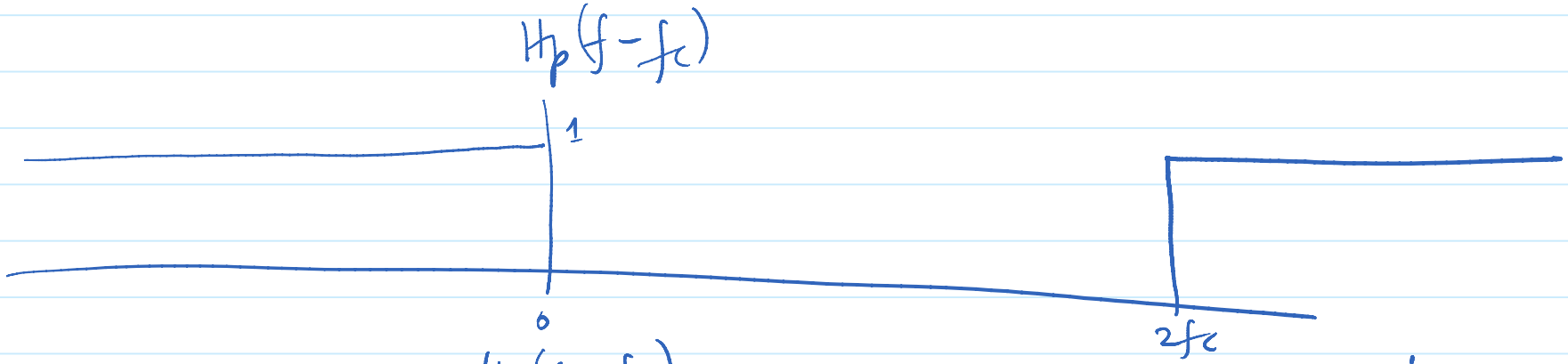
08 September 2017 13:17

$$H_p(f-f_c) + H_p(f+f_c) = \text{constant for } |f| \le B$$

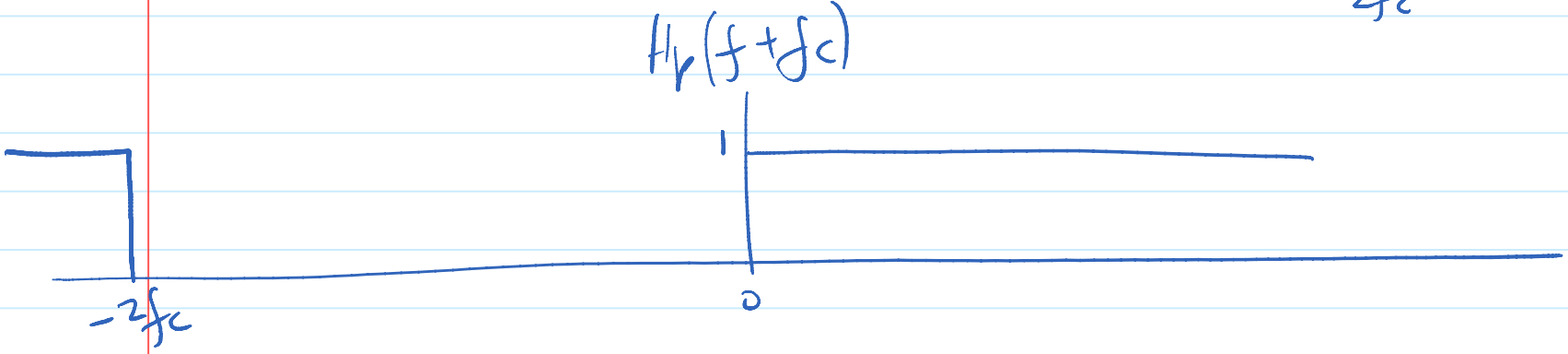
Examples:

1) USB





+



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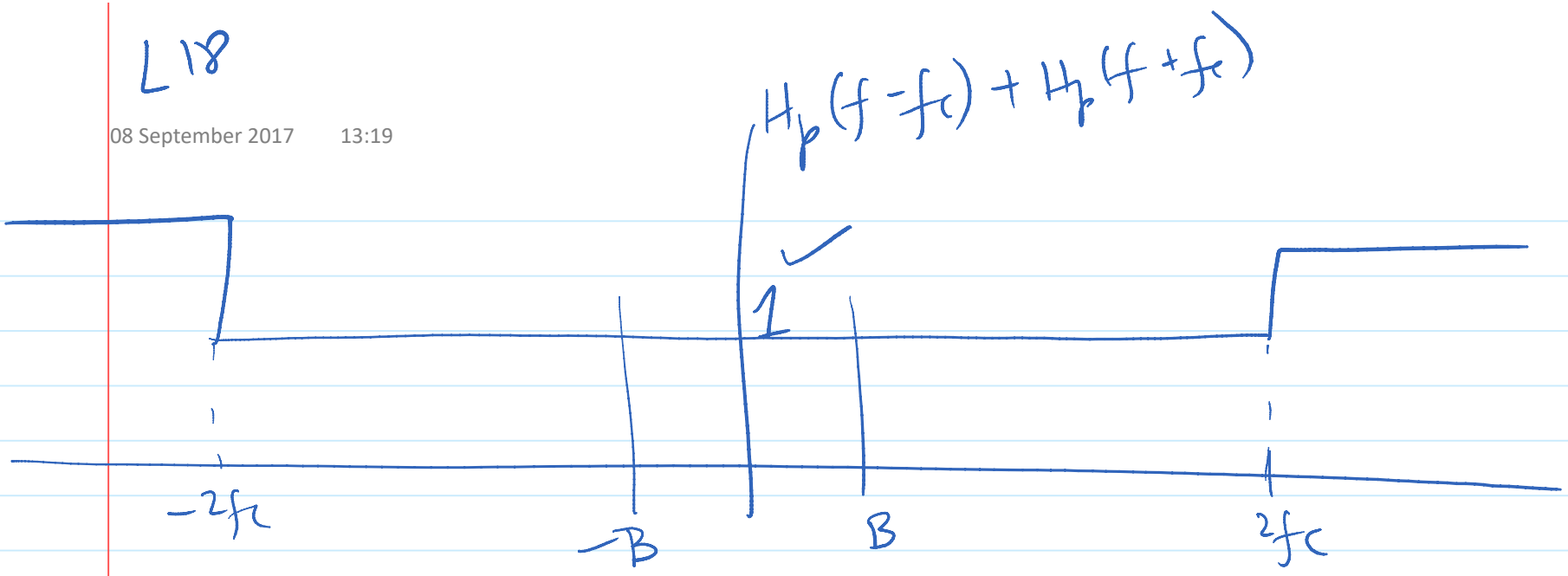
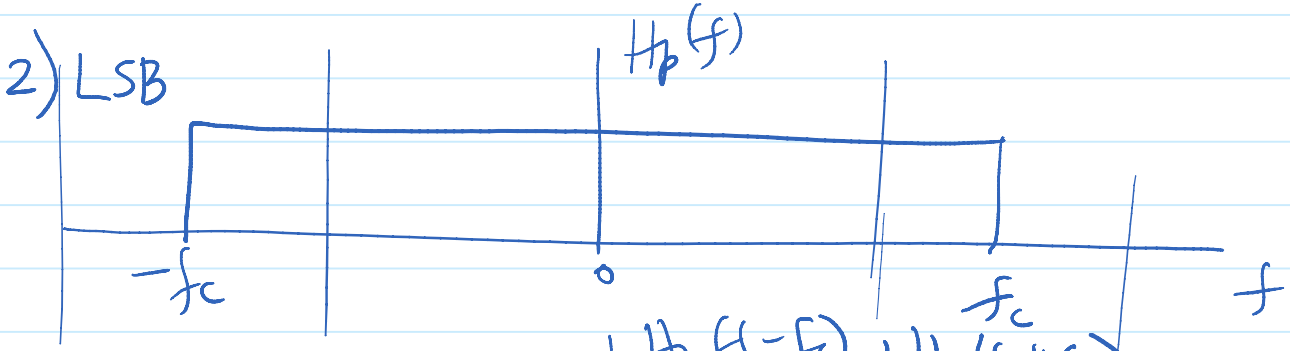
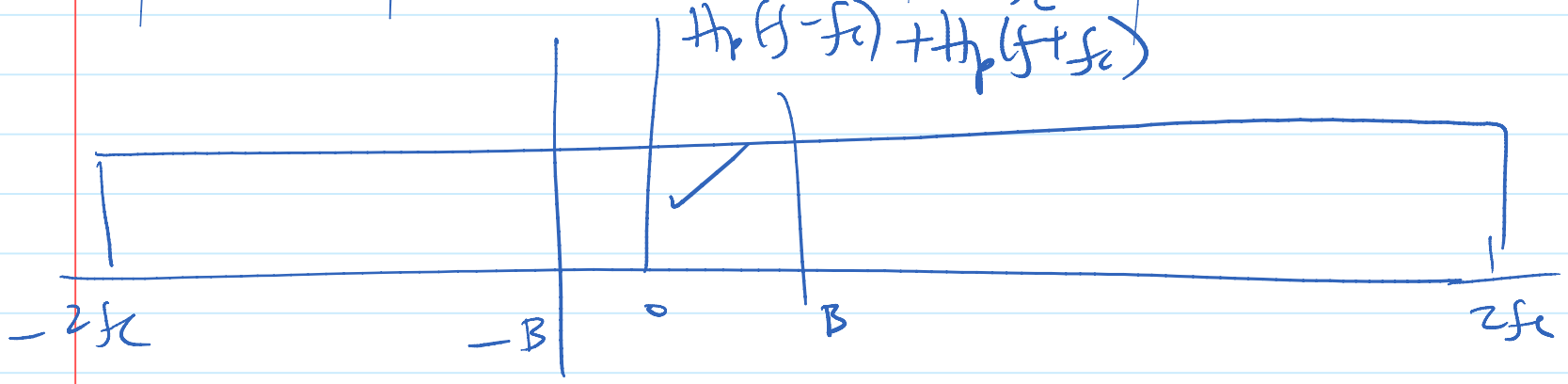


2) LSB





L18

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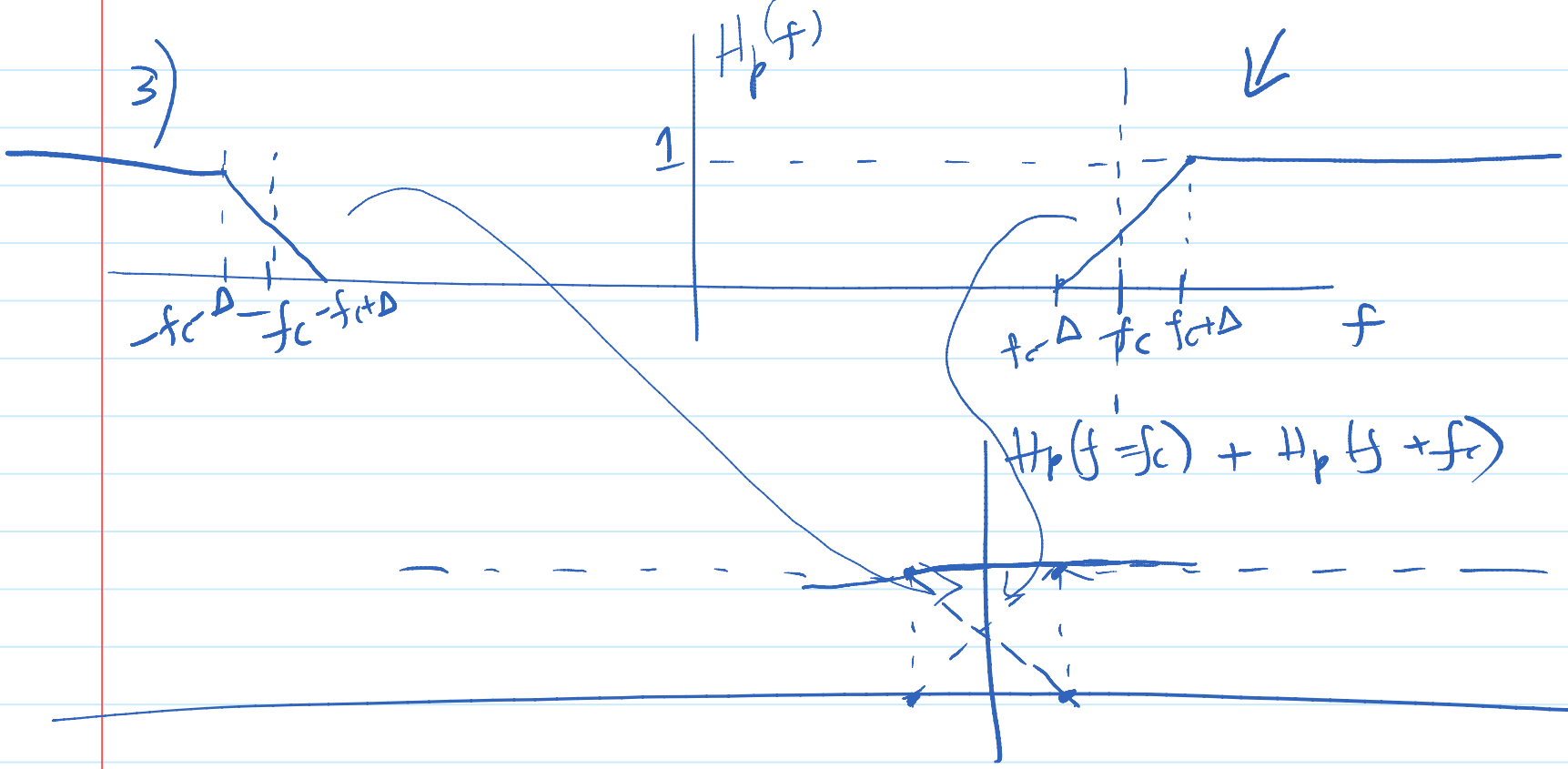


BW of VSB signal

$B+\Delta$

Application of VSB AM : Broadcast TV (analog)

L18

08 September 2017 13:33

Quadrature Amplitude Modulation (QAM)

Two message signals of BW B $\{m_c(t), m_s(t)\}$ ⟶ One passband signal BW 2B

$$m_c(t)\cos 2\pi f_c t - m_s(t)\sin 2\pi f_c t$$

* As efficient in spectrum usage as SSB AM.

⟶ {DSB-SC, Conventional AM}

SSB (USB/LSB)

VSB

QAM

L18

08 September 2017 13:44

Angle modulation:

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

$$= e(t)\cos\left(2\pi f_c t + \theta(t)\right)$$

$e(t)$: constant

$\theta(t)$: Message

Lecture 19

11 September 2017 09:58

Angle modulation

Any real passband signal $u_p(t)$ can be expressed as

$$u_p(t) = \underbrace{e(t)}_{\text{envelope}} \cos\left(2\pi f_c t + \underbrace{\theta(t)}_{\text{phase}}\right)$$

In angle modulation, $\theta(t)$ is a function of the message signal $m(t)$. $e(t)$ is usually a constant (A_c).

$$u_p(t) = A_c \cos\left(2\pi f_c t + \theta(t)\right)$$

$\theta(t)$ fn of $m(t)$

Two types

1) Phase modulation (PM)
2) Frequency modulation (FM).

L19

11 September 2017 10:05

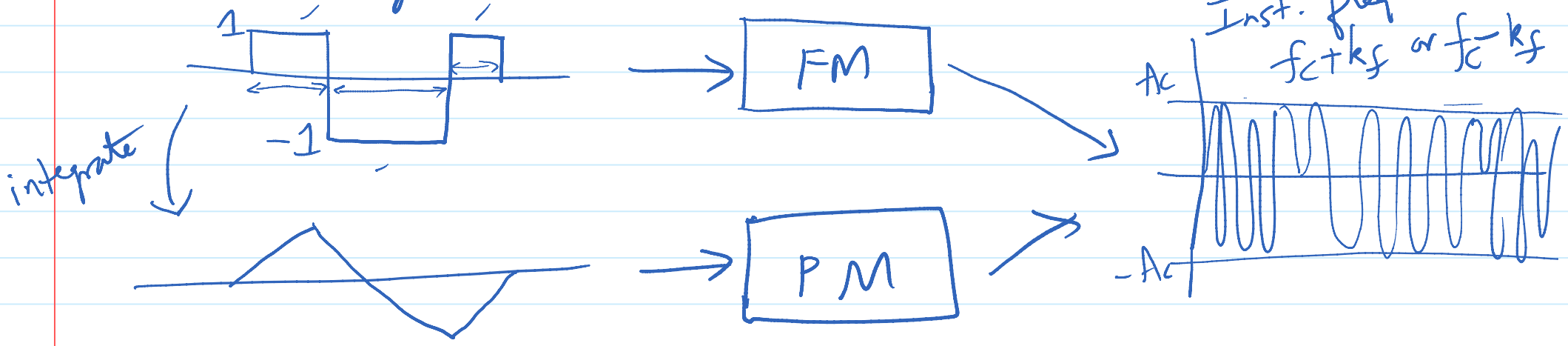
$u_p(t) = A_c \cos(2\pi f_c t + \theta(t))$

Instantaneous freq: $\frac{1}{2\pi}\frac{d}{dt}(\,)$

PM: $\theta(t) = k_p\, m(t)$

FM: $\theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau = \underbrace{2\pi k_f \int_{0}^{t} m(\tau)\,d\tau + \theta(0)}$

* Observe that FM is equivalent to PM with the integral of the message. Similarly, for differentiable message signals, PM can be interpreted as FM with the input being the derivative of the message



L19

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$$\text{Instantaneous freq} = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt}$$

$$\text{Instantaneous freq offset} = \frac{1}{2\pi}\frac{d}{dt}\theta(t)$$

$$\text{FM: } \frac{1}{2\pi}\frac{d}{dt}(\theta(t)) = \frac{1}{2\pi}\frac{d}{dt}\left[2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right] = k_f m(t).$$

$$\text{PM: } \frac{1}{2\pi}\frac{d\theta(t)}{dt} = \frac{1}{2\pi} k_p \frac{d}{dt} m(t).$$

Suppose message has abrupt changes $\longrightarrow$ PM likely to result in higher BW usage than FM

For smooth signals $\longrightarrow$ PM, FM similar

L19

11 September 2017 10:23

Definitions:

Frequency deviation: The maximum deviation in instantaneous frequency due to the message $m(t)$ is given by

$$\Delta f_{max} = k_f \max_t |m(t)| \quad (FM)$$

Modulation index: (β)

$$\beta = \frac{\Delta f_{max}}{B} \quad \text{where } B: \text{BW of } m(t)$$

If $\beta < 1$, we call it narrowband FM.
If $\beta > 1$, we call it wideband FM.

Eg: $m(t) = A_m \cos(2\pi f_m t) \longrightarrow \Delta f_{max} = A_m k_f \quad B = f_m$

$$\beta = \frac{A_m k_f}{f_m}$$

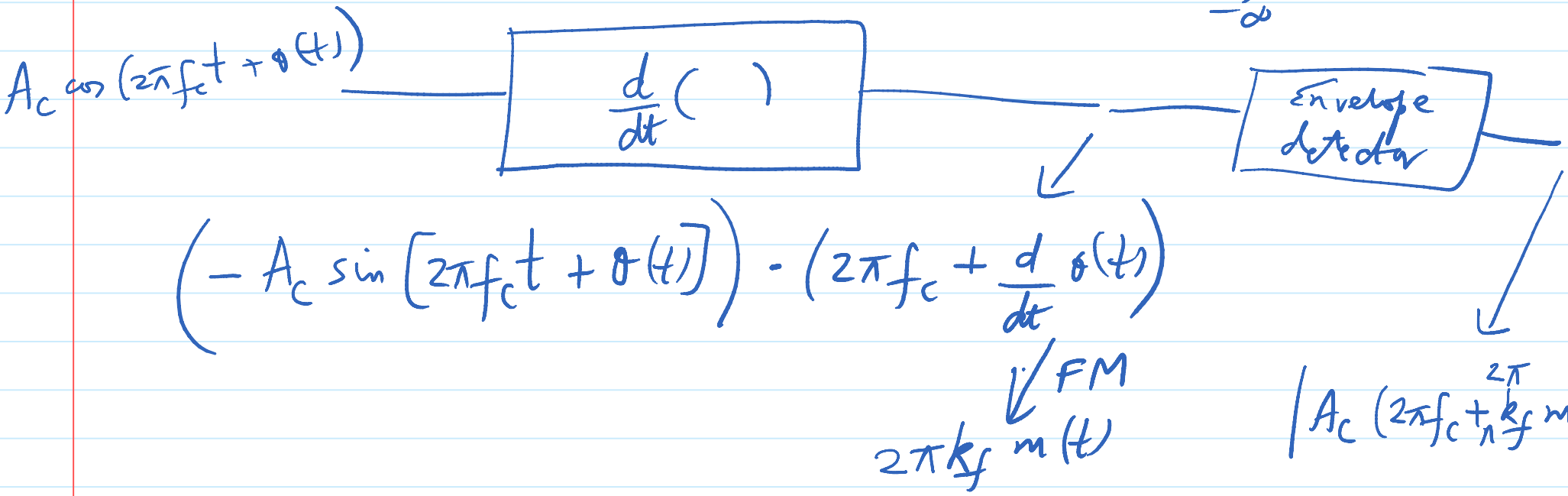
L19

11 September 2017 10:32

FM modulation: Using a voltage-controlled oscillator (VCO)

FM demodulation: $A_c \cos(2\pi f_c t + \theta(t))$

$\theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau$.



L19

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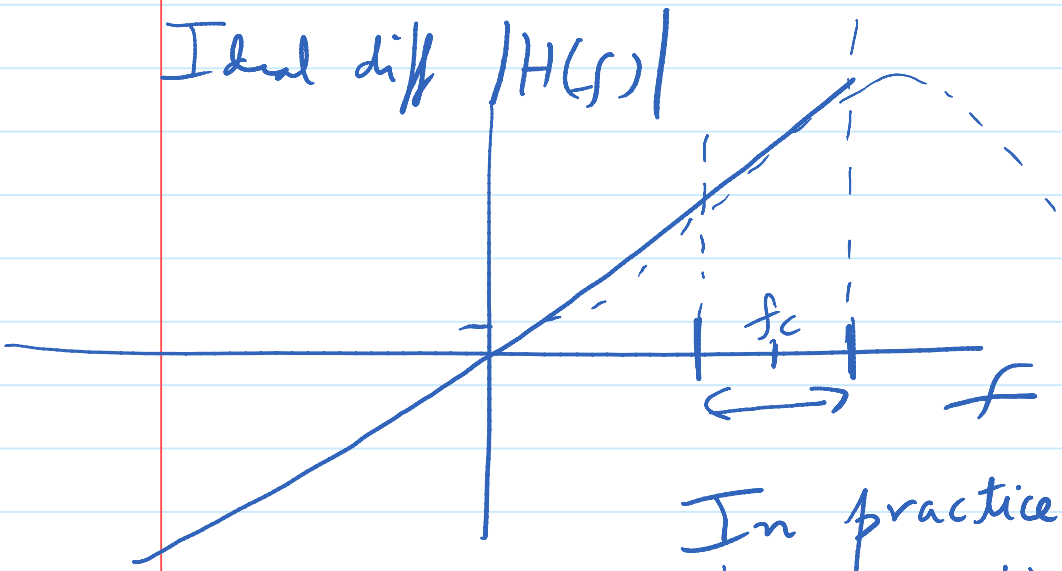
$$\left| A_c \left(2\pi f_c + 2\pi k_f m(t)\right) \right| = 2\pi A_c \left| f_c + k_f m(t) \right|$$

$$\left(f_c + k_f m(t) > 0 \right) = 2\pi A_c \left(f_c + k_f m(t) \right)$$

Remove dc component



Scaled version of $m(t)$

(Assume $m(t)$ does not have a dc component)

In practice, we need this freq. response only over the band occupied by the FM signal.

Lecture 20

12 September 2017 08:58

FM spectrum:

$$u_{FM}(t) = A_c \cos\left(2\pi f_c t + \underbrace{\theta(0) + 2\pi k_f \int_0^t m(\tau)\,d\tau}_{\theta(t)}\right)$$

* For a general $m(t)$, it is not possible to analytically evaluate $U_{FM}(f)$.

* Under the strict defn. of BW, BW of $u_{FM}(t)$ is infinite. We need to use approximate BW notions.

OWT

Plan

* Example: Let $m(t) = A_m \cos(2\pi f_m t)$. What is $U_{FM}(f)$?

* Approximate formula for BW of $u_{FM}(t)$ for a general $m(t)$.

L 20

12 September 2017 09:09

FM spectrum for a sinusoidal message

$$m(t) = A_m \cos(2\pi f_m t)$$

$$\theta(t) = \theta(0) + 2\pi k_f \int_0^t m(\tau)\, d\tau$$

$$= \theta(0) + 2\pi k_f A_m \frac{\sin(2\pi f_m t)}{2\pi f_m}$$

$$= \theta(0) + \boxed{\frac{k_f A_m}{f_m}} \sin(2\pi f_m t) = \beta \sin(2\pi f_m t)$$

Take 0

$\boxed{\frac{k_f A_m}{f_m}} \triangleq \beta$

$$\boxed{u_{FM}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)}.$$

L 20

12 September 2017 09:13

Consider the complex baseband equivalent of $u_{FM}(t)$ with ref f_c.

$$u(t) = A_c e^{j\beta \sin(2\pi f_m t)}$$

→ $u(t)$ is periodic ; Period $= \frac{1}{f_m}$

$$u(t) = \sum_{n=-\infty}^{\infty} u[n] e^{j 2\pi n f_m t}$$

$$u[n] = \frac{1}{(1/f_m)} \int_{-\frac{1}{2f_m} \leftarrow -T/2}^{\frac{1}{2f_m} \leftarrow T/2} u(t) e^{-j 2\pi n f_m t} dt$$

T : period.

L 20

12 September 2017 09:18

$$= f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt$$

(Let $x = 2\pi f_m t$)

$$= f_m \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} \frac{dx}{2\pi f_m}$$

(besselj)

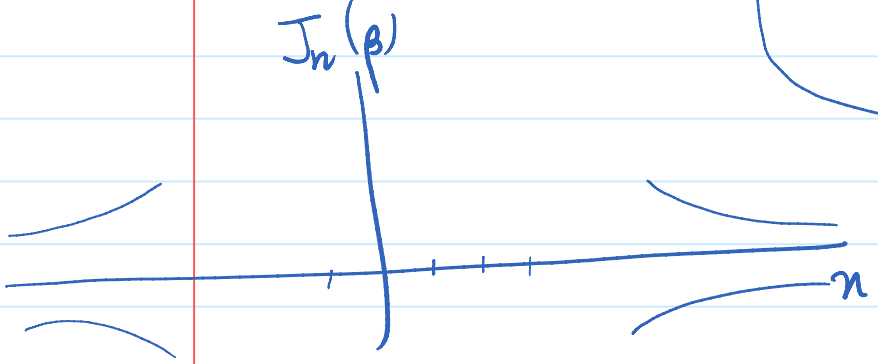
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx. \triangleq J_n(\beta)$$

Bessel fn of nth order of the first kind.

L 20

12 September 2017 09:23

$$e^{j\beta \sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j 2\pi n f_m t}$$



$$\longleftrightarrow \sum_{n=-\infty}^{\infty} J_n(\beta)\, \delta(f - n f_m)$$

(Not exactly bandlimited)

Some facts:

1) For integer n, $J_n(\beta) = (-1)^n J_{-n}(\beta) = (-1)^n J_n(-\beta)$

2) $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$ (Parseval's thm)

3) $J_n(\beta) \to 0$ as $n \to \infty$

L 20

12 September 2017 09:31

$$e^{j\beta \sin(2\pi f_m t)} \approx \sum_{n=-K}^{K} J_n(\beta)\, e^{j 2\pi n f_m t}$$

such that $\sum_{n=-K}^{K} J_n^2(\beta) \geq \alpha$ (Energy containment α)

Fact → Say $\alpha = 0.99$. $BW_{FM} \approx \frac{2B}{f_m}(\beta+1)$ for integer values of β $\beta = 1, 2, \ldots, 10$.

Carson's rule: To estimate BW of an FM signal

$$B_{FM} \approx 2B + 2\Delta f_{max} \approx 2B(\beta+1)$$

βB

L20

12 September 2017 09:43

Special case ($\beta \ll 1$)

$\theta(t) = \beta \sin(2\pi f_m t)$

$$u_{FM}(t) = A_c \cos(2\pi f_c t + \theta(t))$$

$$= A_c \cos\theta(t) \cos 2\pi f_c t - A_c \sin\theta(t) \sin 2\pi f_c t$$

$\cos\theta(t) \approx 1$, $\sin\theta(t) \approx \theta(t)$

NARROWBAND FM

$$\approx A_c \cos 2\pi f_c t - A_c \theta(t) \sin 2\pi f_c t$$

(similar conventional AM)

$BW \approx 2B$

$\beta > 1 \rightarrow$ WIDEBAND FM.

Lecture 21

13 September 2017 08:02

So far, we assumed that the received signal is exactly the modulated signal that we are interested in.

eg.



Filter (1) Other signals occupy neighbouring frequency bands

Tuning (2) We need to be able to demodulate different radio stations at different times

Amplifiers (3) Amplification is also needed

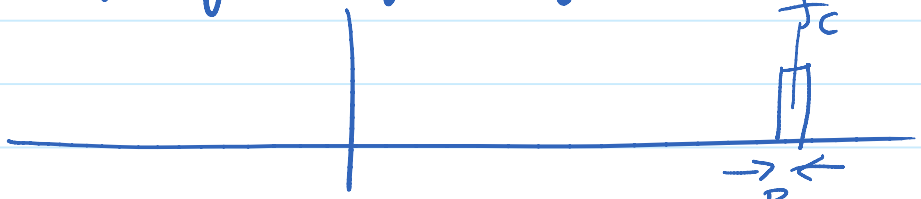
Need to design filters, tuners and amplifiers.

L 21

13 September 2017 08:09

Issues

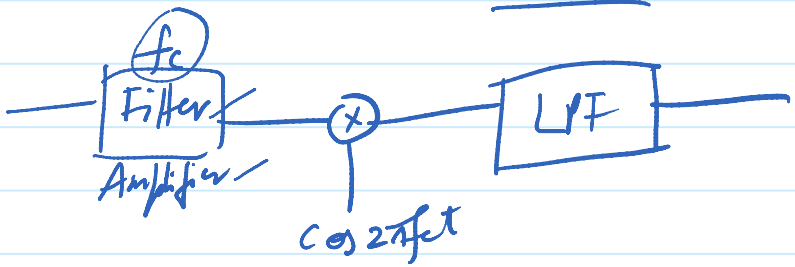
* It may be difficult to design a tunable filter of same quality across a wide range of f_c.
* If $f_c \gg B$, designing a filter of BW B at f_c is difficult.



$\left\{\frac{B}{f_c}\right\}$ small.

* Difficult to amplify at very high f_c.

— The receiver that we have already discussed is a <u>direct conversion receiver</u> ($f_c \to 0$)



— <u>Superheterodyne receiver</u>

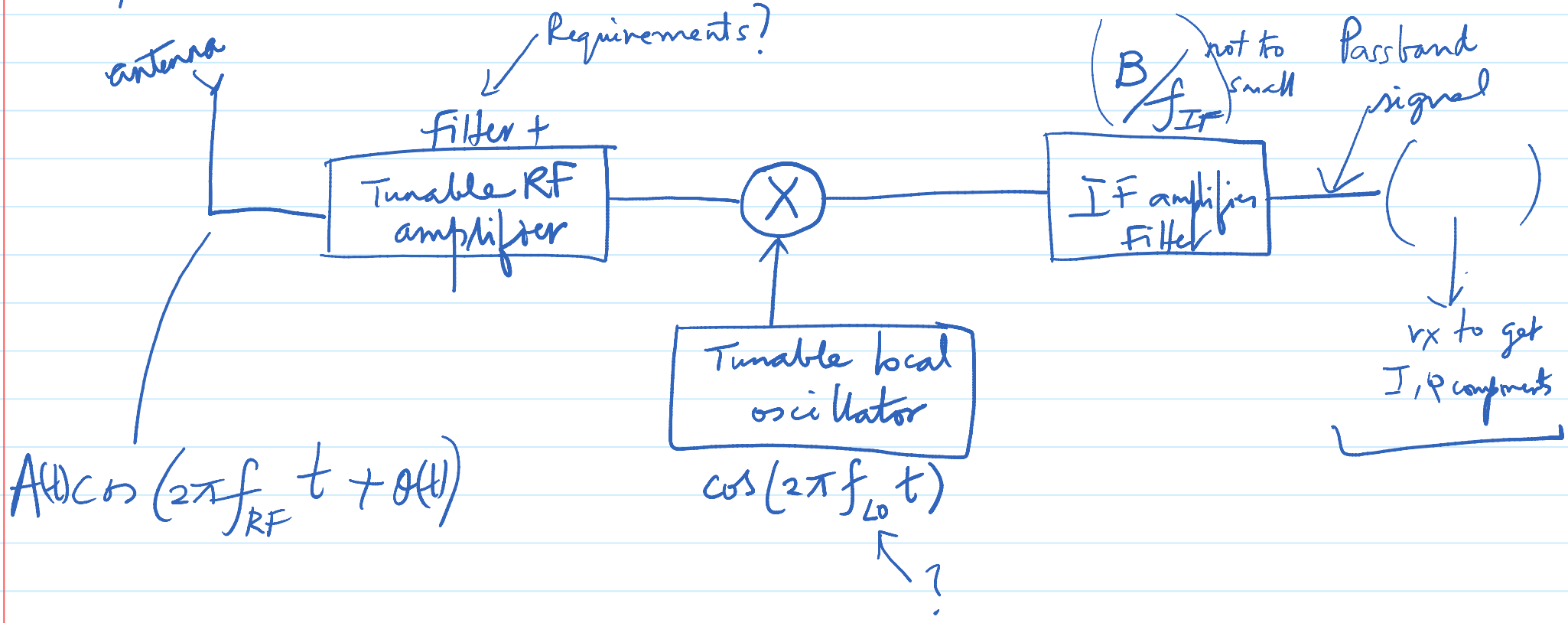
$$f_c \to f_{IF} \to 0$$

↳ Intermediate frequency

L 21

13 September 2017 08:18

Superheterodyne receiver.



$$A(t)\cos(2\pi f_{RF} t + \theta(t))\cos 2\pi f_{LO} t = \frac{A(t)}{2}\cos\left(2\pi (f_{RF} - f_{LO}) t + \theta(t)\right)$$

$$- \frac{A(t)}{2}\cos\left(2\pi (f_{RF} + f_{LO}) t + \theta(t)\right)$$

L21

13 September 2017 08:26

Either $-f_{RF} + f_{LO} = f_{IF}$ or $f_{RF} + f_{LO} = f_{IF}$

$\Rightarrow f_{LO} = f_{RF} - f_{IF}$ or $f_{RF} + f_{IF}$

Fix f_{IF}, f_{LO}.

Is There any requirement on the RF filter BW?

Need $B_{RF\,filter} < 2 f_{IF}$ Image rejection filter

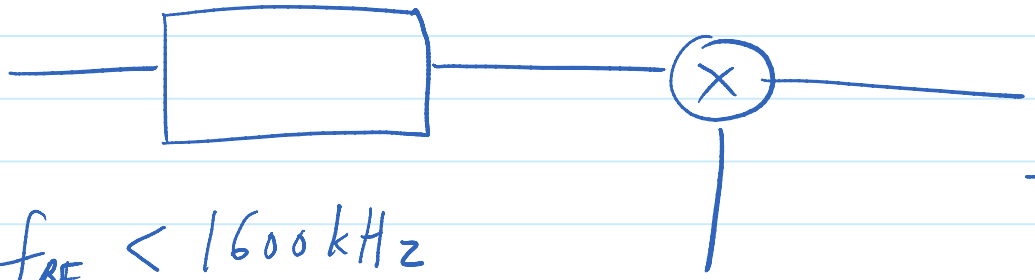
L21

13 September 2017 08:29

$f_{LO} = f_{RF} + f_{IF}$

Example

(AM)

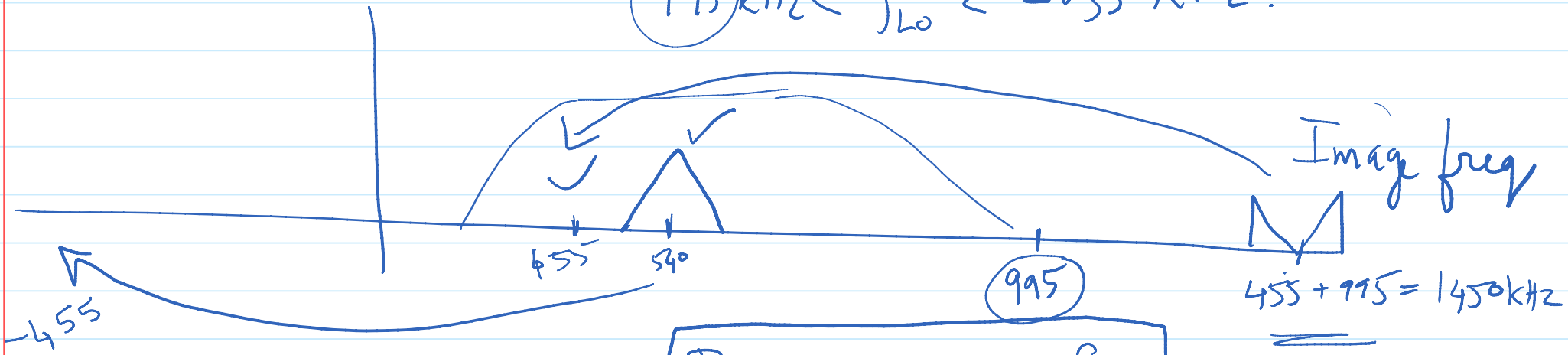


$f_{IF} = 455\ kHz$

$540\ kHz < f_{RF} < 1600\ kHz$

?

$995\ kHz < f_{LO} < 2055\ kHz.$



$455 + 995 = 1450\ kHz$

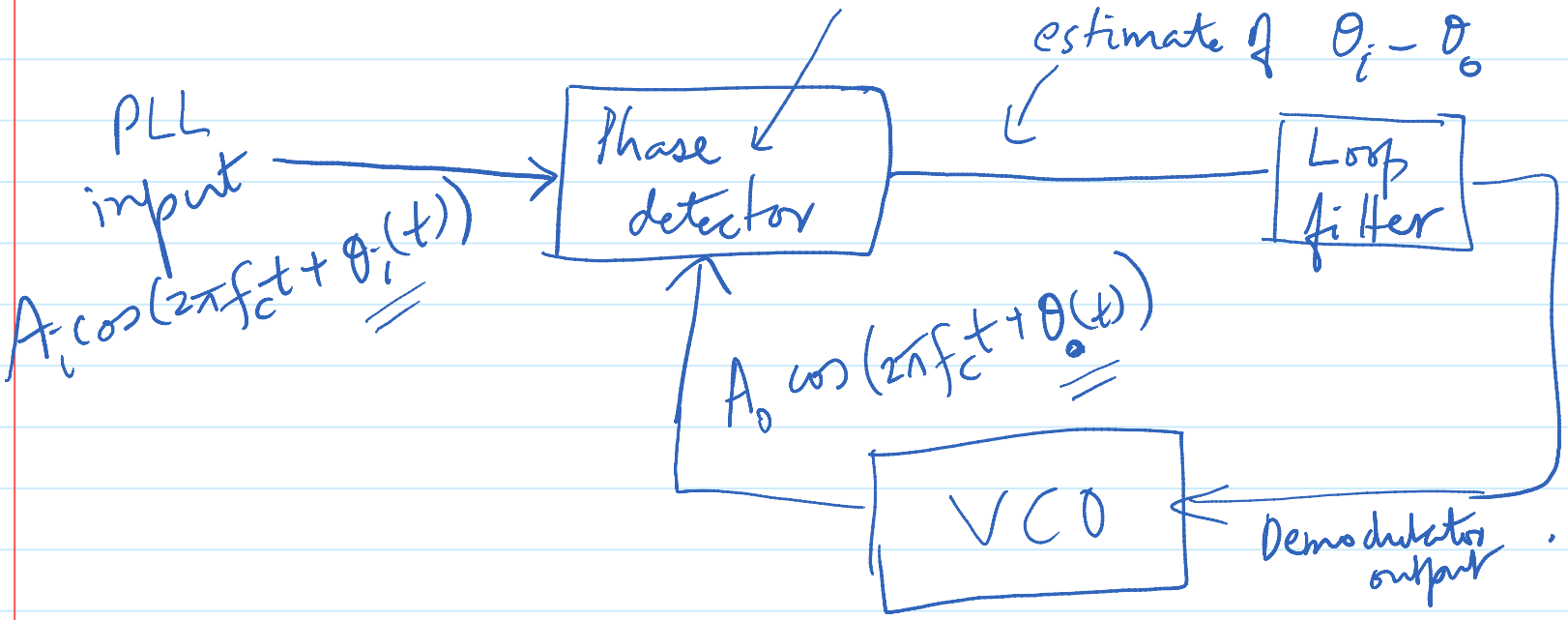
$B_{RF\ filter} < 2 \times f_{IF}$

Lecture 22

14 September 2017 10:05

Phase-locked loop (PLL)

* In the context of FM, it can be used as a demodulator.
* Also has other applications.
* Negative feedback system.



L 22

18 September 2017 10:10

Phase detector:

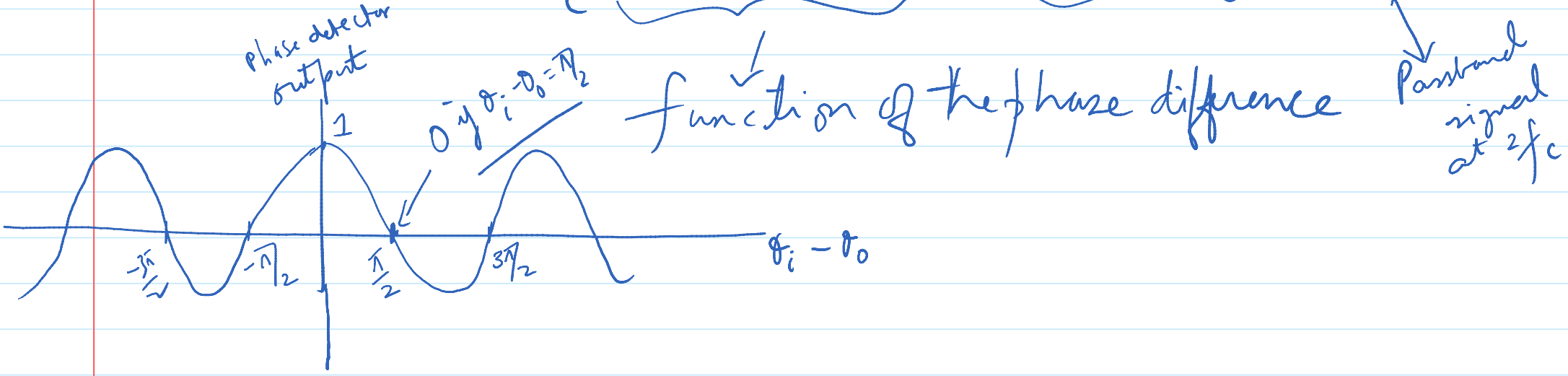
1) Multiplier (or mixer) & LPF

$$\cos(2\pi f_c t + \theta_i(t)) \cdot \cos(2\pi f_c t + \theta_0(t))$$

$$\cos(2\pi f_c t + \theta_i(t)) \cdot \cos(2\pi f_c t + \theta_0(t)) = \frac{1}{2}\left[\cos(\theta_i(t) - \theta_0(t)) + \cos(4\pi f_c t + \theta_i(t) + \theta_0(t))\right]$$

$\cos(\theta_i(t) - \theta_0(t))$ → function of the phase difference

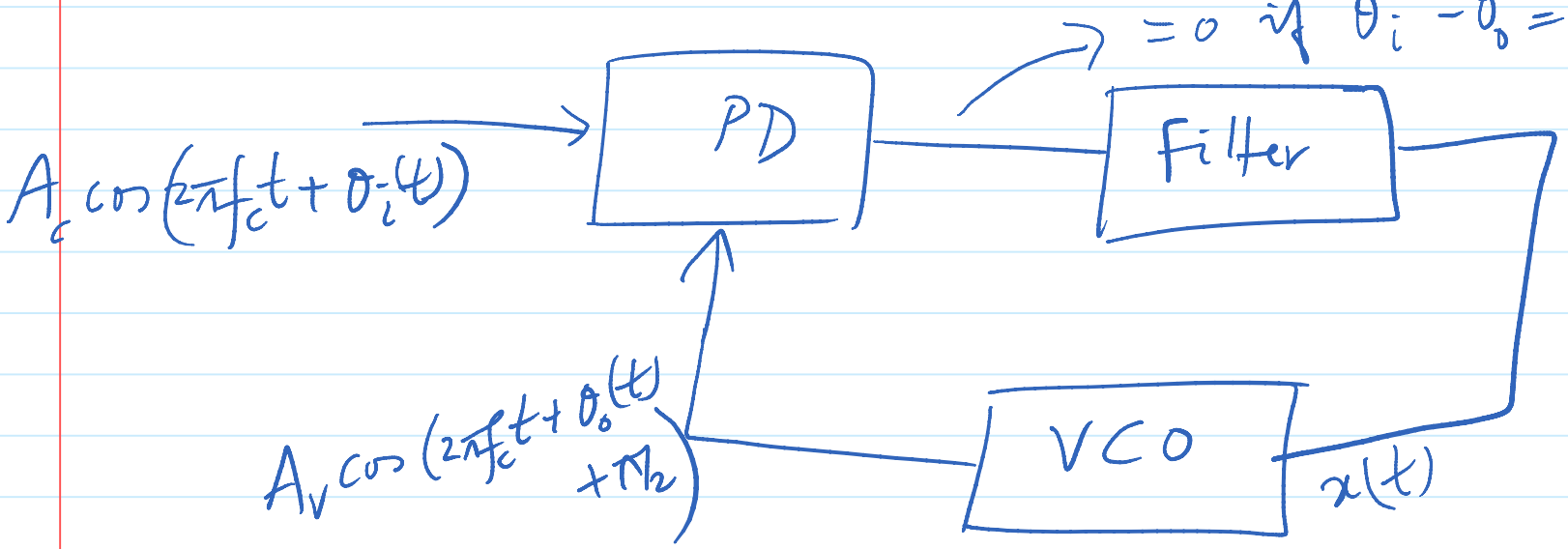
$\cos(4\pi f_c t + \theta_i(t) + \theta_0(t))$ → Passband signal at $2f_c$



L 22

18 September 2017 10:18

Modified description (so that PD output is 0 when $\theta_i - \theta_o = 0$)



Model

Instantaneous freq is the input.

When $\theta_i = \theta_o$, we are in "lock".

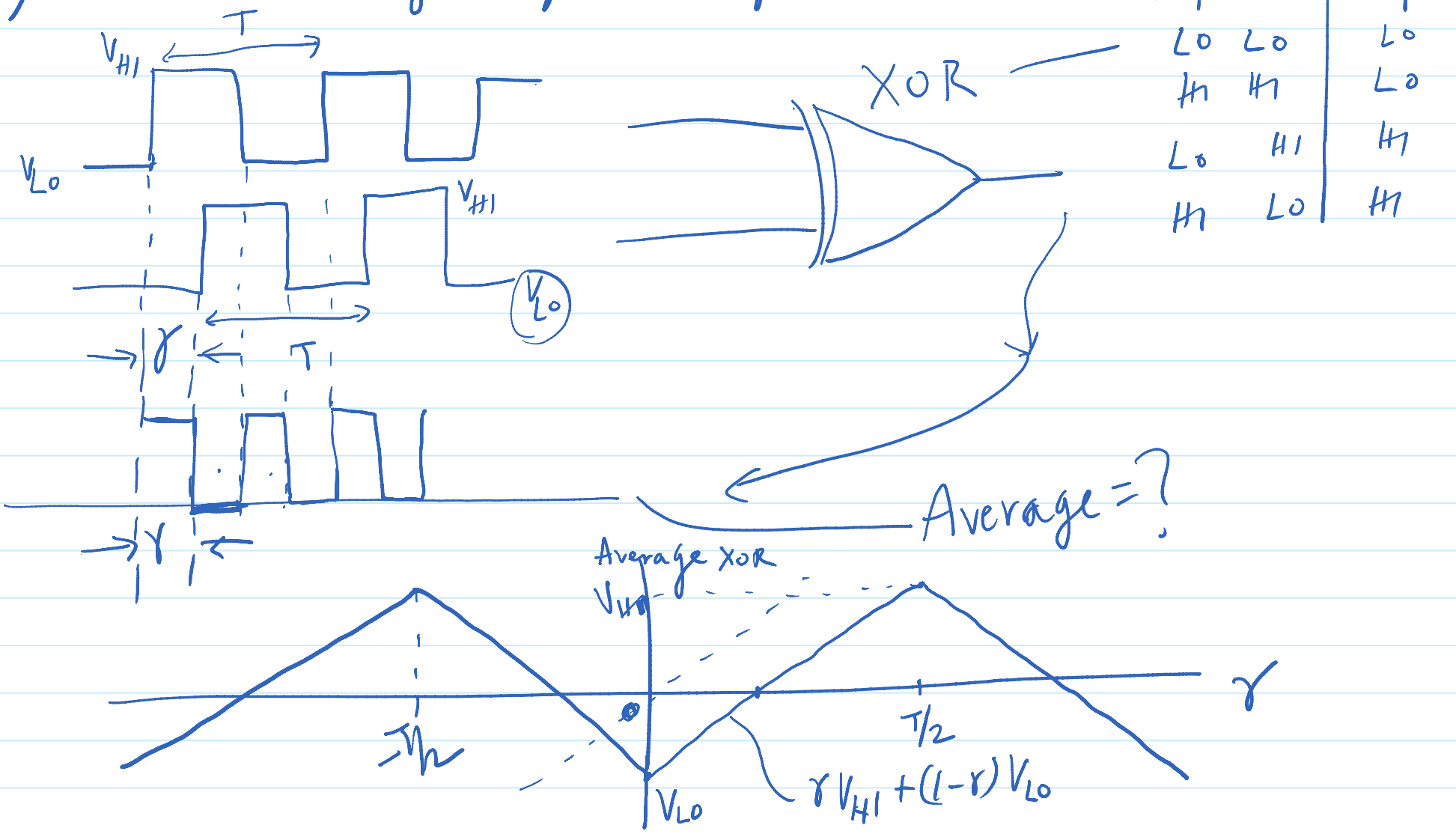
$$\theta_o(s) = \frac{K_V X(s)}{s} \leftarrow \qquad \theta_o(t) = K_V \int_0^t x(\tau)\,d\tau + \theta_o(0)$$

($\theta_o(0) = \pi/2$)

L22

18 September 2017 10:25

2) Phase detector for digital waveforms



L 22

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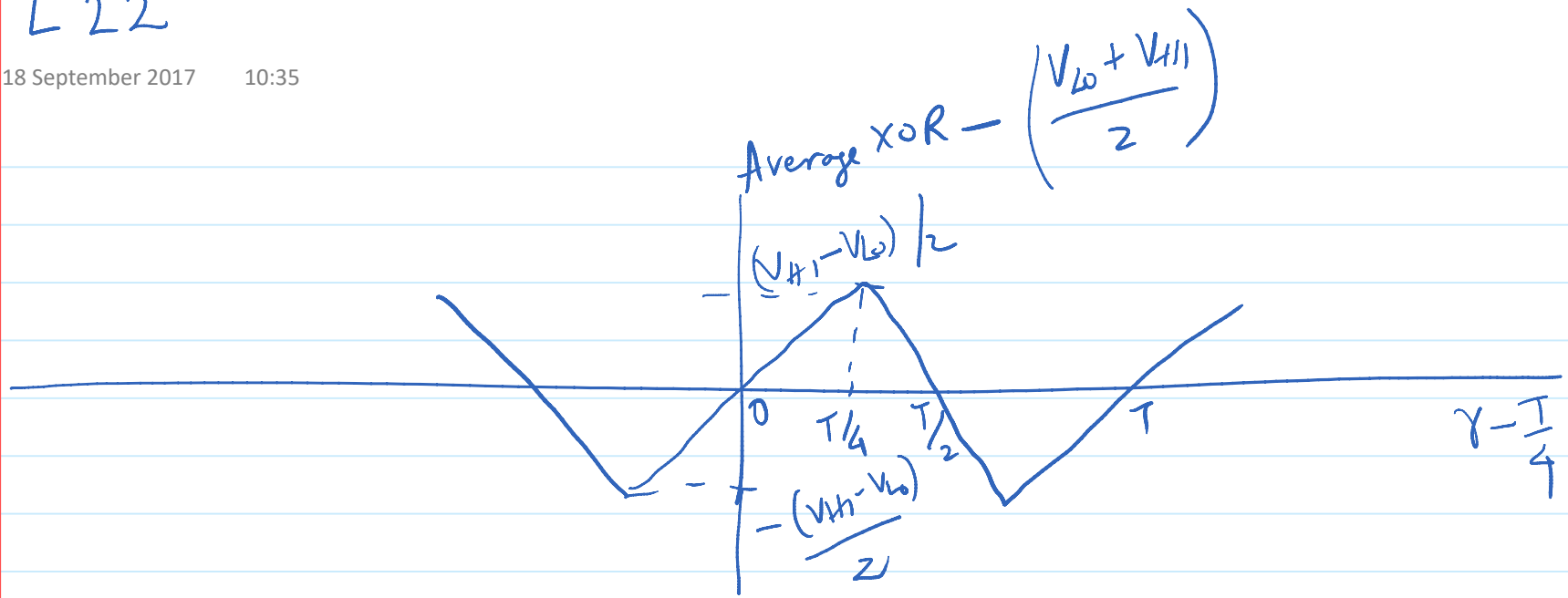
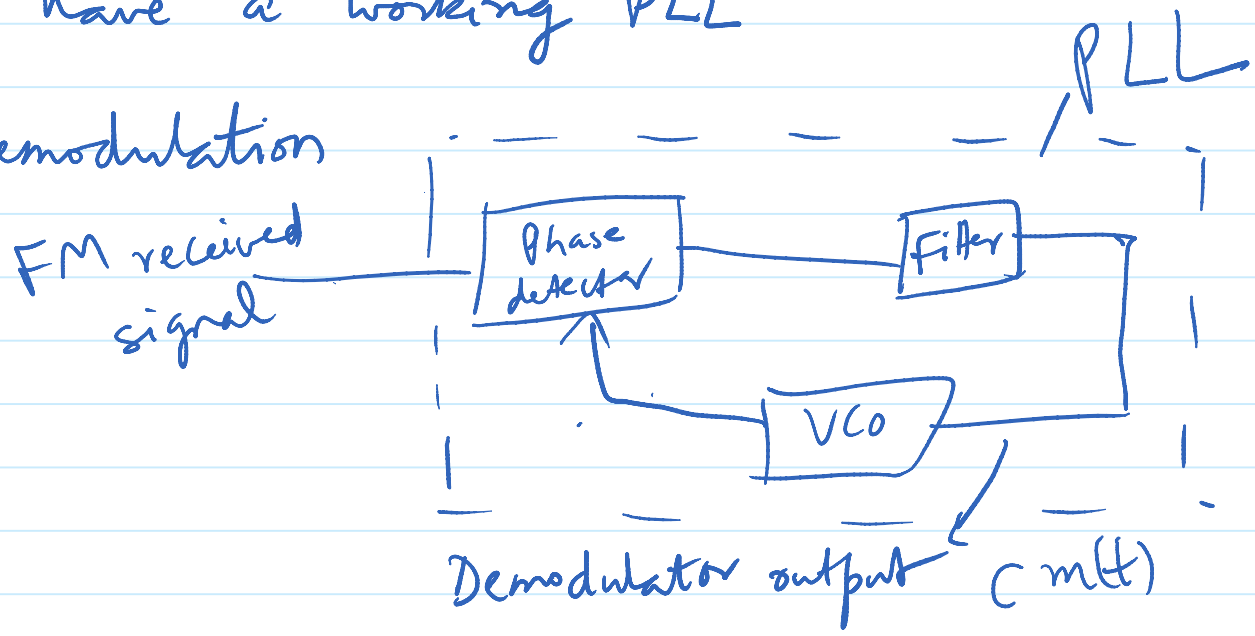


Suppose we have a working PLL

1) FM demodulation



"lock"

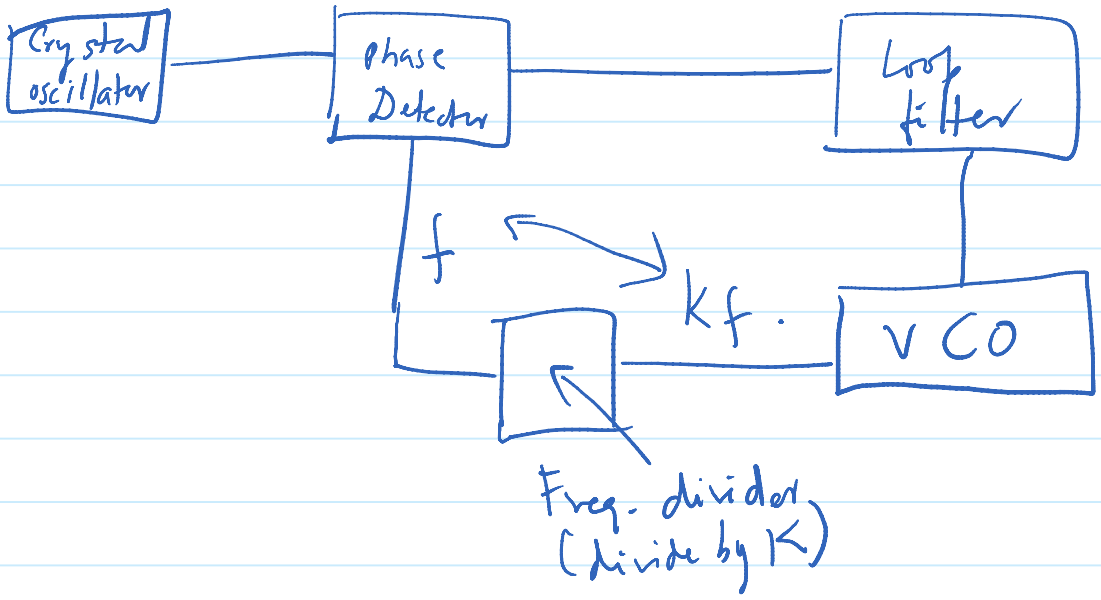
L22

18 September 2017 10:44

2) Frequency synthesis

Eg: We have a crystal oscillator at 40 MHz

Need an oscillator at 1 GHz ← 25 times



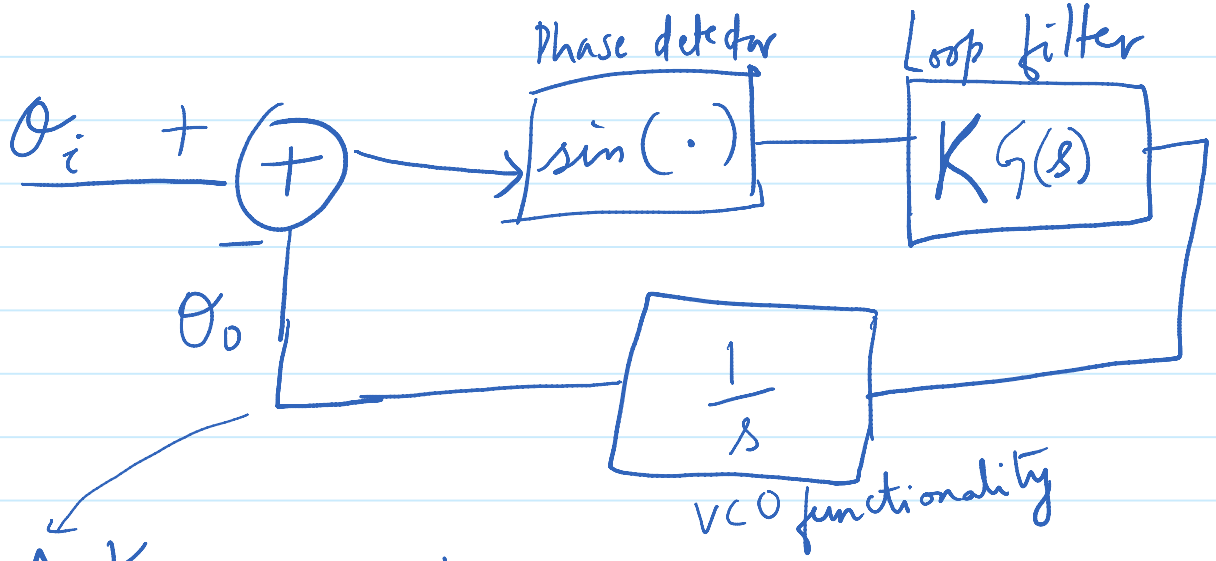
Lecture 23

19 September 2017 08:59

Phase Locked Loop

* Develop a model for the PLL and analyze. error.

Model 1
(Nonlinear)



$$K = \frac{1}{2} A_c A_v K_v$$

K_v ← VCO constant

— $2f_c$ term at the output of the phase detector ignored.

L23

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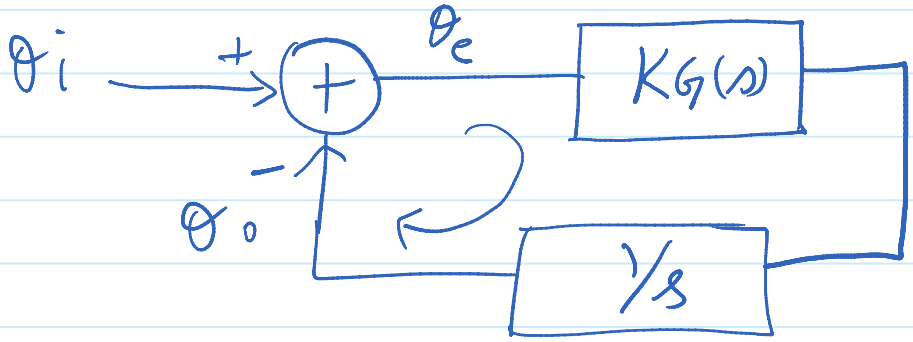
Linearized model

When $\theta_i - \theta_0$ is small (PLL is in "tracking mode")

$$\sin(\theta_i - \theta_0) \approx \theta_i - \theta_0$$



$$\theta_0(s) = \theta_e(s)\frac{KG(s)}{s}$$

$$= \left(\theta_i(s) - \theta_0(s)\right)\frac{KG(s)}{s}$$

$$\theta_0(s)\left[1 + \frac{KG(s)}{s}\right] = \theta_i(s)\frac{KG(s)}{s}$$

$$H(s) \triangleq \frac{\theta_0(s)}{\theta_i(s)} \qquad H_e(s) \triangleq \frac{\theta_e(s)}{\theta_i(s)}$$

L23

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$$H(s) = \frac{\theta_o(s)}{\theta_i(s)} = \frac{KG(s)/s}{1 + \frac{KG(s)}{s}} = \frac{KG(s)}{s + KG(s)}.$$

$$H_e(s) = \frac{\theta_e(s)}{\theta_i(s)} = \frac{\theta_i(s) - \theta_o(s)}{\theta_i(s)} = 1 - H(s)$$

$$= \frac{s}{s + KG(s)}.$$

Instantaneous frequency offset

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} \longrightarrow F_i(s) = \frac{s\,\theta_i(s)}{2\pi}$$

$$f_o(t) = \frac{1}{2\pi}\frac{d\theta_o(t)}{dt} \longrightarrow F_o(s) = \frac{s\,\theta_o(s)}{2\pi}$$

$$f_e(t) = f_i(t) - f_o(t) \longrightarrow F_e(s) = \frac{s\,\theta_e(s)}{2\pi}$$

$$\frac{F_o(s)}{F_i(s)} = H(s)$$

$$\frac{F_e(s)}{F_i(s)} = H_e(s)$$

L23

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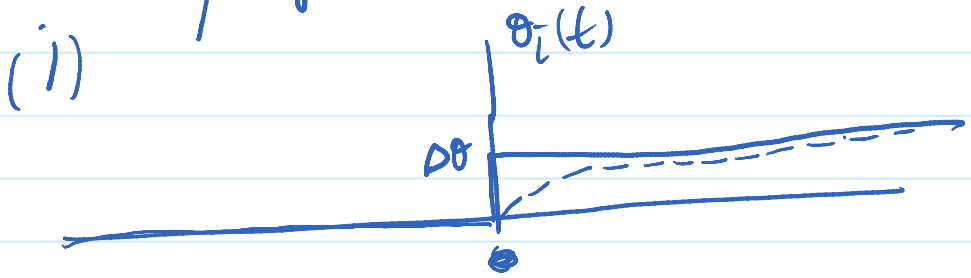
Case 1 $G(s) = 1$ (First-order PLL)

$$H(s) = \frac{K}{s+K} \qquad H_e(s) = \frac{s}{s+K}.$$

(Stable for $K > 0$) ← ensures negative feedback.

Some specific scenarios

(1)



$$\theta_i(s) = \frac{\Delta\theta}{s}$$

$$\theta_o(s) = H(s)\,\theta_i(s) = \frac{K}{s+k}\,\frac{\Delta\theta}{s}$$

$$= \frac{\Delta\theta}{s} - \frac{\Delta\theta}{s+K}$$

$$\theta_o(t) = \Delta\theta\left(1 - e^{-Kt}\right) I_{[0,\infty)}(t)$$

As $t \to \infty$, $\theta_e(t) \to 0$. First order PLL can track (exponentially fast), a sudden change in phase at the input.

L23

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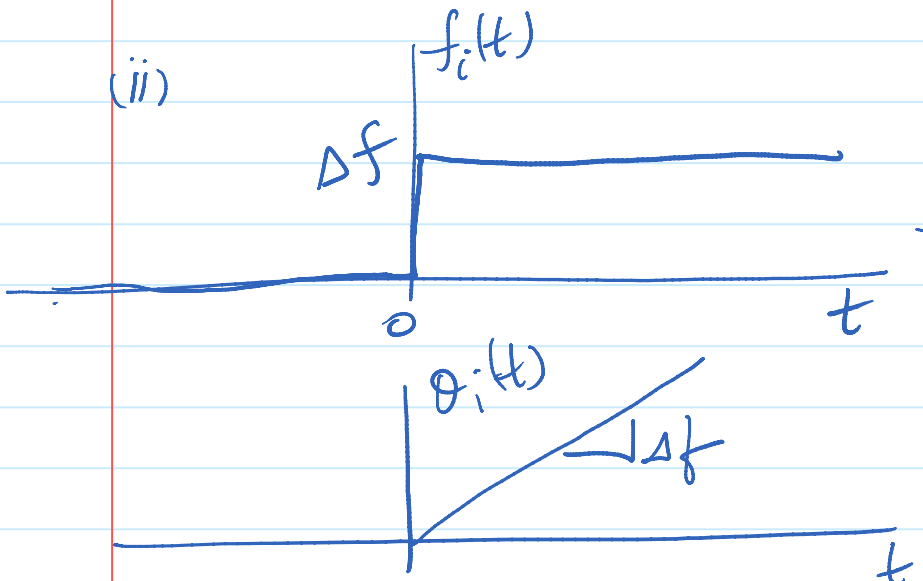
(ii)



$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

$$F_i(s) = \frac{\Delta f}{s}$$

$$\theta_i(t) = 2\pi \int_0^t f_i(\tau)\, d\tau \quad (\theta_i(0) \overset{\text{Assume}}{=} 0).$$

$$\theta_i(s) = \frac{2\pi}{s} F_i(s) = \frac{2\pi \Delta f}{s^2}$$

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\theta_e(s)$$

$$= \lim_{s\to 0} s \frac{s}{s+K} \cdot \frac{2\pi\Delta f}{s^2} = \frac{2\pi\Delta f}{K} \neq 0$$

$$\lim_{t\to\infty} f_e(t) = 0$$

PLL can track jump in instantaneous freq offset, but there is steady-state phase error equal to $2\pi\Delta f/K$.

Lecture 24

20 September 2017 08:00

Case 2: (Second order PLL) $G(s) = 1 + \frac{a}{s}$. $(a>0)$.

Proportional feedback

Integral feedback

$$\frac{\theta_o(s)}{\theta_i(s)} = H(s) = \frac{KG(s)}{s + KG(s)} = \frac{K(s+a)/s}{s + K(s+a)/s} = \frac{K(s+a)}{s^2 + K(s+a)}.$$

$$\frac{\theta_e(s)}{\theta_i(s)} = H_e(s) = \frac{s}{s + KG(s)} = \frac{s^2}{s^2 + K(s+a)}.$$

Poles $s^2 + K(s+a) = 0$ $\quad s = \frac{-K \pm \sqrt{K^2 - 4aK}}{2}$

if $K^2 - 4Ka > 0$ $(K > 4a)$, $K^2 - \underline{4a}K = \underline{(K-4a)}K < K^2$

Poles in LHP.

L24

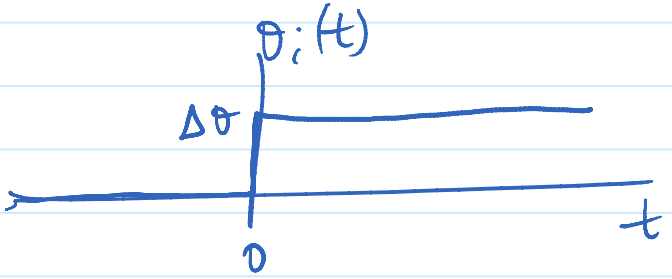
20 September 2017 08:10

if $K^2 - 4aK < 0$ $(K < 4a)$, poles are in LHP if $\underline{K > 0}$.

$\Rightarrow$ When $K > 0$, poles are in LHP

For stability, need $K > 0$.

(i) Suppose $\theta_i(t) = \Delta\theta \, I_{[0,\infty)}(t)$



$$\theta_e(s) = \theta_i(s) H_e(s) = \frac{\Delta\theta}{s} \cdot \frac{s^2}{s^2 + k(s+a)} = \frac{\Delta\theta \, s}{s^2 + k(s+a)}.$$

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\,\theta_e(s) = \lim_{s\to 0} \frac{s^2 \Delta\theta}{s^2 + k(s+a)} = 0$$

Final value theorem

L24

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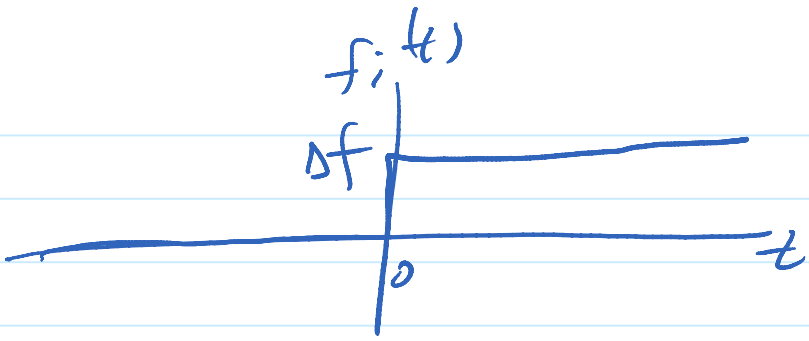
(ii) Suppose $f_i(t) = \Delta f \, I_{[0,\infty)}(t)$



$\lim_{t\to\infty} f_e(t) = ?$ $\qquad$ $\lim_{t\to\infty} v_e(t) = ?$

$$\lim_{s\to 0} s F_e(s) = \lim_{s\to 0} s F_i(s) H_e(s)$$

$$= \lim_{s\to 0} s \frac{\Delta f}{s} \frac{s^2}{s^2 + K(s+a)} = 0$$

L24

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$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\theta_e(s) = \lim_{s\to 0} s \cdot \theta_i(s) \cdot H_e(s)$$

$$= \lim_{s\to 0} s \cdot \frac{\Delta f}{s^2} \cdot \frac{s^2}{s^2 + K(s+a)} = 0$$

For the second order PLL, both $\theta_e(t)$ & $f_e(t) \to 0$ as $t \to \infty$ (for $f_i(t) = \Delta f \, I_{[0,\infty)}(t)$)

So for Linearized model ⟶ Transfer fn ⟶ Steady state error for some inputs.

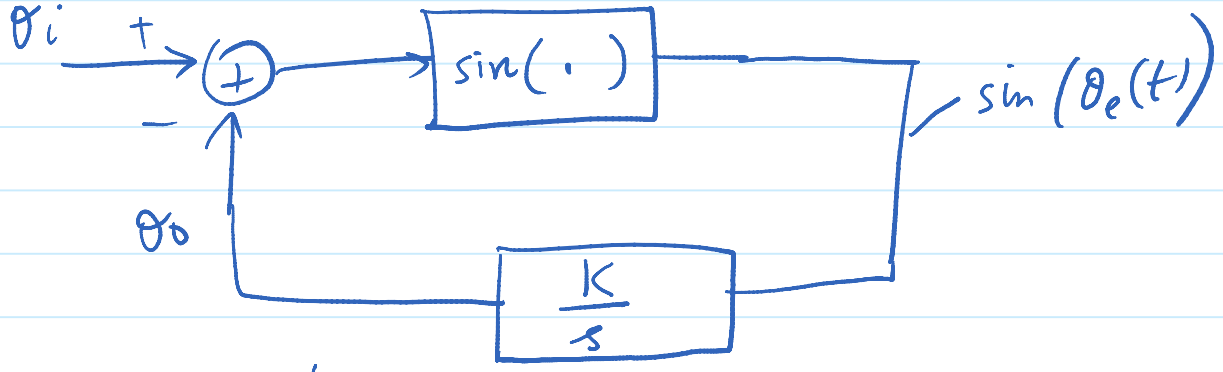
(First, second order)

L24

20 September 2017 08:23

Nonlinear model for the first order PLL

$G(s) = 1$.



$$\begin{cases} \theta_o(t) = \int_0^t K \sin(\theta_e(\tau))\, d\tau \\ \theta_e(t) = \theta_i(t) - \theta_o(t). \end{cases}$$

(Assume $\theta_o(0) = 0$)

$$\int_0^t K \sin(\theta_e(\tau))\, d\tau = \theta_i(t) - \theta_e(t).$$

L24

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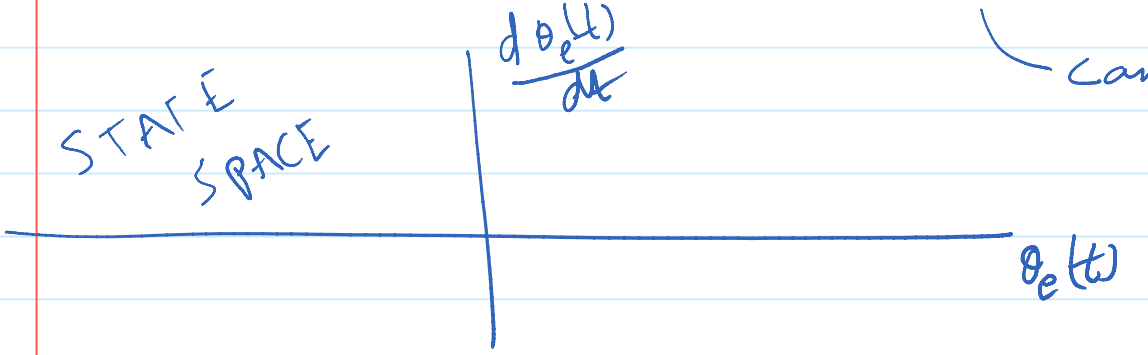
$$K \sin(\theta_e(t)) = \frac{d\theta_i(t)}{dt} - \frac{d\theta_e(t)}{dt}$$

* Consider step change in $f_i(t)$ $\left(f_i(t) = \Delta f \, \mathbb{I}_{[0,\infty)}(t) \right)$.

$$\frac{d\theta_i(t)}{dt} = 2\pi \Delta f \quad \left(\text{for } t > 0 \right).$$

$$\boxed{K \sin(\theta_e(t)) = 2\pi \Delta f - \frac{d\theta_e(t)}{dt}}$$

↳ cannot solve analytically.



$\left(\text{Want } \frac{d\theta_e(t)}{dt} = 0 \text{ in steady state} \right)$

L24

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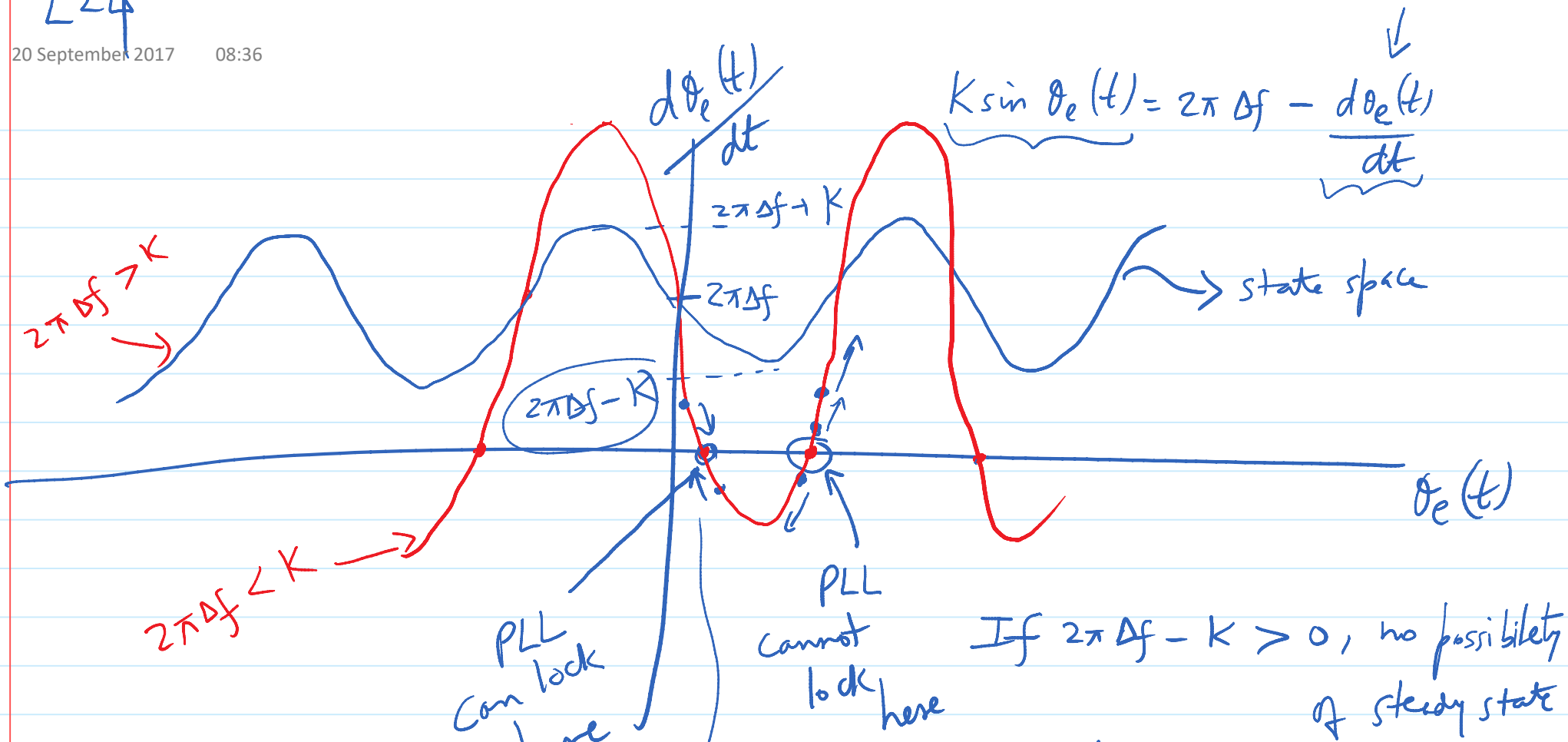


$K \sin\theta_e(t) = 2\pi\Delta f - \frac{d\theta_e(t)}{dt}$

If $2\pi\Delta f - K > 0$, no possibility of steady state

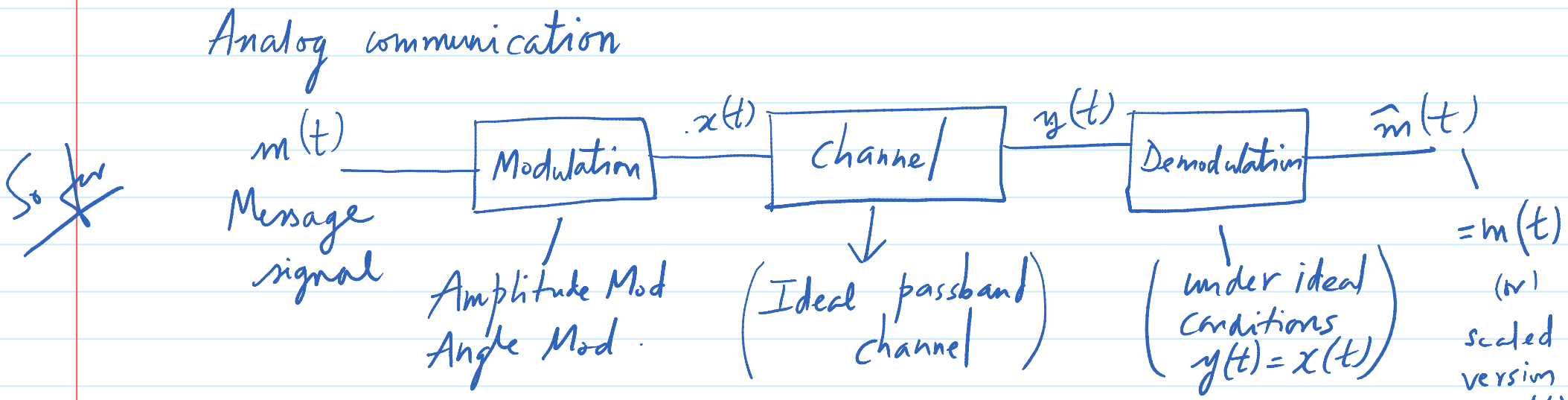
Steady state phase error $= \sin^{-1}\left(\frac{2\pi\Delta f}{K}\right)$ (Need $K > 2\pi\Delta f$)

(Example 3.5.1 in the book)

Lecture 25

22 September 2017 13:02

Analog communication



Suppose $y(t) = x(t) + n(t)$ (Noise) $\rightarrow \hat{m}(t) \neq m(t)$

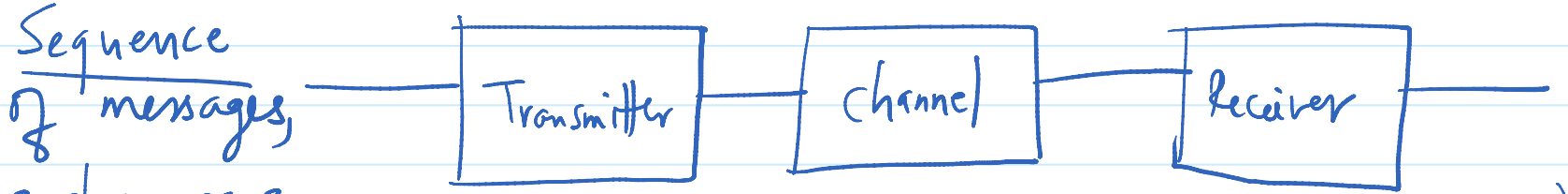
Performance measure $\int |m(t) - \hat{m}(t)|^2 dt$

AM: I & Q of Noise gets added

L 25

25 September 2017 10:11

Digital communication



Sequence of messages, each message takes one of a finite no. of possibilities.

eg. 0 1 1 0 . . . .

a b c d c b . . .

Digital source eg. Files stored on a computer

Analog source eg. Speech $\xrightarrow{\text{(source coding)}}$ Sequence of messages

↓ approximation

(sampling, quantization, coding)

L25

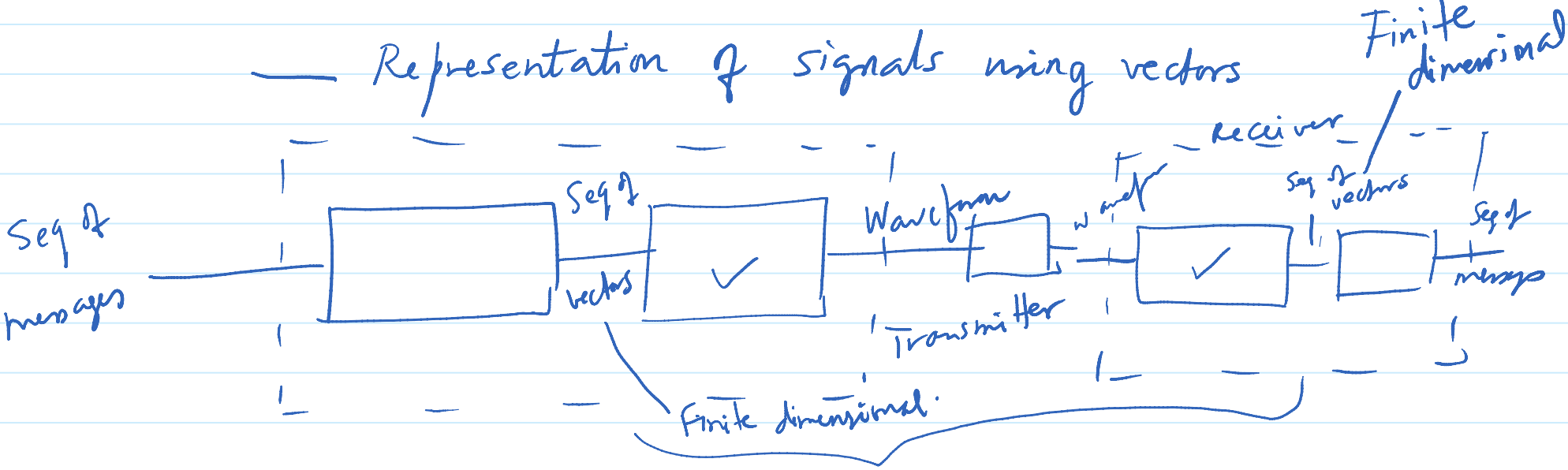
25 September 2017 10:19

Metrics

- Probability of error
- Rate: How many messages / sec ?
- Bandwidth

Basics of digital communication

— Representation of signals using vectors



L25

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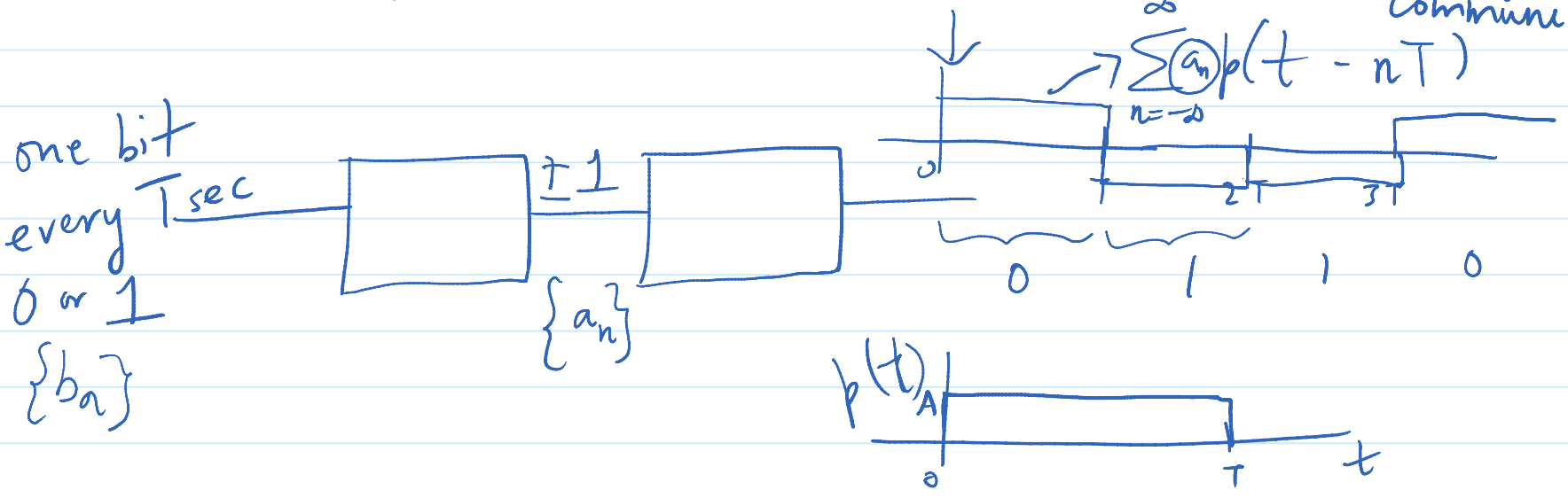
— Noise model : Gaussian noise

— Equivalent vector channel model

— Hypothesis testing (Receiver design) : Binary digital communication system



L25

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Representation of signals as vectors:

Finite number of signals $s_1(t), s_2(t), \dots, s_K(t)$ $\longrightarrow$ $\underline{s}_1, \underline{s}_2, \dots, \underline{s}_K$.

$$\underline{s}_i = [s_{i1}\ s_{i2}\ \dots\ s_{ik}]$$

$$s_i(t) = \sum_{j=1}^{k} s_{ij}\, \phi_j(t)$$

Lecture 26

26 September 2017 09:05

Given $f_1(t), f_2(t), \ldots, f_k(t)$ that are orthonormal, i.e.,

$$\int f_i(t) f_j^*(t)\, dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(each f_i has unit energy, f_i, f_j orthogonal for $i \neq j$)

What is the best approximate representation of $s(t)$ in terms of these fns.?

Let $\hat{s}(t) = \sum_{i=1}^{k} s_k f_i(t)$

Minimize $\int |s(t) - \hat{s}(t)|^2 dt$ over all choices of $\{s_k\}$

L26

26 September 2017 09:11

Solution: optimal $s_k = \int s(t) f_k^*(t)\,dt \triangleq c_k$

(or) $\hat{s}(t) = \sum_{i=1}^{k} c_i f_i(t)$ minimizes $\int |s(t) - \hat{s}(t)|^2 dt$

Proof:

$$|s(t) - \hat{s}(t)|^2 = \left(s(t) - \sum_{k=1}^{k} s_k f_k(t)\right)\left(s(t) - \sum_{k=1}^{k} s_k f_k(t)\right)^*$$

$$= |s(t)|^2 - \sum_{k=1}^{k} s_k^* s(t) f_k^*(t) - \sum_{k=1}^{k} s_k s^*(t) f_k(t)$$

$$+ \sum_{k=1}^{k}\sum_{l=1}^{k} s_k s_l^* f_k(t) f_l^*(t)$$

$$\int |s(t) - \hat{s}(t)|^2 dt = ?$$

L26

26 September 2017 09:18

$$\int |s(t) - \hat{s}(t)|^2 dt = E_s - \sum_{k=1}^{K} s_k^* c_k - \sum_{k=1}^{K} s_k c_k^* + \sum_{k=1}^{K} |s_k|^2$$

$$\min_{\{s_k\}} (\quad)$$

$$= E_s - \sum_{k=1}^{K} |c_k|^2 + \sum_{k=1}^{K} |c_k|^2 - \sum_{k=1}^{K} s_k^* c_k - \sum_{k=1}^{K} s_k c_k^* + \sum_{k=1}^{K} |s_k|^2$$

$$= E_s - \sum_{k=1}^{K} |c_k|^2 + \sum_{k=1}^{K} |c_k - s_k|^2 \geq 0$$

$E_s - \sum_{k=1}^{K} |c_k|^2$: Does not depend on s_k

L 26

26 September 2017 09:26

Optimal $s_k = c_k$

Energy of the error for this choice = $E_s - \sum_{k=1}^{K} |c_k|^2$

Given $s_1(t), s_2(t), \ldots, s_M(t)$. Find orthonormal fns $f_1(t), f_2(t), \ldots, f_N(t)$ that can represent $s_1(t), \ldots, s_M(t)$ exactly. $(N \leq M)$

– Gram-schmidt procedure.

L26

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step 1: $f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$ where $E_1 = \int |s_1(t)|^2 dt$

step 2: $f_2(t) = \frac{s_2(t) - c_{21} f_1(t)}{\sqrt{E_n (s_2(t) - c_{21} f_1(t))}}$ $\perp f_1(t)$

$E_2 - |c_{21}|^2$

$\hat{s}_2(t) = c_{21} f_1(t)$

$\rightarrow c_{21} = \int s_2(t) f_1^*(t) dt$

$\int (s(t) - \hat{s}(t)) f_i(t) dt$

$= c_i - c_i = 0.$

for $i = 2$ to M

$$f_i(t) = \frac{s_i(t) - \sum_{k=1}^{i-1} c_{ik} f_i(t)}{\sqrt{E_n (s_i(t) - \sum_{k=1}^{i-1} c_{ik} f_i(t))}} \quad (\text{if} \neq 0)$$

L26

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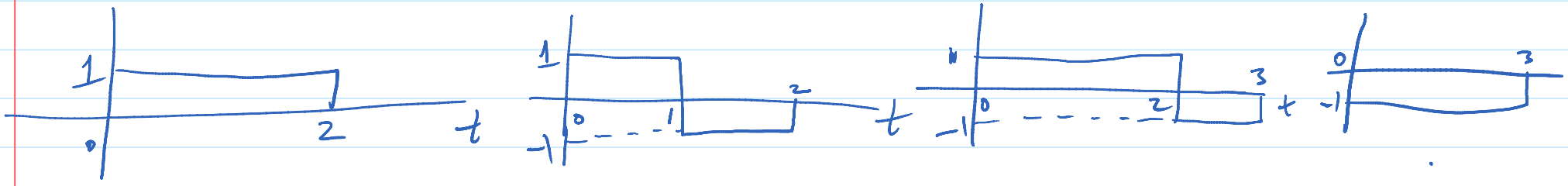
- Can choose any order (any permutation of $1, \ldots, M$) for the G-S procedure

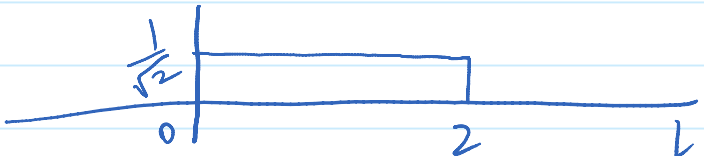
Example:

$s_1(t)$ $s_2(t)$ $s_3(t)$ $s_4(t)$



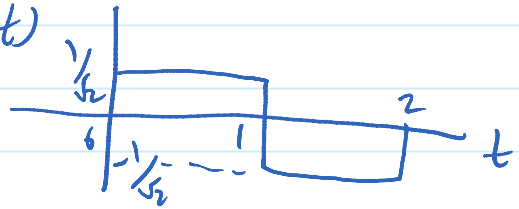
Using G-S procedure (with order 1, 2, 3, 4)

$f_1(t)$



$c_{21} = 0$

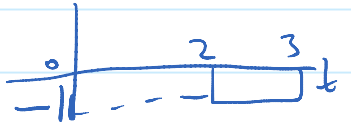
$f_2(t)$



$c_{31} = \sqrt{2}$

$c_{32} = 0$

Error



L26

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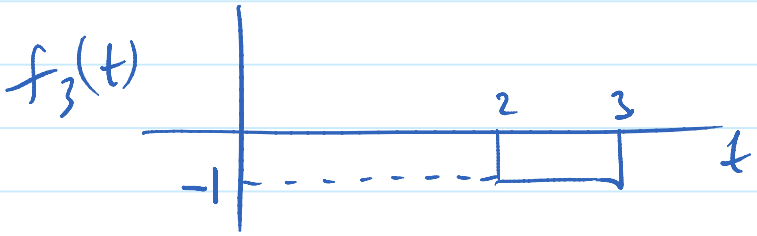
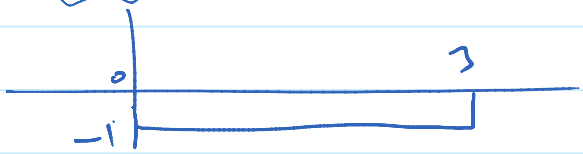


$$c_{41} = -\sqrt{2}$$

$$c_{42} = 0$$

$$c_{43} = 1$$

$$c_{41} f_1(t) + c_{42} f_2(t) + c_{43} f_3(t)$$

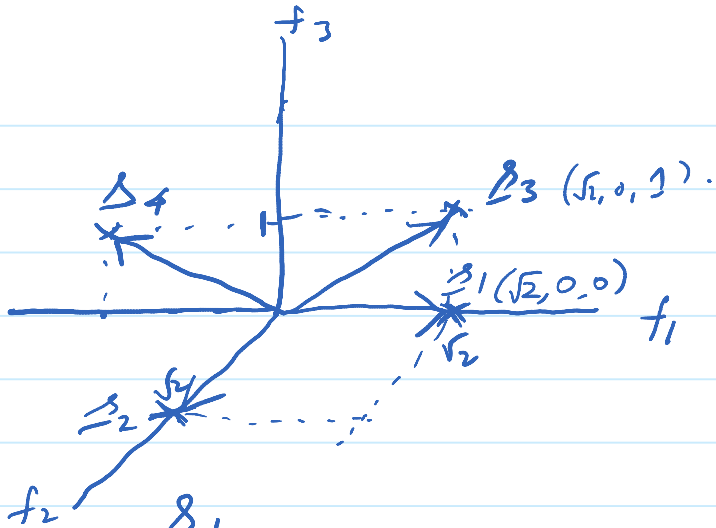


$f_1(t), f_2(t), f_3(t)$ are the required orthonormal fns.

Lecture 27

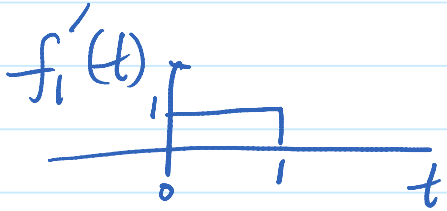
26 September 2017 10:23

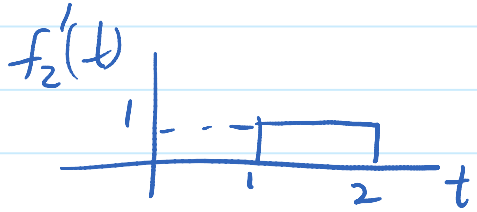
$$s_1(t) = \sqrt{2}\, f_1(t)$$
$$s_2(t) = \sqrt{2}\, f_2(t)$$
$$s_3(t) = \sqrt{2}\, f_1(t) + f_3(t)$$
$$s_4(t) = -\sqrt{2}\, f_1(t) + f_3(t)$$

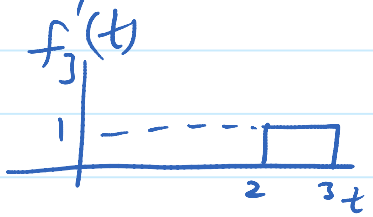


$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

Another choice of basis functions

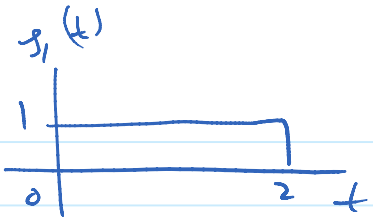


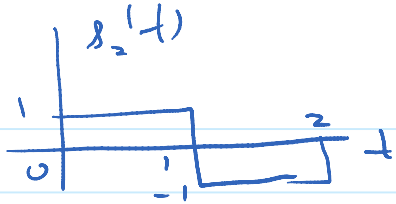


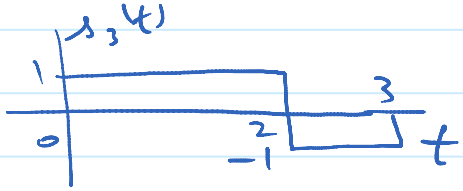


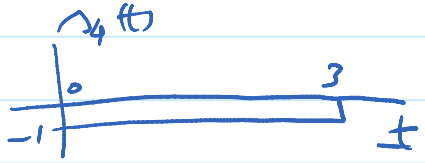
L27

27 September 2017 08:14





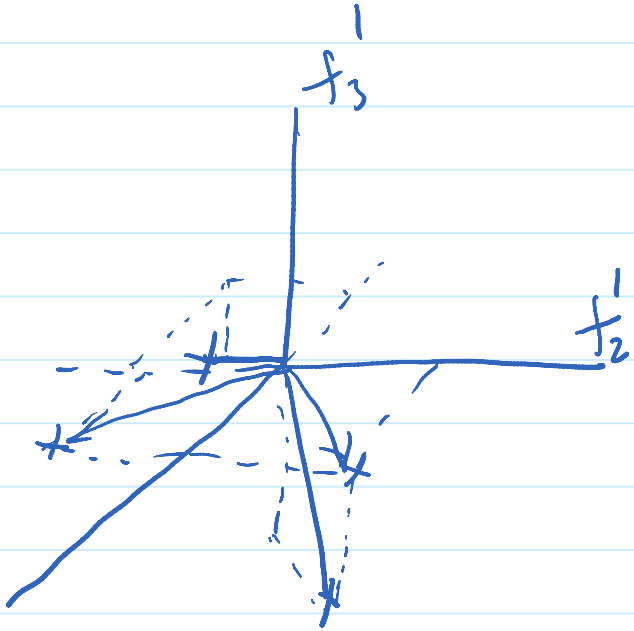




$$s_1(t) = f_1'(t) + f_2'(t)$$
$$s_2(t) = f_1'(t) - f_2'(t)$$
$$s_3(t) = f_1'(t) + f_2'(t) - f_3'(t)$$
$$s_4(t) = -f_1'(t) - f_2'(t) - f_3'(t)$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$



L 27

27 September 2017 08:24

$$\text{Energy of } s_i(t) = \int |s_i(t)|^2 \, dt = \int \left| \sum_{k=1}^{K} c_k f_k(t) \right|^2 dt$$

$$= \sum_{k=1}^{K} |c_k|^2 \checkmark$$

$$\langle s_i(t), s_j(t) \rangle = \int \left(\sum_{k=1}^{K} c_{ik} f_k(t) \right) \overline{\left(\sum_{\ell=1}^{K} c_{j\ell} f_\ell(t) \right)^*} \, dt$$

$$= \sum_{k=1}^{K} c_{ik} c_{jk}^*$$

Irrespective of the choice of basis fns., length of the vectors & their inner products are going to be the same.

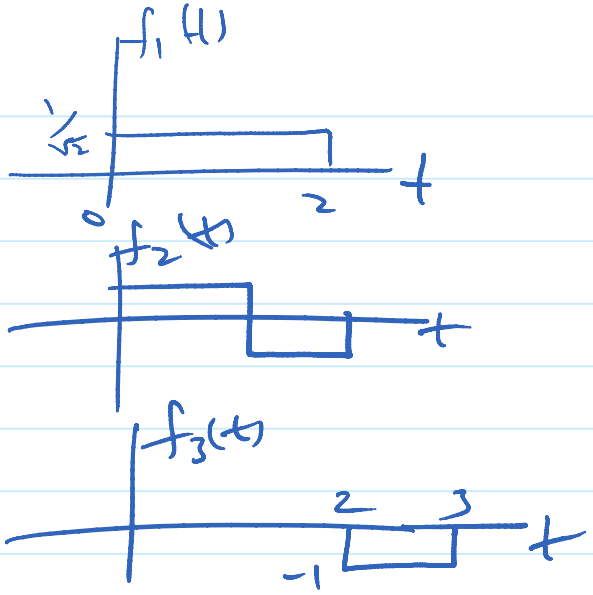
L27

27 September 2017 08:31

$$f_1(t) = \frac{1}{\sqrt{2}} f_1'(t) + \frac{1}{\sqrt{2}} f_2'(t)$$

$$f_2(t) = \frac{1}{\sqrt{2}} f_1'(t) - \frac{1}{\sqrt{2}} f_2'(t)$$

$$f_3(t) = -f_3'(t)$$

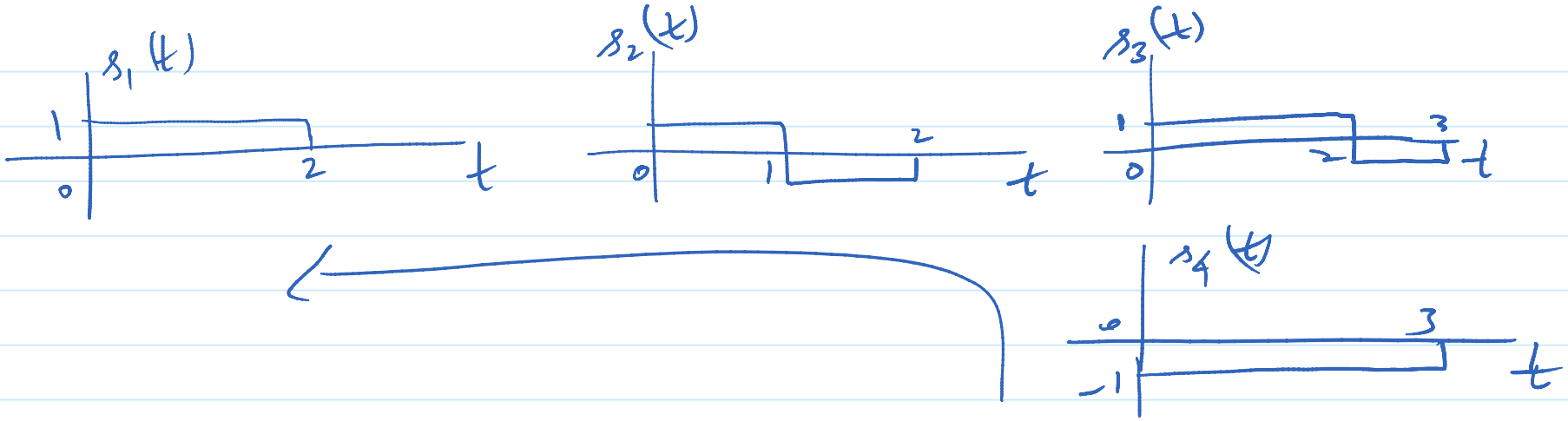


$$\begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & (-1) \end{bmatrix}}_{\text{Unitary}} \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

L27

27 September 2017 08:37



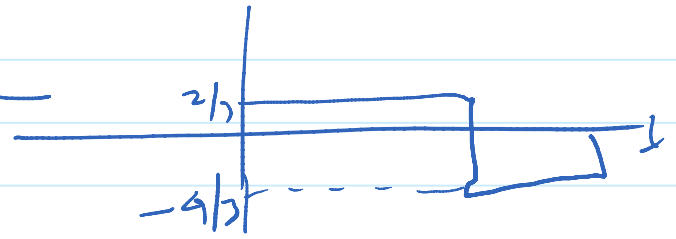
$$\begin{cases} g_1(t) = \frac{1}{\sqrt{3}} s_4(t) \\ g_2(t) = (\quad) \\ g_3(t) = \frac{1}{\sqrt{2}} s_2(t) \end{cases}$$

$\langle s_3, g_1 \rangle = -1/\sqrt{3}$

$s_3(t) - \left(\frac{-1}{\sqrt{3}} g_1(t)\right)$



$\frac{1}{\sqrt{8/3}}$



$$\begin{bmatrix} ( ) & ( ) & ( ) \\ 0 & 0 & \sqrt{2} \\ -1/\sqrt{2} & ( ) & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}$$

Lecture 28

03 October 2017 08:39

- Representing signals as vectors
  - * Need a set of orthonormal basis fns
  - * Using this set, we can get a vector representation.

$\{s_1(t), \ldots, s_K(t)\}$ — Set of K signals — $\xrightarrow{\text{Gram-Schmidt procedure}}$ $\{f_1(t), \ldots, f_N(t)\}$ $N \leq K$ orthonormal basis fns.

$$s_i(t) = \sum_{j=1}^{N} c_{ij} f_j(t)$$

$s_i(t)$: Signal

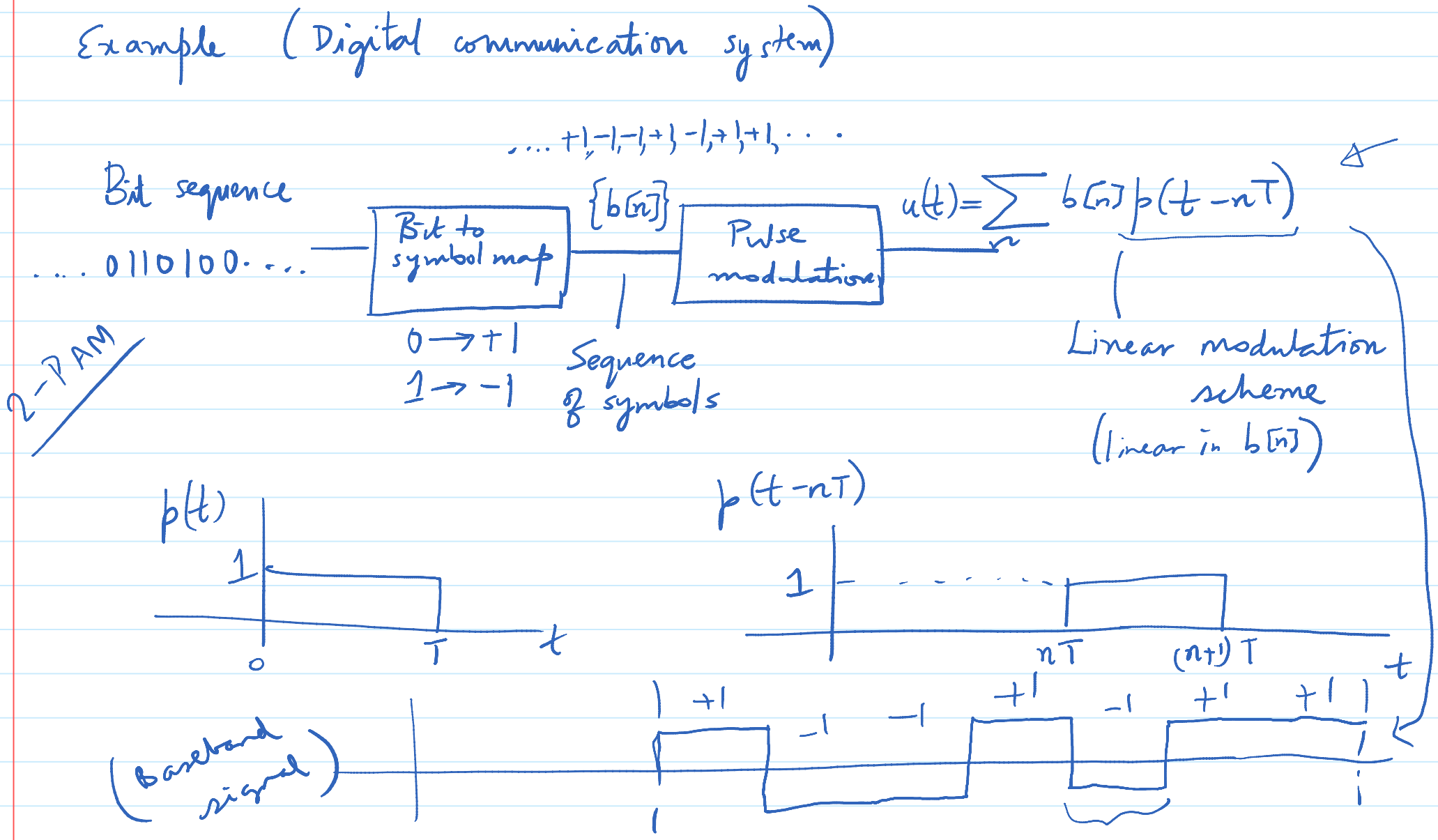
$c_{ij} \rightarrow [c_{i1}\ c_{i2} \cdots c_{iN}] \triangleq \underline{s}_i$ Vector representation of $s_i(t)$

$$c_{ij} = \int s_i(t) f_j^*(t)\, dt$$

L 28

03 October 2017 09:12

Example (Digital communication system)



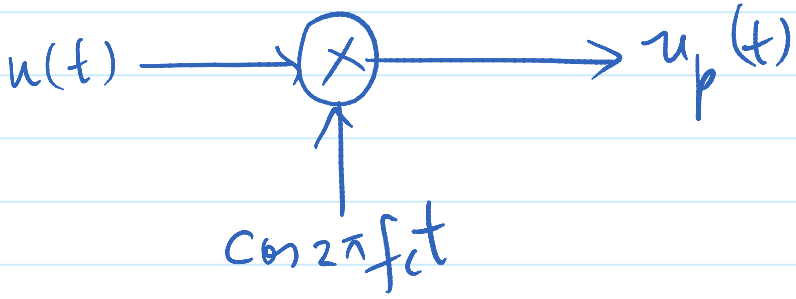
L 28

03 October 2017 09:22

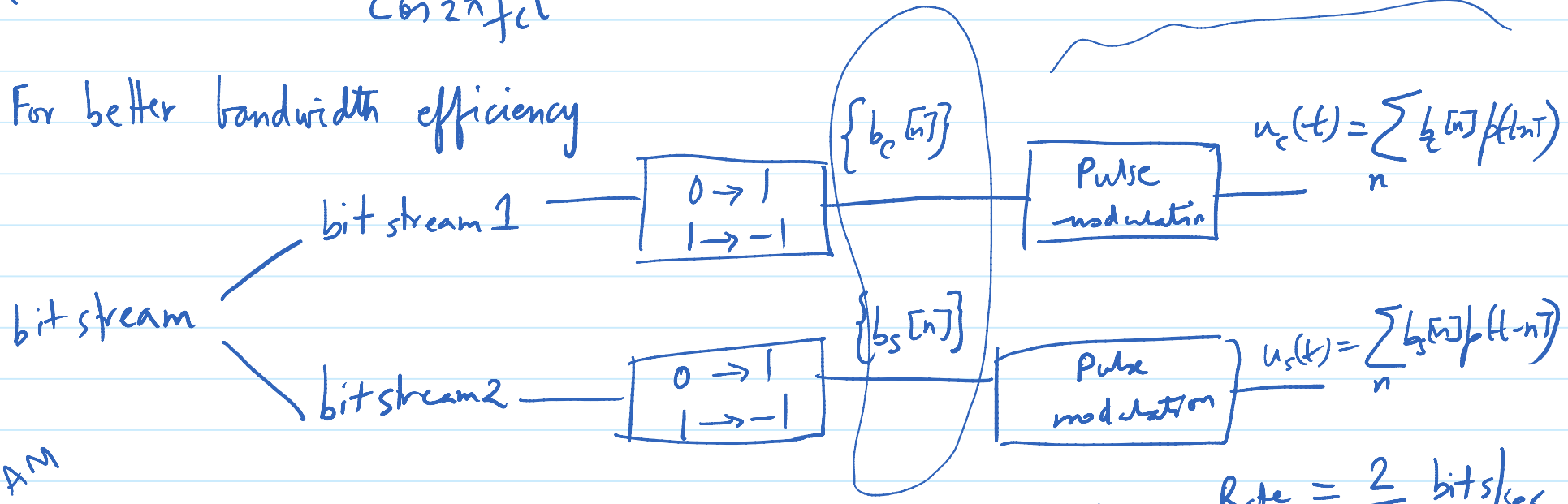
In case we have a passband channel.

BW related to $\frac{1}{T}$ ($\approx \frac{k}{T}$)

Rate of communication $= \frac{1}{T}$ bits/sec



For better bandwidth efficiency



4-QAM

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

Rate $= \frac{2}{T}$ bits/sec

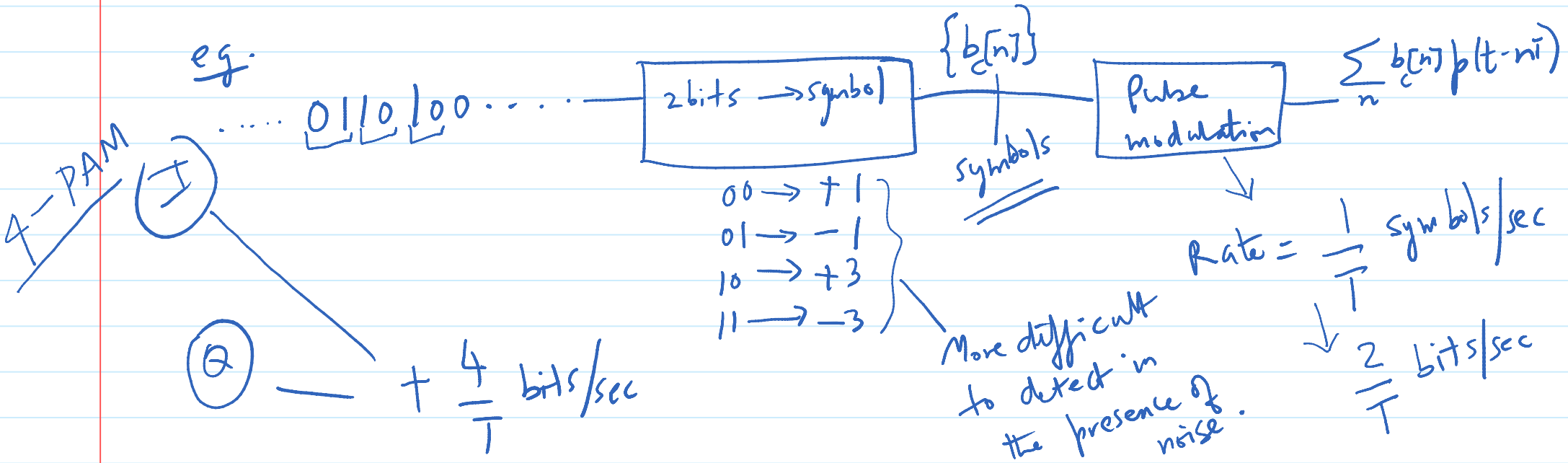
L28

03 October 2017 09:28

Given a BW for the system, there will an upper limit on $\frac{1}{T}$.

$\rightarrow$ upper limit on rate ($\frac{2}{T}$ bits/sec for the binary modulation scheme above)

Rate can also be increased by increasing the # possible symbols.



L28

03 October 2017 09:37

Vector representation of symbols:

Binary (I)



2 possible vectors

$\{+1, -1\}$

2-PAM

Signal constellations

(I & Q)

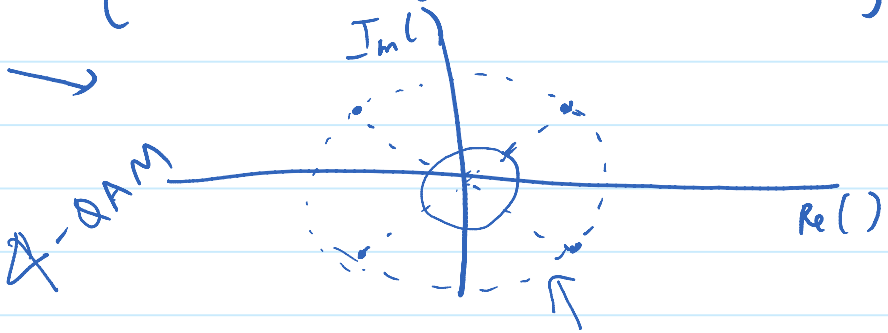
complex baseband signal

$u_c(t) + j u_s(t)$

$= \sum_n (b_c[n] + j b_s[n]) p(t - nT)$

4 possible vectors

$\{1+j, 1-j, -1+j, -1-j\}$



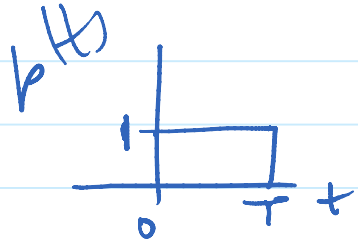
L28

03 October 2017 09:44

Passband signal $u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$

$= \mathrm{Re}\left[u(t)\, e^{j2\pi f_c t}\right]$

$= \mathrm{Re}\left[\sum_n (b_c[n] + j\, b_s[n])\, p(t-nT)\, e^{j2\pi f_c t}\right]$



$\mathrm{Re}\left((b_c[n] + j\, b_s[n])\, e^{j2\pi f_c t}\right)$

Passband signal in $[nT, (n+1)T)$

QPSK
4-PSK
4-phase shift keying

$u_p(t) = c\cos(2\pi f_c t + \theta[n])$

$c\, e^{j\theta[n]}$ ← 4 possible $\theta[n]$

$1+j \quad 1-j \quad -1+j \quad -1-j$

$\sqrt{2}\, e^{j\pi/4} \quad \sqrt{2}\, e^{j\frac{3\pi}{4}} \quad \sqrt{2}\, e^{-j\pi/4} \quad \sqrt{2}\, e^{-j\frac{3\pi}{4}}$

Lecture 29

04 October 2017 07:48

So far

1) Pulse Amplitude Modulation (PAM) M-PAM

possible values for each input symbol

pts in signal constellation

2) M-QAM

pts in signal constellation

3) M-PSK (Phase shift Keying)

No noise case

Rx Passband $\longrightarrow$ I/Q $\xrightarrow{\text{Inner product / Projection}}$ vector representation $\downarrow$ bits

Same as Analog comm. sys.

L29

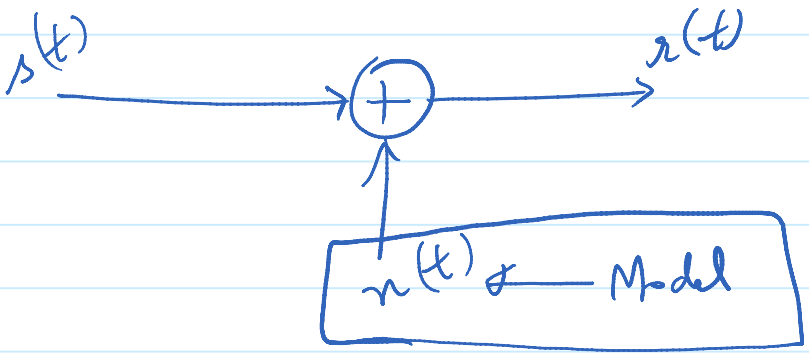
04 October 2017 08:14

- Noise models
- Receiver design for a given noise model

} ⟸ Probability and random processes.

(Conditional probability, Bayes rule, Gaussian random variables, Gaussian random processes)

Channel ⟸ Uncertainty, need to model



L 2.9

04 October 2017 08:28

Probability basics

Notion of an experiment → Outcome not deterministic

Set of possible outcomes → (Sample space Ω)

Throw a dice

$\{1, 2, 3, 4, 5, 6\}$

Event: Any subset of Ω e.g. $\{1\}$, $\{2, 4, 6\}$ — #subsets 2^6

$\{1\}$: Basic events

→ set operations (complement, union, intersection, difference ...)

Probability measure (model): Assigns probabilities to events $P(A) = (\ )$

A: event; $(\ )$: real number in $[0, 1]$

L29

04 October 2017 08:38

Assign probabilities according to the following rules.

1) $0 \leq P(A) \leq 1$

2) $P(\phi) = 0$, $P(\Omega) = 1$

3) If two events A and B are disjoint (or mutually exclusive), i.e. $A \cap B = \phi$, then

$$P(A \cup B) = P(A) + P(B).$$

Ω : discrete with 'n' elements ⟶ Assign prob. to each basic event according to 1), 2).

Prob. of any of the 2^n events easily using 3)

All 2^n events can be assigned valid probabilities.

L29

04 October 2017 08:45

eg $\Omega : \mathbb{R}$

- All possible subsets of $\mathbb{R}$ are possible events
  - May not be able to assign prob. to all these subsets while satisfying 1), 2) & 3).
  - Assign prob to subsets $(-\infty, a]$ $\forall a \in \mathbb{R}$.

✓ Events for which we can calculate prob are unions & intersections of $(-\infty, a]$

Lecture 30

06 October 2017 12:48

Based on the 3 axioms in the last lecture, we can derive some other properties.

For example:

1) $P(A^c) = 1 - P(A)$ [Use $A \cup A^c = \Omega$, $A \cap A^c = \emptyset$]

2) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

[Use $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$

Use $A_2 = (A_1 \cap A_2) + (A_2 \setminus A_1)$]

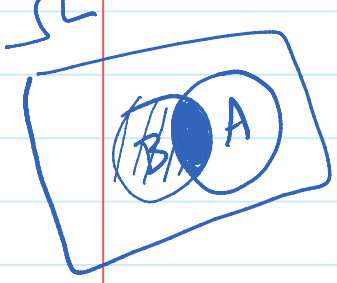
L30

06 October 2017 13:10

Conditional probability:

$P(A/B)$: Conditional probability of A given B

(Prob. of A given that we know outcome is in B)



$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{(Assume } P(B) > 0\text{)}$$

[Conditional probabilities behave just like regular probabilities.

$$P(A^c/B) = 1 - P(A/B)$$

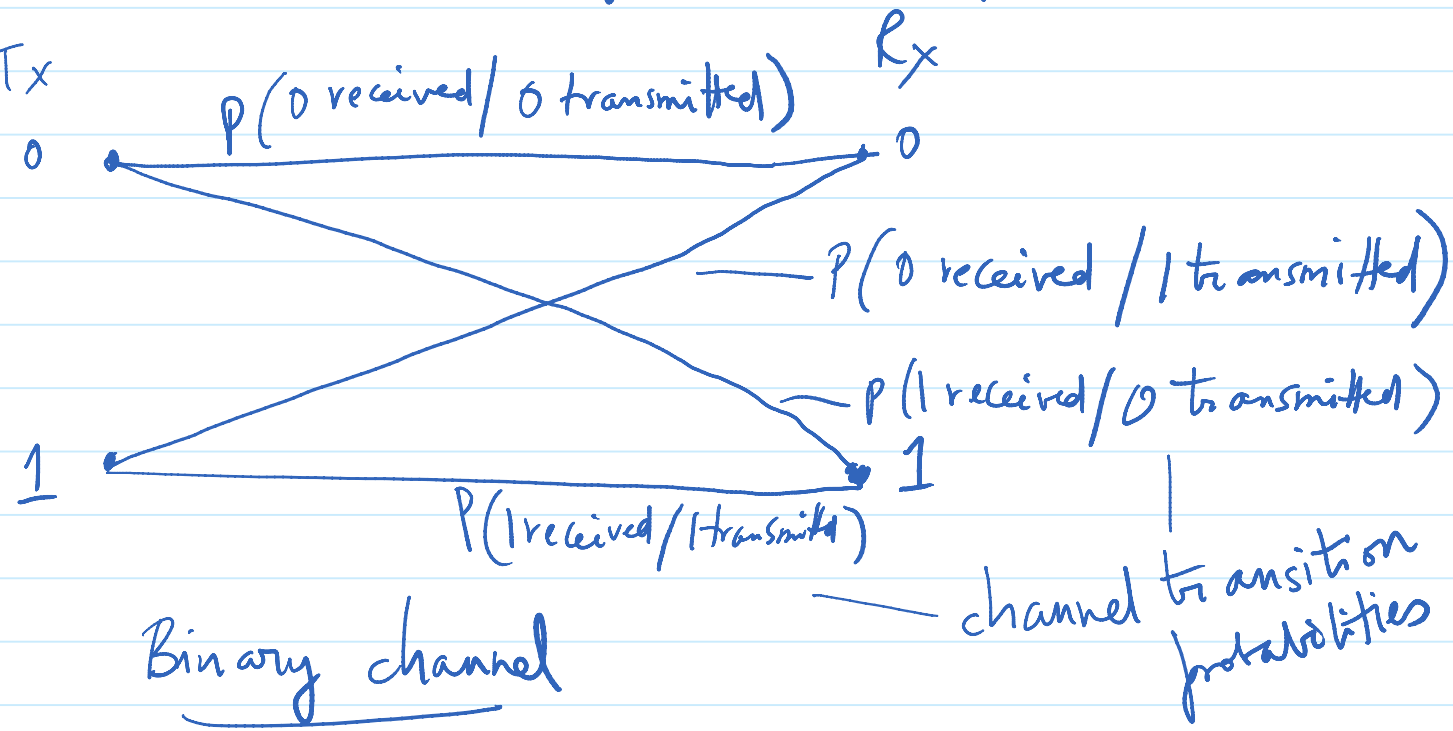
$$P(A_1 \cup A_2 / B) = P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B)]$$

L30

06 October 2017 13:15

Communication channel model using conditional probabilities:

Eg



Binary channel

L30

06 October 2017 13:20

Law of total probability

$$P[A] = P[A \cap B] + P[A \cap B^c]$$

$B \cup B^c = \Omega$

$B_1, B_2, \dots, B_n$: $B_i \cap B_j = \phi \ \forall i \neq j$

Partition of Ω

$B_1 \cup B_2 \cdots \cup B_n = \Omega$

$$P(A) = \sum_i P(A \cap B_i)$$

Bayes rule

$$P(B/A) = \frac{P(A/B)\,P(B)}{P(A)}$$

channel

Need at $\forall$

L31

10 October 2017 09:10

Conditional density

X, Y are 2 r.v.'s.

(Discrete) $p_{Y|X}(y|x) = P[Y=y \mid X=x]$

↳ conditional PMF of Y given $X=x$.

(Continuous)

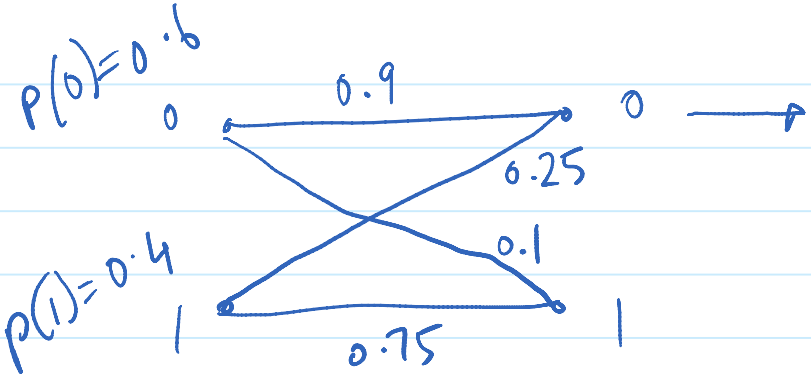
$$\left\{ \lim_{\Delta x, \Delta y \to 0} \frac{P[Y \in [y, y+\Delta y) \mid X \in [x, x+\Delta x)]}{\Delta y} \right\} \triangleq p_{Y|X}(y|x)$$

↳ conditional PDF.

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \longleftarrow \frac{\frac{P[X \in [x, x+\Delta x), Y \in [y+\Delta y)]}{\Delta x \Delta y}}{\frac{P[X \in [x, x+\Delta x)]}{\Delta x}}$$

L 30

06 October 2017 13:26



$$P(\text{0 transmitted} \mid \text{0 received}) = \frac{P(\text{0 received} \mid \text{0 transmitted})\, P(\text{0 transmitted})}{P(\text{0 received})}$$

$$= \frac{(0.9)(0.6)}{(0.9)(0.6) + (0.25)(0.4)}$$

$$= \frac{0.54}{0.54 + 0.1} = \frac{54}{64} = \frac{27}{32}.$$

$P(\text{0 received}) \downarrow$

$P(\text{0 rx}, \text{0 tx}) + P(\text{0 rx}, \text{1 tx})$

L30

06 October 2017 13:31

$$P(1 \text{ transmitted} / 1 \text{ received}) = \frac{P(1 \text{ received} / 1 \text{ tr}) P(1 \text{ tr})}{P(1 \text{ rx})}$$

$$= \frac{(0.75)(0.4)}{(0.75)(0.4) + (0.1)(0.6)} = \frac{0.3}{0.3 + 0.06}$$

$$= \frac{5}{6} \quad \text{..}$$

Suppose Rx takes incoming bits as they are

$$P(\text{error}) = P(\text{error}, 1 \text{ rx}) + P(\text{error}, 0 \text{ rx})$$

$$= P(\text{error} / 1 \text{ rx}) P(1 \text{ rx}) + P(\text{error} / 0 \text{ rx}) P(0 \text{ rx})$$

$$= \frac{1}{6}(0.36) + \frac{5}{32}(0.64) = 0.06 + 0.1 = \underline{0.16}$$

L 30

06 October 2017 13:37

Independence of events A_1 and A_2

Events A_1 and A_2 are independent if

$$P(A_1 \cap A_2) = P(A_1)P(A_2).$$

Conditional independence of A_1 and A_2 given B

$$P(A_1 \cap A_2 \mid B) = P(A_1 \mid B)P(A_2 \mid B)$$

Useful bounds (easily derived from axioms)

(1) $$P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P(A_i)$$ (Union Bound)

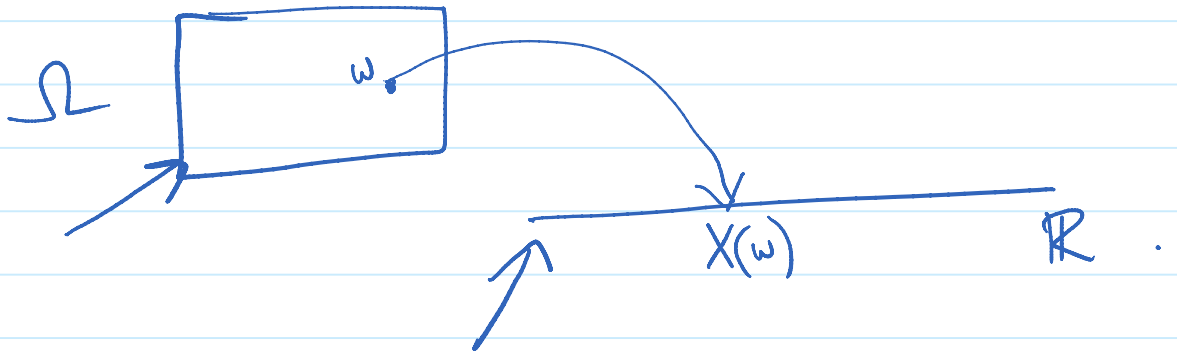
(2) If event $A \Rightarrow$ event B $(A \subseteq B)$, $P(A) \leq P(B)$.

L30

06 October 2017 13:44

Random variables

Mapping from Ω to $\mathbb{R}$.



$\omega \in \Omega$

$X(\omega)$

Lecture 31

09 October 2017 10:01

$X : \Omega \longrightarrow \mathbb{R}$

Discrete random variable X:

- takes finite / countable number of possible values $x_1, x_2, x_3, \ldots$
- characterized completely by specifying

$$P[X = x_i] \quad \forall i$$

($X = x_i$: Basic events)

$p_X(x_i) \triangleq P[X = x_i]$: Probability mass function (PMF)

- Eg: Bernoulli r.v.

$$X = \begin{cases} 0 & \text{w.p. } 1-p \\ 1 & \text{w.p. } p \end{cases}$$

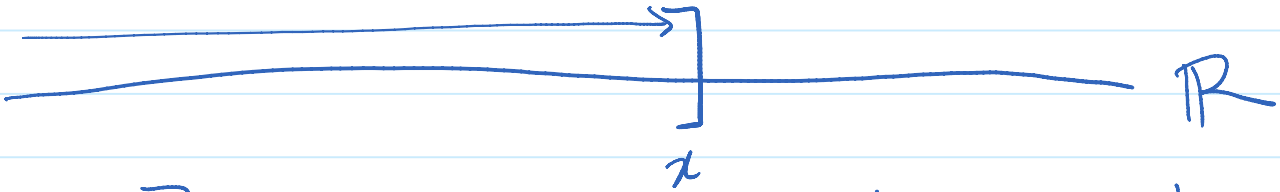
L31

09 October 2017 10:06

Continuous random variable X

— takes values in a continuum
(uncountably infinite # of possible values)

— Specified by $P[X \leq x] \quad \forall x \in \mathbb{R}$.



$F_X(x) \triangleq P[X \leq x]$: cumulative distribution function (CDF)

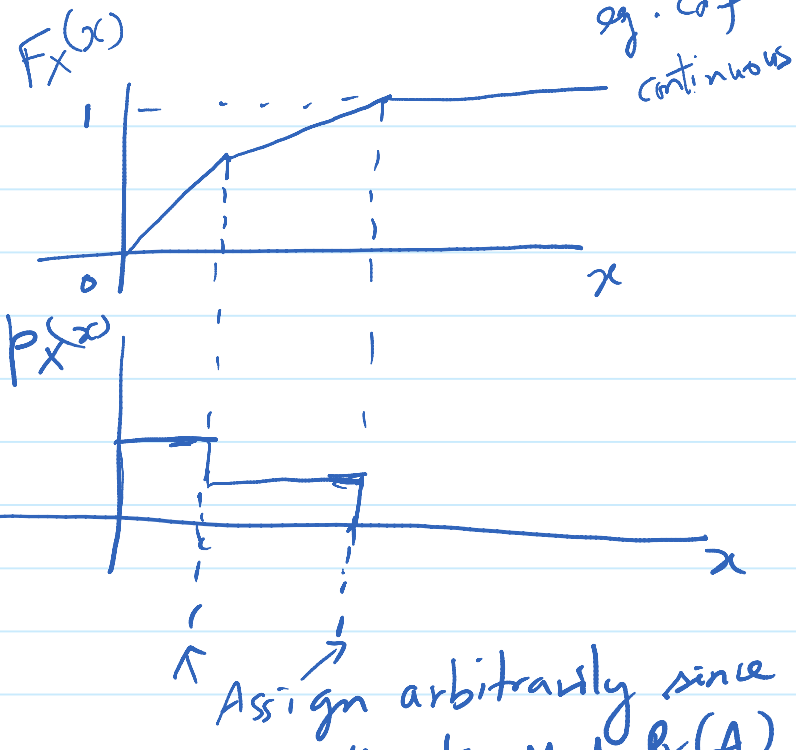
— If $F_X(x)$ is continuous in x, then we can define a probability density fn. (PDF)

L31

09 October 2017 10:12

$$p_X(x) = \frac{d}{dx} F_X(x)$$

$$Pr(A) = \int_{x \in A} p_X(x)\, dx \qquad (A \subseteq \mathbb{R})$$



$$P[\underbrace{x_1 < X \leq x_2}_{A}] = P[X \leq x_2] - P[X \leq x_1]$$

$$= F_X(x_2) - F_X(x_1)$$

$P[X \leq x] - P[X < x]$

$$Pr[X = x] = 0 \quad (F_X(x) \text{ is continuous}) \quad F_X(x) - F_X(x^-)$$

L 31

09 October 2017 10:22

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} p_X(x)\,dx$$

Properties of CDF $F_X(x)$:

- $F_X(x)$ is nondecreasing in x

  [i.e. $x_1 \le x_2$, $F_X(x_1) \le F_X(x_2)$]

[
- $F_X(-\infty) = 0$ ($\leftarrow \lim_{x \to -\infty} F_X(x)$)
- $F_X(\infty) = 1$ ($\leftarrow P(\Omega) = 1$)
]



Complementary CDF: $F_X^c(x) = 1 - F_X(x) = Pr[X > x]$

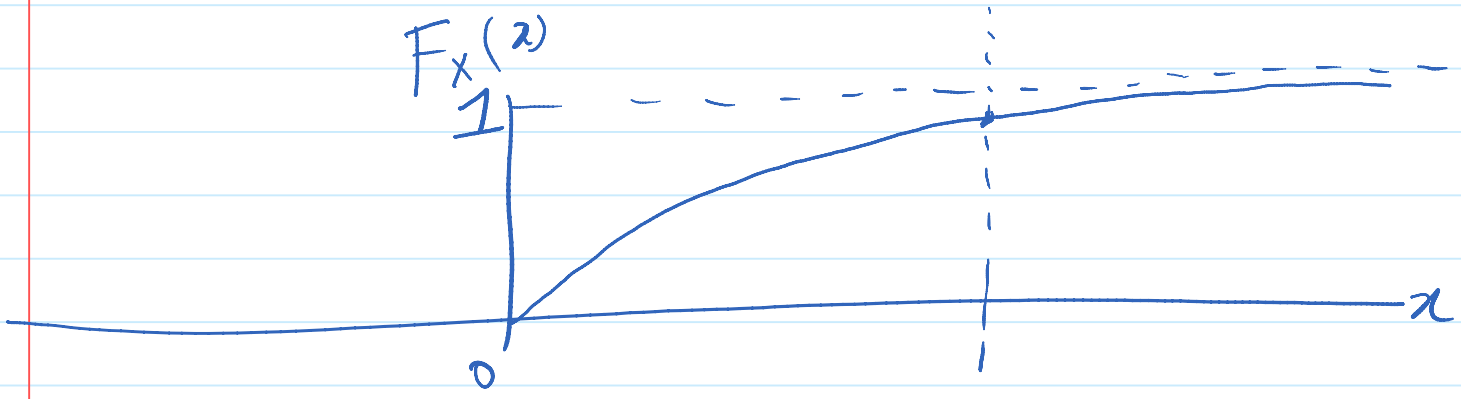
L31

09 October 2017 10:28

Eg Exponential r.v. with parameter λ X

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & x < 0 . \end{cases}$$





$$F_X(x) = \int_{-\infty}^{x} p_X(x)\, dx$$

$$= \int_{0}^{x} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \frac{e^{-\lambda x}}{-\lambda} \Big|_0^x$$

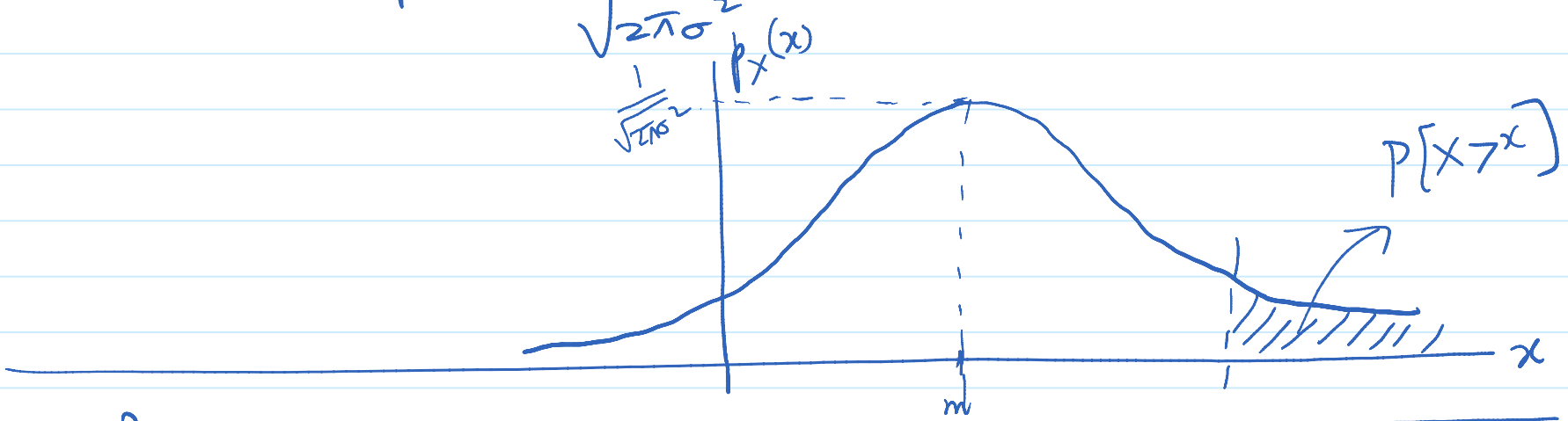
$$= 1 - e^{-\lambda x}$$

L31

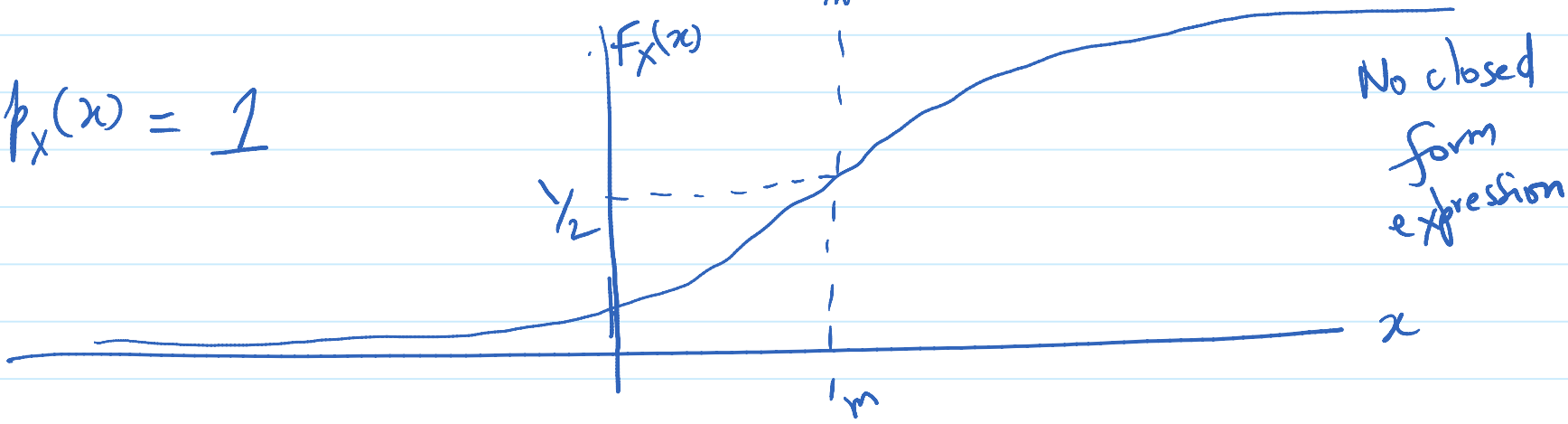
09 October 2017 10:32

Gaussian random variable X with parameters m and σ^2

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} \qquad \forall\, x \in \mathbb{R}.$$



$$F_X(\infty) = \int_{-\infty}^{\infty} f_X(x) = 1$$



L31

09 October 2017 10:38

— Binomial r.v. (discrete) with parameters n, p.

(n: pos. integer; $p \in [0,1]$)

$$X \in \{0, 1, \dots, n\}$$

$$p_X(k) = P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \dots, n.$$



— Poisson r.v. (discrete) with parameter $\lambda > 0$.

$$P[X = k] = \frac{\lambda^k e^{-k}}{k!} \qquad k = 0, 1, 2, \dots.$$

L31

09 October 2017 10:43

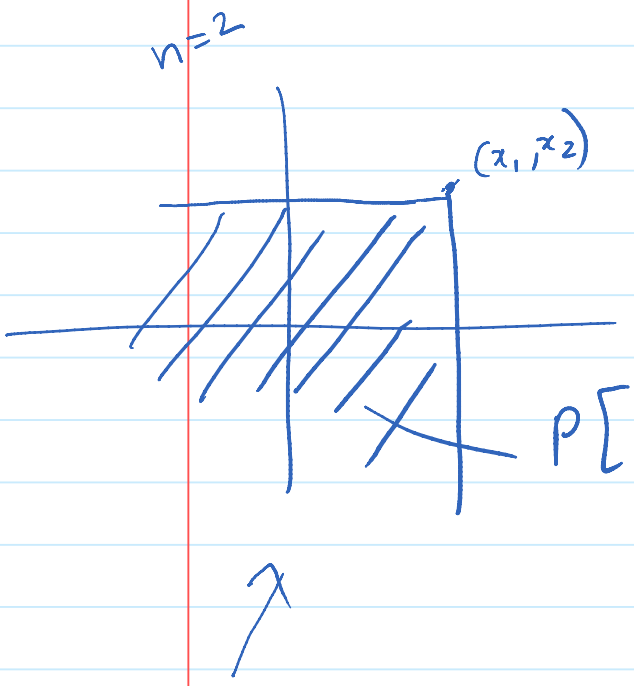
Multiple random variables.

$X_1, X_2, \dots, X_n$

random vector $\rightarrow \underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

specified by $F_{\underline{X}}(\underline{x}) = P[X_1 \le x_1, X_2 \le x_2 \dots X_n \le x_n]$

Joint CDF



Discrete → Joint PMF

$P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]$

Continuous → Joint PDF

$p_{\underline{X}}(\underline{x}) = \frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n} F_{\underline{X}}(\underline{x})$

Lecture 32

10 October 2017 09:03

Joint CDF from Joint PDF

$$F_{\underline{X}}(\underline{x}) = F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_1} p_{\underline{X}}(\underline{u})\, du_1\, du_2 \cdots du_n$$

Marginal distributions

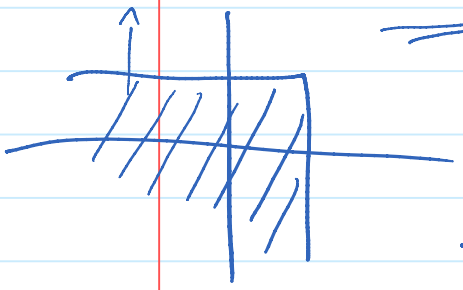
$\underline{X} \longrightarrow$ Joint CDF $F_{\underline{X}}(\underline{x})$

$X_i \longrightarrow$ Marginal CDF? $F_{X_i}(x_i)$

$$F_{X_i}(x_i) = P[X_i \le x_i]$$

$$F_{\underline{X}}(\underline{x}) = P[X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n]$$

Let $x_j \to \infty$ for all $j \neq i$



Just knowing all marginal CDFs is not enough to find joint CDF

L32

10 October 2017 09:19

$$\rightarrow p_{X,Y}(x,y) = p_{Y|X}(y|x)\, p_X(x) = p_{X|Y}(x|y)\, p_Y(y) \Big\} \xrightarrow{\text{combine}} \text{Bayes' rule}$$

chain rule $$p_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = p_{X_1}(x_1)\, p_{X_2|X_1}(x_2|x_1) \cdots p_{X_n|X_{n-1},\cdots,X_1}(x_n|x_{n-1},\cdots,x_1)$$

order $1,2,\cdots,n$

Works for any permutation

$$p_{Y|X}(y|x) = \frac{p_{X|Y}(x|y)\, p_Y(y)}{p_X(x)}$$

Bayes' rule

L32

10 October 2017 09:23

X discrete, Y continuous

$$p_{Y|X}(y|x) = \frac{\overbrace{P[X=x|Y=y]}^{p_{X|Y}(x|y)}\, p_Y(y)}{\underbrace{P[X=x]}_{p_X(x)}}$$

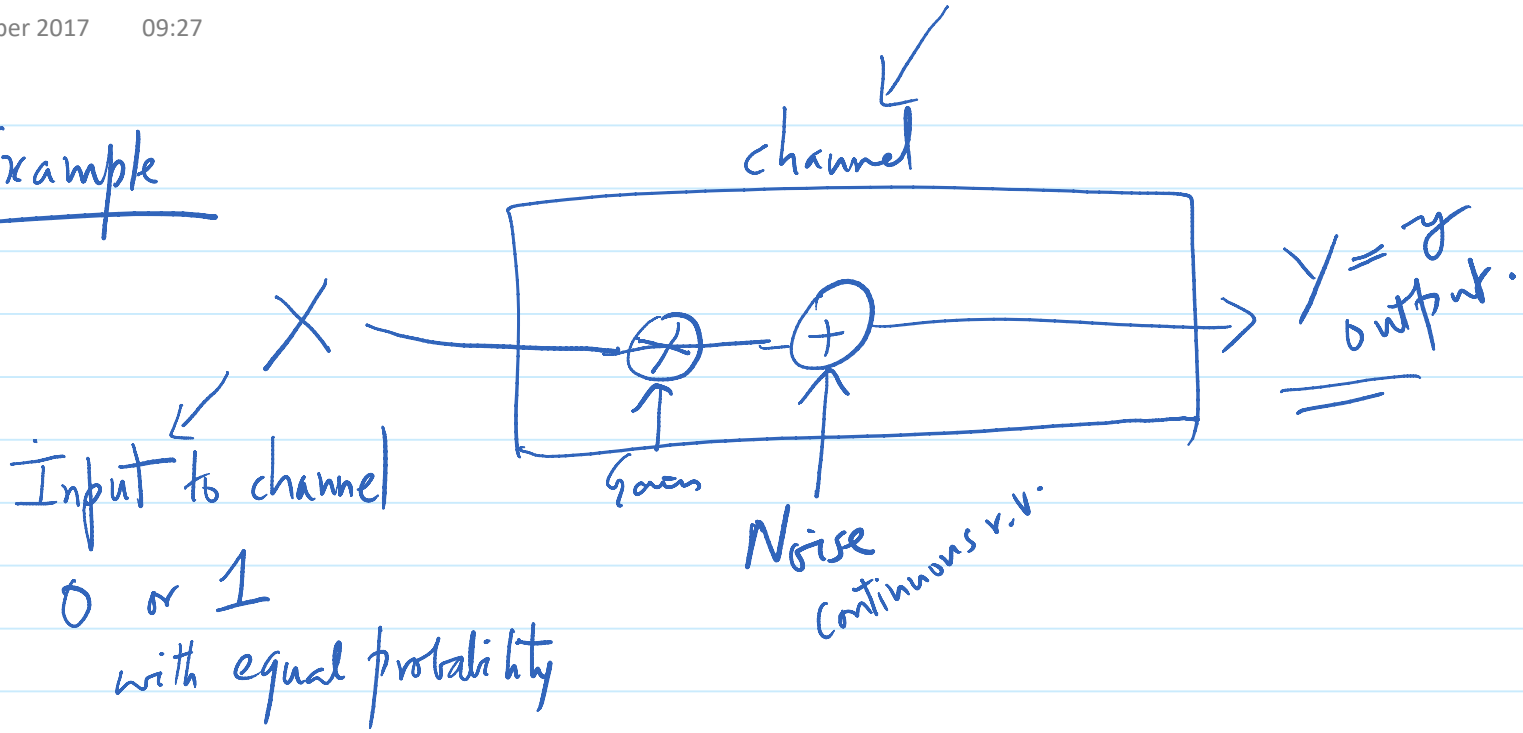
$$\left(\frac{P[Y \in [y, y+\Delta y) | X=x]}{\Delta y} = \frac{P[X=x \mid Y \in [y, y+\Delta y)]\, P[Y \in [y, y+\Delta y)]}{P[X=x]\ \Delta y} \right)$$

let $\Delta y \to 0$

L32

10 October 2017 09:27

Example



Rx has to decide whether 1 was transmitted or 0 was transmitted.

channel model

$p_{Y|X=0}(y|0)$: Gaussian $(0, 4)$

$p_{Y|X=1}(y|1)$: Gaussian $(10, 4)$

L 32

10 October 2017 09:31

Receiver observes $Y = y$ and decides 0 or 1.

$P[\text{Error} : \text{Decision} \neq \text{transmitted bit} \mid Y = y]$ to be minimized.

If dec is 0, $P(\text{error} \mid Y=y) = P(X=1 \mid Y=y)$

If dec is 1, $P(\text{error} \mid Y=y) = P(X=0 \mid Y=y)$

$$P[X=1 \mid Y=y] = \frac{p_{Y|X}(y \mid X=1)\, P[X=1]}{p_Y(y)}$$

$p_{Y|X}(y \mid X=1)$: Gaussian $(10, 4)$

$P[X=0] = 1/2$ ✓

$P[X=1] = 1/2$ ✓

$p_Y(y) \rightarrow p_{Y|X}(y \mid 1)\, P[X=1] + p_{Y|X}(y \mid 0)\, P[X=0]$

$p_{Y|X}(y \mid 0)$: Gaussian $(0, 4)$

L32

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$$P[X=1|Y=y] = \frac{\overbrace{\frac{1}{\sqrt{2\pi(4)}} e^{-\frac{(y-10)^2}{2(4)}}}^{f_{Y|X}(y|1)} \cdot \frac{1}{2}}{\frac{1}{\sqrt{2\pi(4)}} e^{-\frac{(y-10)^2}{2(4)}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2\pi(4)}} e^{-\frac{y^2}{2(4)}} \cdot \frac{1}{2}}$$

$$= \frac{e^{-(y-10)^2/8}}{e^{-(y-10)^2/8} + e^{-y^2/8}}$$

$$P[X=0|Y=y] = \frac{e^{-y^2/8}}{e^{-(y-10)^2/8} + e^{-y^2/8}}$$

$P[X=1|Y=y] > P[X=0|Y=y]$ if $e^{-(y-10)^2/8} > e^{-y^2/8}$

L 32

10 October 2017 09:41

$$\frac{(y-10)^2}{8} < \frac{y^2}{8} \quad (\times) \quad (y-10)^2 < y^2.$$

Let $y = 7$

$(7-10)^2 = 9$ ✓

$(7)^2 = 49$

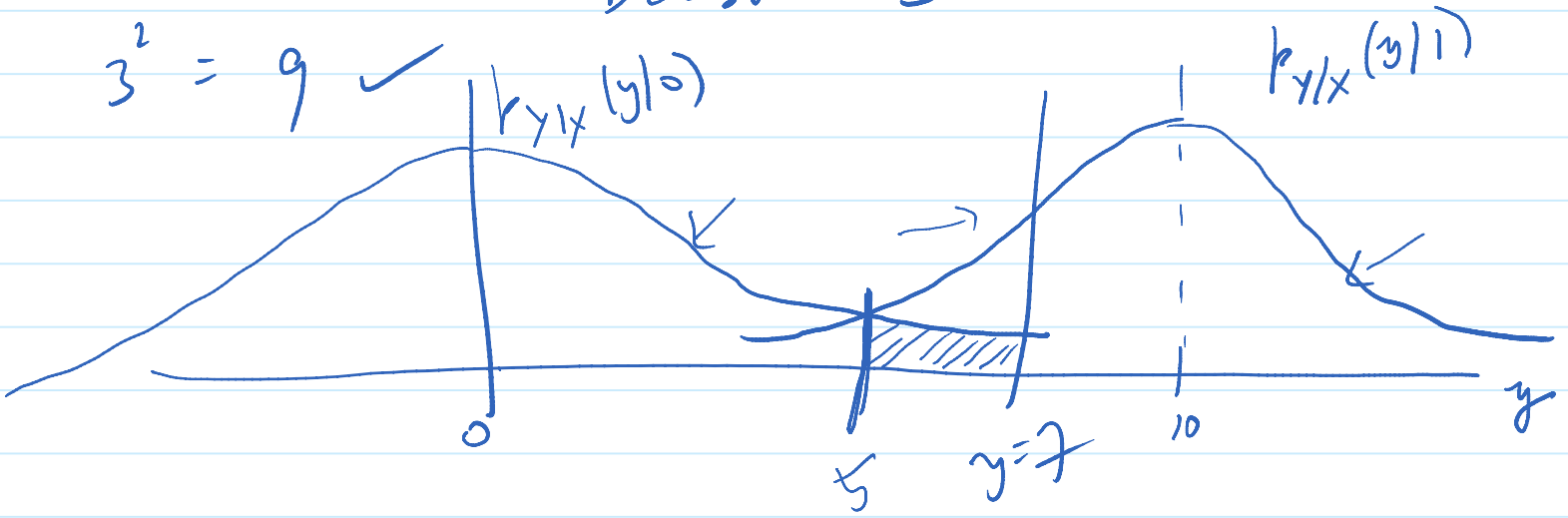
Decision = 1.

$y = 3$

$(7)^2 = 49$

$3^2 = 9$ ✓

Decision = 3.



Lecture 33

11 October 2017 08:05

Independent random variables

$X_1, X_2, \ldots, X_n$ are independent if

Independent events A, B

$P(A \cap B) = P(A)\,P(B)$

$$P\left[X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\right] = P\left[X_1 \in A_1\right] P\left[X_2 \in A_2\right] \cdots P\left[X_n \in A_n\right]$$

✓ for any $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$

Then

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = F_{X_1}(x_1)\,F_{X_2}(x_2)\cdots F_{X_n}(x_n)$$

Joint distribution is product of marginals

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1)\,p_{X_2}(x_2)\cdots p_{X_n}(x_n)$$

L 33

11 October 2017 08:11

Independent and identically distributed random variables (i.i.d)

$X_1, X_2, \ldots, X_n$ are i.i.d. if

- $X_1, X_2, \ldots, X_n$ are independent
- have the same marginal distributions.

Eg: i.i.d. Bernoulli r.v.'s.

$X_1, X_2, \ldots, X_n$

$\forall i \quad X_i \sim \text{Bernoulli}(p) \qquad P[X_i = 1] = p, \; P[X_i = 0] = 1 - p$

L33

11 October 2017 08:16

Consider $Y = \sum_{i=1}^{n} X_i$

X_i: Bernoulli(p) i.i.d.

Y: distribution?

Y can take the following values $0, 1, 2, \cdots, n$

$$P[Y \le y]$$

(Binomial r.v)

$$P[Y = y] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & y = 0, 1, 2, \cdots, n \\ 0 & \text{else} \end{cases}$$

$$P[Y=1] = P[\text{one of the } X_i\text{'s is 1 \& others are 0}]$$

$$= P\left[\{X_1=1, X_2=0, \cdots, X_n=0\} \cup \{X_1=0, X_2=1, X_3=0, \cdots X_n=0\} \cup \cdots \{X_1=0, X_2=0, \cdots, X_{n-1}=0, X_n=1\}\right]$$

w.p. $p(1-p)^{n-1}$

$$= \binom{n}{1} p (1-p)^{n-1}$$

L33

11 October 2017 08:26

Sum of independent r.v.'s X_1, X_2 independent

1) Discrete $p_{X_1}(x_1), p_{X_2}(x_2)$

pmf of $Y = X_1 + X_2$? Δ

$$P[Y=y] = P[X_1+X_2=y]$$

$$= \sum_k P[\underbrace{X_1+X_2=y}_{A}, \underbrace{X_2=k}_{B_k}]$$

$$\overset{\text{independence}}{=} \sum_k P[X_1=y-k, X_2=k]$$

$$= \sum_k p_{X_1}(y-k)\, p_{X_2}(k)$$ ← convolution (·) of $p_{X_1}(\cdot)$ & p_{X_2}

L33

11 October 2017 08:34

$X_1, X_2, \ldots, X_n$ indep $\qquad p_Y(y) = p_{X_1}(y) * p_{X_2}(y) * \cdots * p_{X_n}(y)$

$Y = \sum_{i=1}^{n} X_i$

2) Continuous r.v.'s X_1, X_2.

$Y = X_1 + X_2$

$p_Y(y) = p_{X_1}(y) * p_{X_2}(y)$ [Proof: —]

---

* X_1, X_2 indep

$$P[X_1 \in A_1 / X_2 \in A_2] = \frac{P[X_1 \in A_1, X_2 \in A_2]}{P[X_2 \in A_2]}$$

$$= \frac{P[X_1 \in A_1]\, P[X_2 \in A_2]}{P[X_2 \in A_2]}$$

L 33

11 October 2017 08:41

[Sum $Y = X_1 + X_2 + \cdots + X_n$]

Functions of random variables

$$Y = g(X)$$

Find the distribution of Y. Distribution of X known.

Method 1

Try to find $P[Y \le y] = P[g(X) \le y] = P[X \in A(y)]$

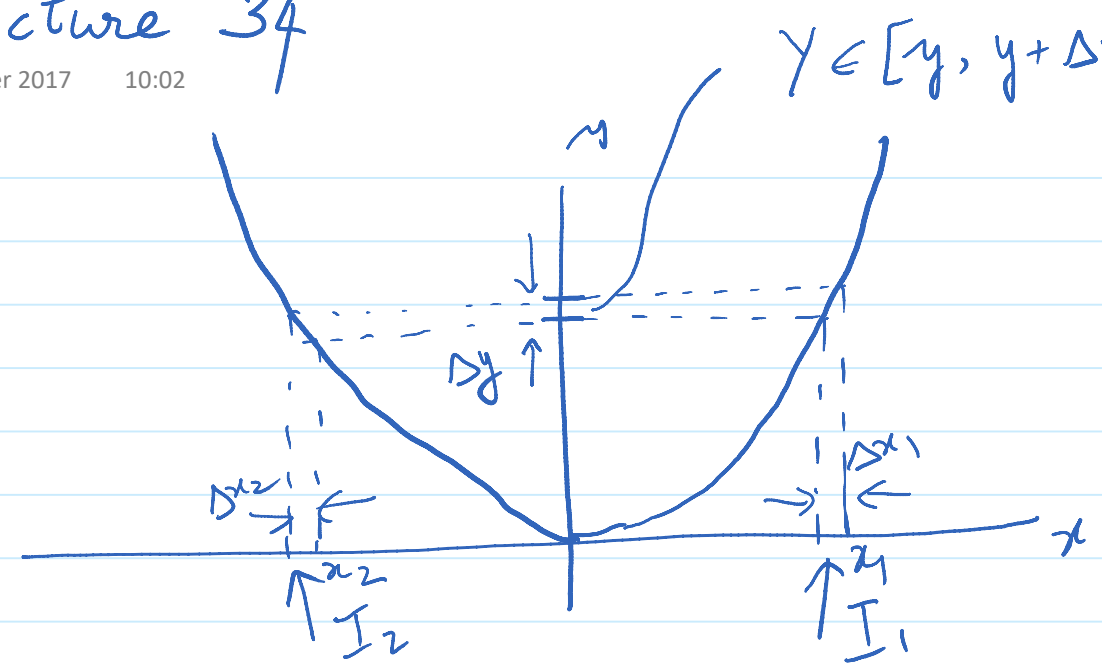
$A(y) = \{x : g(x) \le y\}$

Method 2 (For continuous r.v.'s, differentiable $g(\cdot)$)

Directly relate pdf of Y to pdf of X

Lecture 34

16 October 2017 10:02



$$Y = g(X) = X^2$$

$$\{x : g(x) = y\}$$

$$\rightarrow = \{x_1, x_2\}$$

$$P[Y \in [y, y+\Delta y]] = P[X \in I_1] + P[X \in I_2]$$

$$\downarrow$$

$$p_Y(y)|\Delta y| = p_X(x_1)|\Delta x_1| + p_X(x_2)|\Delta x_2|$$

$$p_Y(y) = \frac{p_X(x_1)}{|\Delta y/\Delta x_1|} + \frac{p_X(x_2)}{|\Delta y/\Delta x_2|}$$

L34

16 October 2017 10:09

$$p_Y(y) = \sum_{i=1}^{m} \frac{p_X(x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}$$ where x_i's satisfy $g(x_i) = y$.

(See examples 5.4.1 & 5.4.2)

Generalization to random vectors. (method 2):

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \qquad \underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \qquad \underline{Y} = g(\underline{X}) = \begin{pmatrix} g_1(\underline{X}) \\ g_2(\underline{X}) \\ \vdots \\ g_n(\underline{X}) \end{pmatrix}$$

eg (X_1, X_2)

$$Y_1 = \sqrt{X_1^2 + X_2^2}, \quad Y_2 = \tan^{-1}\left(\frac{X_2}{X_1}\right)$$

$g_1(X_1, X_2)$ $\qquad$ $g_2(X_1, X_2)$

L34

16 October 2017 10:16

$$p_{\underline{Y}}(\underline{y}) \underbrace{[\text{infinitesimal volume}]}_{\Delta y_1 \Delta y_2 \cdots \Delta y_n} = \sum_{i=1}^{k} p_{\underline{X}}(\underline{x}^i) \cdot [\Delta x_1 \Delta x_2 \cdots \Delta x_n]$$

$\underline{x}^i$'s are such that $g(\underline{x}^i) = \underline{y}$.

$$\text{Jacobian matrix} = J(\underline{y}; \underline{x}) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

$$\left(\frac{\Delta y_1 \, \Delta y_2 \cdots \Delta y_n}{\Delta x_1 \, \Delta x_2 \cdots \Delta x_n} \right) \longrightarrow \left| \det \left(J(\underline{y}; \underline{x}) \right) \right|$$

L34

16 October 2017 10:24

$$p_{\underline{Y}}(\underline{y}) = \sum_{i=1}^{m} \frac{p_{\underline{X}}(\underline{x}^i)}{|\det(J(\underline{y};\underline{x}^i))|}$$ where $\underline{x}^i$'s satisfy $g(\underline{x}^i) = \underline{y}$.

eg Linear transformation A, invertible

$$\underline{Y} = A\underline{X} \qquad Y_i = A_{i1}X_1 + \cdots + A_{in}X_n$$

$$J(\underline{y};\underline{x}) = A$$

$$\boxed{p_{\underline{Y}}(\underline{y}) = \frac{p_{\underline{X}}(A^{-1}\underline{y})}{|\det(A)|}}$$

$$\boxed{\underline{Y} = A\underline{X} + \underline{b}} \qquad p_{\underline{Y}}(\underline{y}) = \frac{p_{\underline{X}}(A^{-1}(\underline{y}-\underline{b}))}{|\det(A)|}$$

L 34

16 October 2017 10:33

Expectation

Given a r.v. X with pdf $p_X(x)$

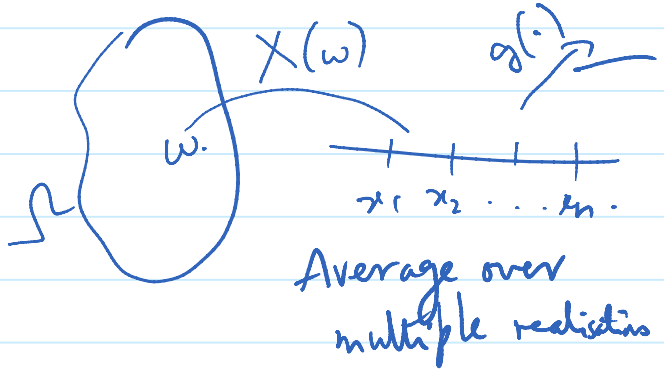
$$E[g(X)] = \int g(x) p_X(x) dx .$$

[discrete r.v. X with pmf $p_X(x)$

$$E[g(X)] = \sum_x g(x) p_X(x) .]$$

Mean of X (or) first moment

$$E[X] = \int x p_X(x) dx \text{ or } \sum_x x p_X(x)$$



L34

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$$E[X^2] = \int x^2 p_X(x)\,dx \quad \text{Second moment}$$

$E[X^n]$: n^{th} moment

$E[(X - E[X])^n]$: n^{th} central moment

↳ $n=2$ Variance $E[(X-E[X])^2]$.

Show $E[X^2] = E[(X - E[X])^2] + (E[X])^2$

(Use Expectation is a linear operator

$E[a_1X_1 + a_2X_2 + b] = a_1E[X_1] + a_2E[X_2] + b$)

L34

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Eg $Y = aX + b$ $\quad X \sim p_X(x)$, $a \neq 0$.

$p_Y(y) = \frac{1}{|a|} p_X\left(\frac{y-b}{a}\right)$ ⟶ scaling and translation

$E[Y] = a\,E[X] + b$

$\mathrm{Var}[Y] = E\left[(Y - E[Y])^2\right] = E\left[(aX + b - aE[X] - b)^2\right]$

$= a^2 E\left[(X - E[X])^2\right]$

$= a^2 \mathrm{Var}(X)$

Lecture 35

17 October 2017 09:02

Expectation for random vectors

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \sim f_{\underline{X}}(\underline{x})$$ Joint pdf or Joint pmf

Mean vector $E[\underline{X}] = \begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix}$

$E[X_i] = \int x_i \cdot f_{\underline{X}}(\underline{x})\, d\underline{x}$ (continuous) $\quad d\underline{x} = dx_1\, dx_2 \ldots dx_n$

n-dim integral

(or)

$$\sum_{\underline{x}} x_i\, p_{\underline{X}}(\underline{x}) \quad \text{(discrete)}$$

$$E[g(\underline{X})] = \int g(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x} \quad \text{(or)} \quad \sum_{\underline{x}} g(\underline{x})\, p_{\underline{X}}(\underline{x})$$

L35

17 October 2017 09:09

Covariance of 2 r.v.'s X_1 and X_2 $\quad Cov(X_1, X_2) = E[(X_1 - E[X_1])(X_2 - E[X_2])]$

$$= E[X_1 X_2] - E[X_1 E[X_2]] - E[E[X_1] X_2] + E[X_1] E[X_2] = E[X_1 X_2] - E[X_1] E[X_2]$$

$$Cov(X, X) = E[(X - E[X])^2] = Var(X)$$

Random vectors $\quad$ Covariance matrix $= \begin{pmatrix} & & \\ & (i,j)^{th} \text{ element} & \\ & & \end{pmatrix}$

$Cov(X_i, X_j)$

$$= E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T]$$

$$= E[\underline{X}\,\underline{X}^T] - E[\underline{X}]E[\underline{X}]^T$$

L35

17 October 2017 09:15

If X_1 and X_2 are independent, then

$$E[g_1(X_1)\, g_2(X_2)] = E[g_1(X_1)]\, E[g_2(X_2)]$$

$$\mathrm{Cov}(X_1, X_2) = E[(X_1 - E[X_1])(X_2 - E[X_2])]$$

$$= \underbrace{E[X_1 - E[X_1]]}_{0}\, \underbrace{E[X_2 - E[X_2]]}_{0}$$

$$E[X_1 X_2] - E[X_1]E[X_2] \longrightarrow = 0$$

$$\left(E[X_1 X_2] = E[X_1]E[X_2] \right) \checkmark$$

(Defn) If $\mathrm{Cov}(X_1, X_2) = 0$, then X_1 and X_2 are uncorrelated.

L35

17 October 2017 09:20

X_1, X_2 Independent $\Rightarrow$ X_1, X_2 Uncorrelated

$\not\Leftarrow$

Not necessarily true.

Eg: 1) $X \sim \text{Gaussian}(0,1)$, $Y = X^2$ $\longrightarrow$ Not independent

but uncorrelated $E[XY] = E[X]E[Y]$

$E[XY] = E[X^3]$

$E[X] = 0$

$E[X^3] = \int_{-\infty}^{\infty} \underbrace{x^3}_{\text{odd}} \underbrace{f_X(x)}_{\text{even}} dx = 0$.

2) If X, Y are "jointly Gaussian", then if X, Y uncorrelated they are also independent.

L35

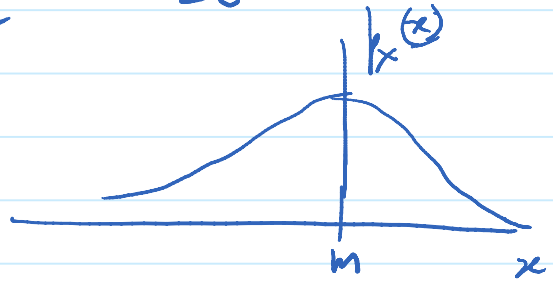
17 October 2017 09:25

Mean vector $E[\underline{X}]$

Covariance matrix $E[(\underline{X}-E[\underline{X}])(\underline{X}-E[\underline{X}])^T] = \underbrace{E[\underline{X}\,\underline{X}^T]}_{\text{Correlation matrix}} - E[\underline{X}]\,E[\underline{X}]^T$

Gaussian random variables

X is Gaussian if $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$



Mean $E[X] = m$

$\mathrm{Var}(X) = \sigma^2$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

Let $\frac{x-m}{\sigma} = t$

$$= \int_{-\infty}^{\infty} \frac{(t\sigma + m)}{\sqrt{2\pi\sigma^2}} e^{-t^2/2} dt\,\sigma$$

L35

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$$= \underbrace{\int_{-\infty}^{\infty} \frac{t\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-t^2/2}\, dt}_{=0} + \underbrace{\int_{-\infty}^{\infty} \frac{m}{\sqrt{2\pi\sigma^2}} e^{-t^2/2}\, dt}_{m\int f_X(x)\,dx = m}$$

$$= m$$

$$\mathrm{Var}(X) = E[(X-m)^2] = \int_{-\infty}^{\infty} \frac{(x-m)^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

$$= \boxed{\int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}} e^{-t^2/2}\, dt}\,\sigma^2$$

$$\frac{(x-m)^2}{\sqrt{\sigma^2}} = t^2 \cdot \sigma$$

$$2\int_0^{\infty} \frac{t^2}{\sqrt{2\pi}} e^{-t^2/2}\, dt = 1$$

(See page 236)

L35

17 October 2017 09:38

Notation: Gaussian distribution with mean m and variance σ^2

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} : N(m, \sigma^2)$$

Standard Gaussian distribution : $N(0,1)$

→ Gaussianity is preserved under scaling and translation

If $X \sim N(m, \sigma^2)$, then $Y = aX + b \sim N(am+b, a^2\sigma^2)$.

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{y-b}{a} - m\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{(y-(am+b))^2}{2a^2\sigma^2}}$$

set $a = \frac{1}{\sigma}$, $b = -\frac{m}{\sigma}$

$$Y = \frac{X-m}{\sigma} \sim N(0,1)$$

L35

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Note $P[X > x] = P\left[\frac{X-m}{\sigma} > \frac{x-m}{\sigma}\right]$

$$\int_{x}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt$$

Cannot be evaluated in closed form.

$$\int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = Q\left(\frac{x-m}{\sigma}\right)$$

$$\left[\int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \triangleq Q(x)\right]$$

Lecture 36

20 October 2017 13:01

Jointly Gaussian random variables.

Defn: $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$ is a jointly Gaussian random vector if $a_1 X_1 + a_2 X_2 + \cdots + a_m X_m$ is a Gaussian random variable for any $a_1, a_2, \ldots, a_m$.

Let $Y = a_1 X_1 + a_2 X_2 + \cdots + a_m X_m = \underline{a}^T \underline{X}$

$$E[Y] = \underline{a}^T E[\underline{X}]$$

$$\mathrm{Var}(Y) = E\left[(Y - E[Y])^2\right] = E\left[\left(\underline{a}^T \underline{X} - \underline{a}^T E[\underline{X}]\right)^2\right]$$

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$$= E\left[\left(\underline{a}^T(\underline{X} - E[\underline{X}])\right)^2\right]$$

$$= E\left[\underline{a}^T(\underline{X} - E[\underline{X}]) \cdot (\underline{X} - E[\underline{X}])^T \underline{a}\right]$$

$$= \underline{a}^T \underbrace{E\left[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T\right]}_{\text{covariance matrix of } \underline{X}} \underline{a}$$

$$= \underline{a}^T C_X \underline{a}$$

$$Y \sim N\left(\underline{a}^T E[\underline{X}],\ \underline{a}^T C_X \underline{a}\right).$$

$$p_Y(y) = \frac{1}{\sqrt{2\pi\,\mathrm{Var}(Y)}}\, e^{-\frac{(y - E[Y])^2}{2\,\mathrm{Var}(Y)}}.$$

L36

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(Problem 5.47)

$$p_{\underline{X}}(\underline{x}) = ?$$

$$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

$$\phi_{\underline{X}}(\underline{w}) \triangleq E\left[e^{j\overbrace{\underline{w}^T\underline{X}}^{Y}}\right]$$

characteristic fn of $\underline{X}$

$$= \int e^{j\underline{w}^T\underline{X}} \cdot p_{\underline{X}}(\underline{x})\, d\underline{x}$$

$\phi_X(\underline{w})$ $\xleftarrow{\text{I.F.T.}}$ & $p_{\underline{X}}(\underline{x})$ are a Fourier transform pair

✓ $\xrightarrow{\text{F.T}}$ ✓

If $\underline{X}$ is jointly gaussian, $\underline{w}^T\underline{X} \sim N(\underline{w}^T\underline{m}_X, \underline{w}^T C_X \underline{w})$

L36

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$$E[e^{jY}] = \int e^{jy} p_Y(y)\, dy$$

$$m = \underline{w}^T \underline{m}_X$$

$$\sigma^2 = \underline{w}^T C_X \underline{w}$$

$$= \int e^{jy} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-m)^2}{2\sigma^2}} dy$$

$$= e^{jm - \sigma^2/2}$$

$$= \boxed{e^{j\underline{w}^T\underline{m}_X - \frac{\underline{w}^T C_X \underline{w}}{2}}} = \phi_{\underline{X}}(\underline{w})$$

characteristic fn of $\underline{X}$

F.T. → can do this if C_X^{-1} exists.

$p_{\underline{X}}(\underline{x})$

L36

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→ If C_X^{-1} exists

$$p_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^m |C_X|}} e^{-\frac{1}{2}(\underline{x}-\underline{m}_X)^T C_X^{-1}(\underline{x}-\underline{m}_X)}$$

$$\sim N(\underline{m}_X, C_X)$$

→ If $\underline{X} \sim N(\underline{m}, C)$, then $\underline{Y} = A\underline{X} + \underline{b}$

($\underline{Y}$: $n\times 1$, A: $n\times m$, $\underline{X}$: $m\times 1$, $\underline{b}$: $n\times 1$)

$$\sim N(A\underline{m} + \underline{b}, ACA^T)$$

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Let $m=2$ $\quad \underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$p_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^2 |C_X|}} e^{-\frac{1}{2}(\underline{x}-\underline{m})^T C_X^{-1} (\underline{x}-\underline{m})}$$

$$C_X = \begin{pmatrix} \sigma_{X_1}^2 & \mathrm{Cov}(X_1,X_2) \\ \mathrm{Cov}(X_1,X_2) & \sigma_{X_2}^2 \end{pmatrix}$$

For which x_1, x_2 is $p_{\underline{X}}(x_1, x_2) = \text{constant } c$?

$(\underline{x}-\underline{m})^T C_X^{-1} (\underline{x}-\underline{m}) = \text{constant} \rightarrow$ ellipse

1) $C_X = \sigma^2 I$ $\quad \frac{1}{\sigma^2}\left[(x_1-m_1)^2 + (x_2-m_2)^2\right] = \text{constant} \rightarrow$ circle

Lecture 37

23 October 2017 10:03

2-dimensional Gaussian PDF

$$p_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)|C|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{m})^T C^{-1}(\underline{x}-\underline{m})}$$

$$C = \begin{bmatrix} \sigma_{X_1}^2 & \mathrm{cov}(X_1,X_2) \\ \mathrm{Cov}(X_1,X_2) & \sigma_{X_2}^2 \end{bmatrix}$$

$$|\mathrm{Cov}(X_1,X_2)| \le \sqrt{\mathrm{Var}(X_1)\cdot\mathrm{Var}(X_2)} \le \sigma_{X_1}\sigma_{X_2}$$

(Cauchy-Schwarz ineq)

$$\frac{\mathrm{cov}(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}} \triangleq \rho \qquad |\rho| \le 1 \quad (-1 \le \rho \le 1)$$

$$C = \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$|C| = \sigma_{X_1}^2\sigma_{X_2}^2 - \rho^2\sigma_{X_1}^2\sigma_{X_2}^2 = \sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2).$$

L37

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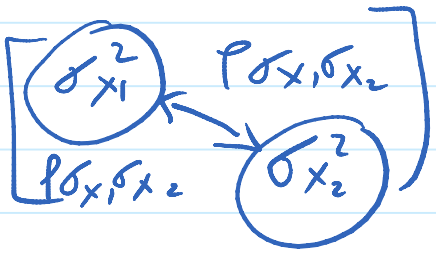
$$|C| = \sigma_{x_1}^2 \sigma_{x_2}^2 (1-\rho^2)$$

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} \sigma_{x_2}^2 & -\rho\sigma_{x_1}\sigma_{x_2} \\ -\rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_1}^2 \end{bmatrix}$$



$$p_{\underline{x}}(\underline{x}) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} e^{-\frac{1}{2|C|}(\underline{x}-\underline{m})^T \begin{bmatrix} \sigma_{x_2}^2 & -\rho\sigma_{x_1}\sigma_{x_2} \\ -\rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_1}^2 \end{bmatrix} (\underline{x}-\underline{m})}$$

$$(\underline{x}-\underline{m})^T \begin{bmatrix} \sigma_{x_2}^2 & -\rho\sigma_{x_1}\sigma_{x_2} \\ -\rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_1}^2 \end{bmatrix} (\underline{x}-\underline{m}) = \begin{bmatrix} x_1-m_1 & x_2-m_2 \end{bmatrix} \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} x_1-m_1 \\ x_2-m_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1-m_1 & x_2-m_2 \end{bmatrix} \begin{bmatrix} \sigma_{x_2}^2(x_1-m_1) - \rho\sigma_{x_1}\sigma_{x_2}(x_2-m_2) \\ -\rho\sigma_{x_1}\sigma_{x_2}(x_1-m_1) + \sigma_{x_1}^2(x_2-m_2) \end{bmatrix}$$

$$= \sigma_{x_2}^2(x_1-m_1)^2 + \sigma_{x_1}^2(x_2-m_2)^2 - 2\rho\sigma_{x_1}\sigma_{x_2}(x_1-m_1)(x_2-m_2)$$

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$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi |C|^{1/2}} e^{-\frac{1}{2}\frac{1}{(1-\rho^2)}\left[\frac{(x_1-m_1)^2}{\sigma_{x_1}^2} + \frac{(x_2-m_2)^2}{\sigma_{x_1}^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_{x_1}\sigma_{x_2}}\right]}$$

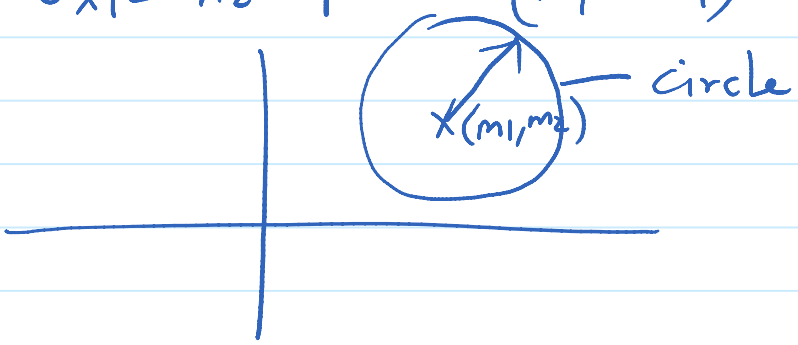
$= \text{constant}$

[$\rho = 0 \Rightarrow X_1, X_2$ independent]

Contours : $\frac{(x_1-m_1)^2}{\sigma_{x_1}^2} + \frac{(x_2-m_2)^2}{\sigma_{x_2}^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_{x_1}\sigma_{x_2}} = \text{constant}$

Special cases

(1) $\rho = 0$, $\sigma_{x_1} = \sigma_{x_2} = 1$ $\quad (x_1-m_1)^2 + (x_2-m_2)^2 = \text{constant}$



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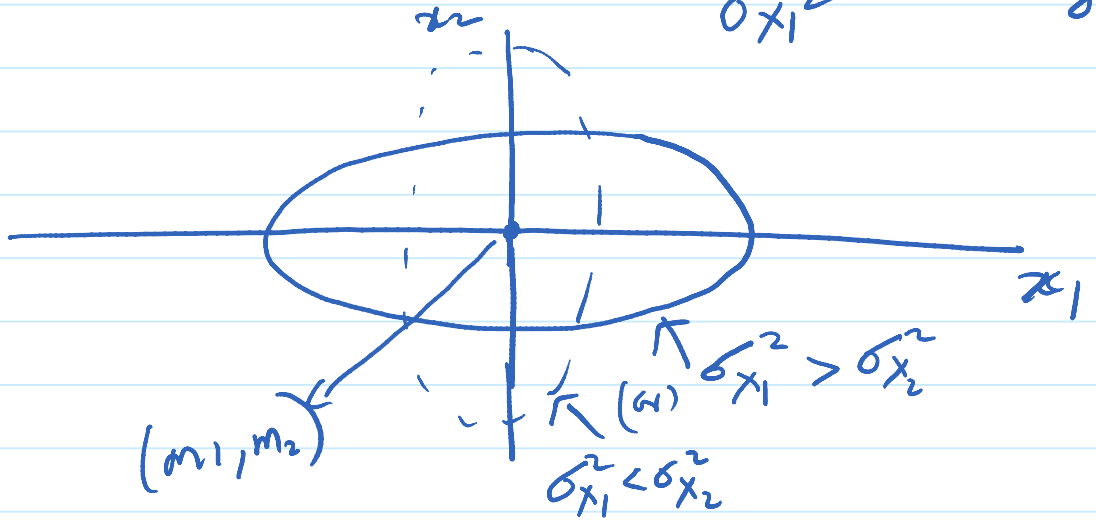
(2) $\rho = 0, \quad \sigma_{x_1} \neq \sigma_{x_2}$ $\qquad \dfrac{(x_1 - m_1)^2}{\sigma_{x_1}^2} + \dfrac{(x_2 - m_2)^2}{\sigma_{x_2}^2} = \text{constant}$



(3) $\sigma_{x_1} = \sigma_{x_2} = 1, \quad \rho > 0$ $\qquad (x_1 - m_1)^2 + (x_2 - m_2)^2 - 2\rho(x_1 - m_1)(x_2 - m_2) = \text{constant}$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = c$$

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$$\frac{\partial}{\partial x_1}\left[(x_1-m_1)^2+(x_2-m_2)^2-2\rho(x_1-m_1)(x_2-m_2)\right]=0$$

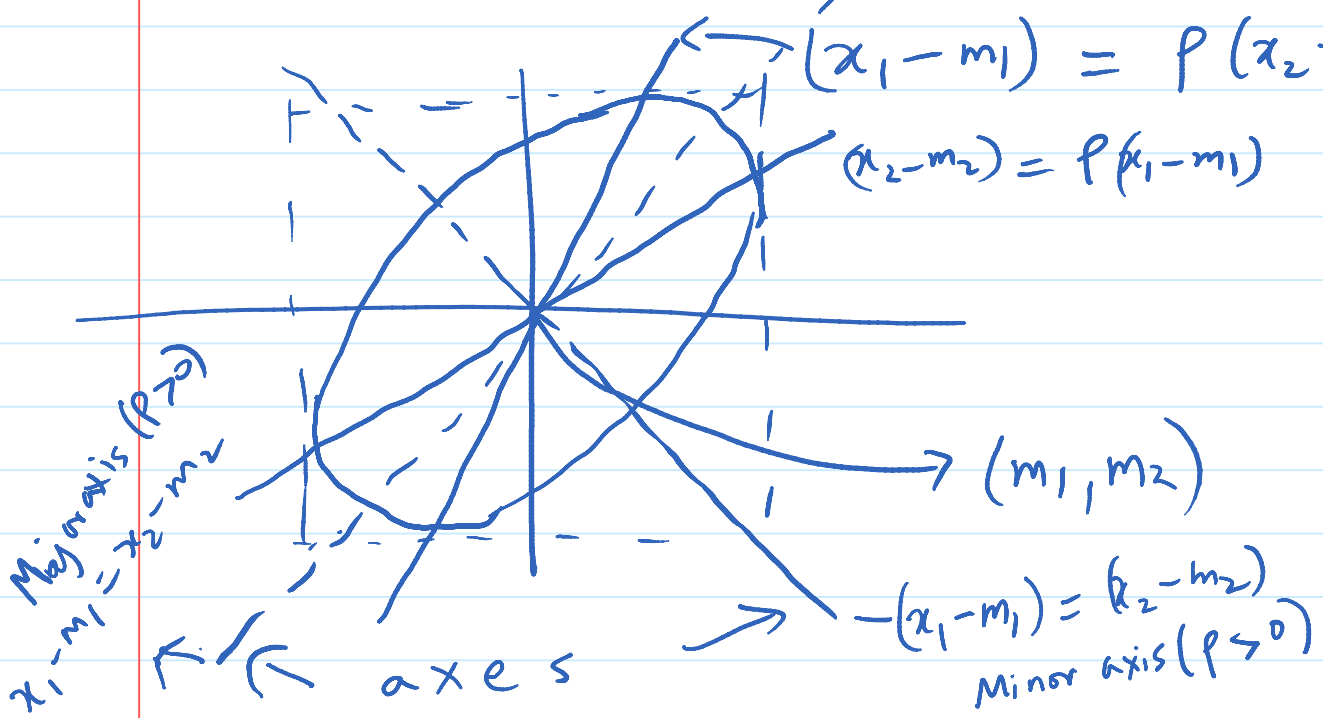
$$2(x_1-m_1)-2\rho(x_2-m_2)=0$$

$$(x_1-m_1)=\rho(x_2-m_2)$$

$$(x_2-m_2)=\rho(x_1-m_1)$$

$$\frac{\partial}{\partial x_2}(\ )=0$$

$$(x_2-m_2)=\rho(x_1-m_1)$$



b) $\rho<0$, $\sigma_{x_1}^2=\sigma_{x_2}^2=1$ Maj. axis $(x_1-m_1)=-(x_2-m_2)$

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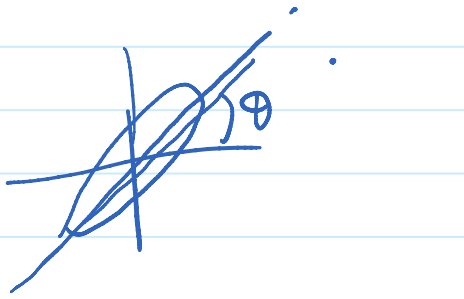
(5) $\sigma_{x_1}^2 \neq \sigma_{x_2}^2$, $\rho \neq 0$.



Rotated ellipse

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_{x_1}\sigma_{x_2}}{\sigma_{x_1}^2 - \sigma_{x_2}^2}\right)$$

$$\frac{(x_1-m_1)^2}{\sigma_{x_1}^2} + \frac{(x_2-m_2)^2}{\sigma_{x_2}^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_{x_1}\sigma_{x_2}} = c$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 - m_1 \\ x_2 - m_2 \end{bmatrix}$$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = k$$

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$$p_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}|C_X|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{m})^T C_X^{-1}(\underline{x}-\underline{m})}$$

If C_X is diagonal (the r.v.'s are uncorrelated),

$$|C_X|^{1/2} = \sigma_{X_1}\sigma_{X_2}\cdots\sigma_{X_n}$$

$$C_X^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \rightarrow \frac{1}{\sigma_{X_i}^2} \quad (i^{th} \text{ diag term})$$

$$p_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma_{X_i}} e^{-\frac{1}{2\sigma_{X_i}^2}(x_i-m_i)^2}$$

$p_{X_i}(x_i) \rightarrow N(m_i, \sigma_{X_i}^2)$

X_i's are independent

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Another property of jointly Gaussian r.v.'s

$$\underline{X} \sim N(\underline{m}, C) \qquad \underline{X} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} \qquad \underline{m} = \begin{bmatrix} \underline{m}^{(1)} \\ \underline{m}^{(2)} \end{bmatrix}$$

$$C = \left[\begin{array}{c|c} C^{11} & C^{12} \\ \hline C^{21} & C^{22} \end{array}\right]$$

$$\underline{X}^{(1)} \sim N(\underline{m}^{(1)}, C^{11}) \quad , \quad \underline{X}^{(2)} \sim N(\underline{m}^{(2)}, C^{22})$$

$$p_{\underline{X}^{(2)} | \underline{X}^{(1)} = \underline{x}^{(1)}}(\underline{x}^{(2)}) \sim N(\underline{m}', C')$$

$$\rightarrow \frac{p_{\underline{X}}(\underline{x})}{p_{\underline{X}^{(1)}}(\underline{x}^{(1)})}$$

$$\underline{m}' = \underline{m}^{(2)} + C^{21} C^{11^{-1}} (\underline{x}^{(1)} - \underline{m}^{(1)})$$

$$C' = C^{22} - C_{21} C_{11}^{-1} C_{12}$$

Lecture 38

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Random vector $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

Random processes $X(t,\omega)$

— Collection of random variables $\{X_t, t \in T\}$

where T is the index set (and can be infinite)

Countably infinite

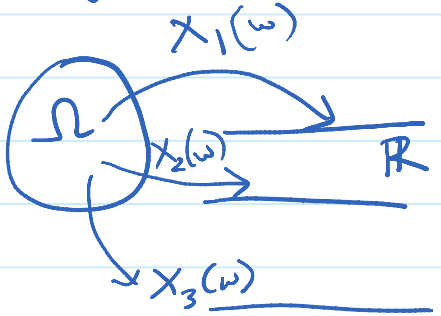
$\ldots, X_{-1}, X_0, X_1, \ldots$

Discrete-time random processes

Uncountably infinite

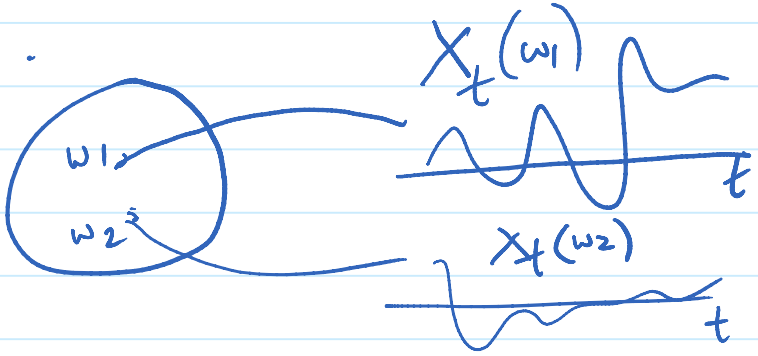
$\{X_t, t \in \mathbb{R}\}$

Continuous-time random processes.



— Collection of sample functions.

$X(t,\omega)$



L 3 8

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Example

$$X_t = X_1 \cos 2\pi f_c t - X_2 \sin 2\pi f_c t$$

where X_1, X_2 are i.i.d. $N(0,1)$

$\sim N(\underline{0}, I)$

Random process

→ For each t, we have a linear combination of X_1 & X_2

$\sim N(0, 1)$ $\left(\cos^2 2\pi f_c t + \sin^2 2\pi f_c t = 1\right)$

$$\left(X_{t_1}, X_{t_2}\right) \sim N\left(\underline{0}, \left[A A^T\right]\right)$$

"Gaussian random process"

$$\begin{pmatrix} X_{t_1} \\ X_{t_2} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos 2\pi f_c t_1 & -\sin 2\pi f_c t_1 \\ \cos 2\pi f_c t_2 & -\sin 2\pi f_c t_2 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

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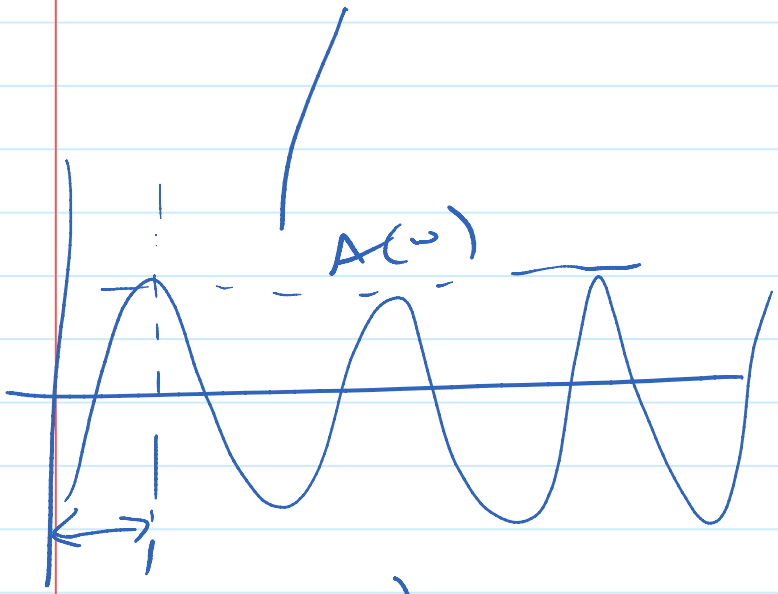
→ For each ω, $X_t(\omega) = X_1(\omega)\cos 2\pi f_c t - X_2(\omega)\sin 2\pi f_c t$

$= A(\omega)\cos(2\pi f_c t + \theta(\omega))$

$A(\omega) = \sqrt{X_1^2(\omega) + X_2^2(\omega)}$

$\theta(\omega) = \tan^{-1}\left(\frac{X_2(\omega)}{X_1(\omega)}\right)$



↳ Sinusoid with random amplitude & phase

$A \sim$ Rayleigh dist $\quad \theta \sim$ Uniform distn.

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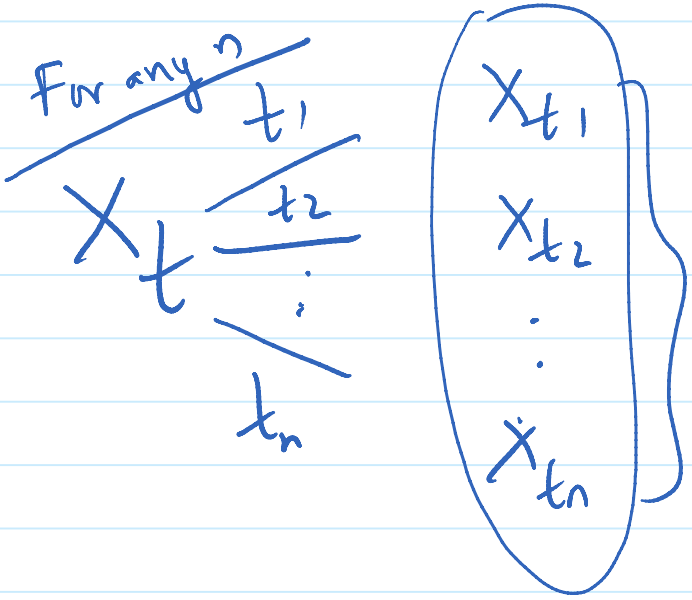
We know the statistics of a random process if we know the joint statistics of an arbitrarily chosen finite collection of samples.



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We may not be able to write down closed-form expressions for all the joint pdfs in general.

Stationary random processes

A random process X_t is stationary if it is statistically indistinguishable from X_{t-d} for any $d \in \mathbb{R}$.

$\triangleq Y_t$

→ for any $n, t_1, t_2, \ldots, t_n$

Joint pdf of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ = Joint pdf of $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$.

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Suppose this property is true only for $n = 1$.

Then, random process is stationary of order 1.

if true for $n \leq k$,

then stationary of order k.

Lecture 39

03 November 2017 22:37

Mean function of random process X_t

$$m_X(t) = E[X_t]$$

Autocorrelation function of process X_t

$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$$

(Autocovariance) $$C_X(t_1, t_2) = E\left[(X_{t_1} - E[X_{t_1}])(X_{t_2} - E[X_{t_2}])\right]$$

$$= R_X(t_1, t_2) - E[X_{t_1}]E[X_{t_2}].$$

Example: $X_t = X_1 \cos 2\pi f_c t - X_2 \sin(2\pi f_c t)$

$m_X(t) = 0$ = constant $R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$

L 39

06 November 2017 10:09

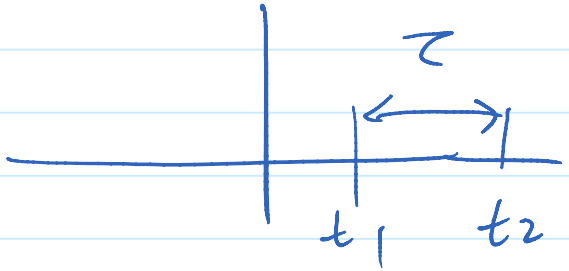
$$R_X(t_1,t_2) = E\left[\left(X_1 \cos 2\pi f_c t_1 - X_2 \sin 2\pi f_c t_1\right)\left(X_1 \cos 2\pi f_c t_2 - X_2 \sin 2\pi f_c t_2\right)\right]$$

$$= \underbrace{E[X_1^2]}_{1} \cos 2\pi f_c t_1 \cos 2\pi f_c t_2 + \overbrace{E[X_2^2]}^{1} \sin 2\pi f_c t_1 \sin 2\pi f_c t_2$$

$$- \underbrace{E[X_1 X_2]}_{E[X_1]E[X_2]=0.} \left(\cos 2\pi f_c t_1 \sin 2\pi f_c t_2 + \sin 2\pi f_c t_1 \cos 2\pi f_c t_2\right)$$

$$= \cos\left(2\pi f_c (t_1 - t_2)\right) \leftarrow \text{fn. of } \underbrace{(t_1 - t_2)}$$

$$R_X(t, t+\tau) = \cos 2\pi f_c \tau.$$



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Wide-sense stationary (W.S.S.) random processes

A random process X_t is W.S.S. if

$$m_X(t) = \text{constant for all } t = m_X(0)$$

and $$R_X(t_1, t_2) = R_X(t_1 - t_2, 0) \text{ for all } t_1, t_2.$$

[Previous example: W.S.S.]

Stationarity $\Rightarrow$ W.S.S.

$\not\Leftarrow$

L39

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Deterministic signals

$x(t) \longleftrightarrow X(f)$ $\quad |X(f)|^2$ Energy spectral density

Periodic signals : Power spectral density

Random process X_t (random signal)

W.S.S. X_t : Power spectral density

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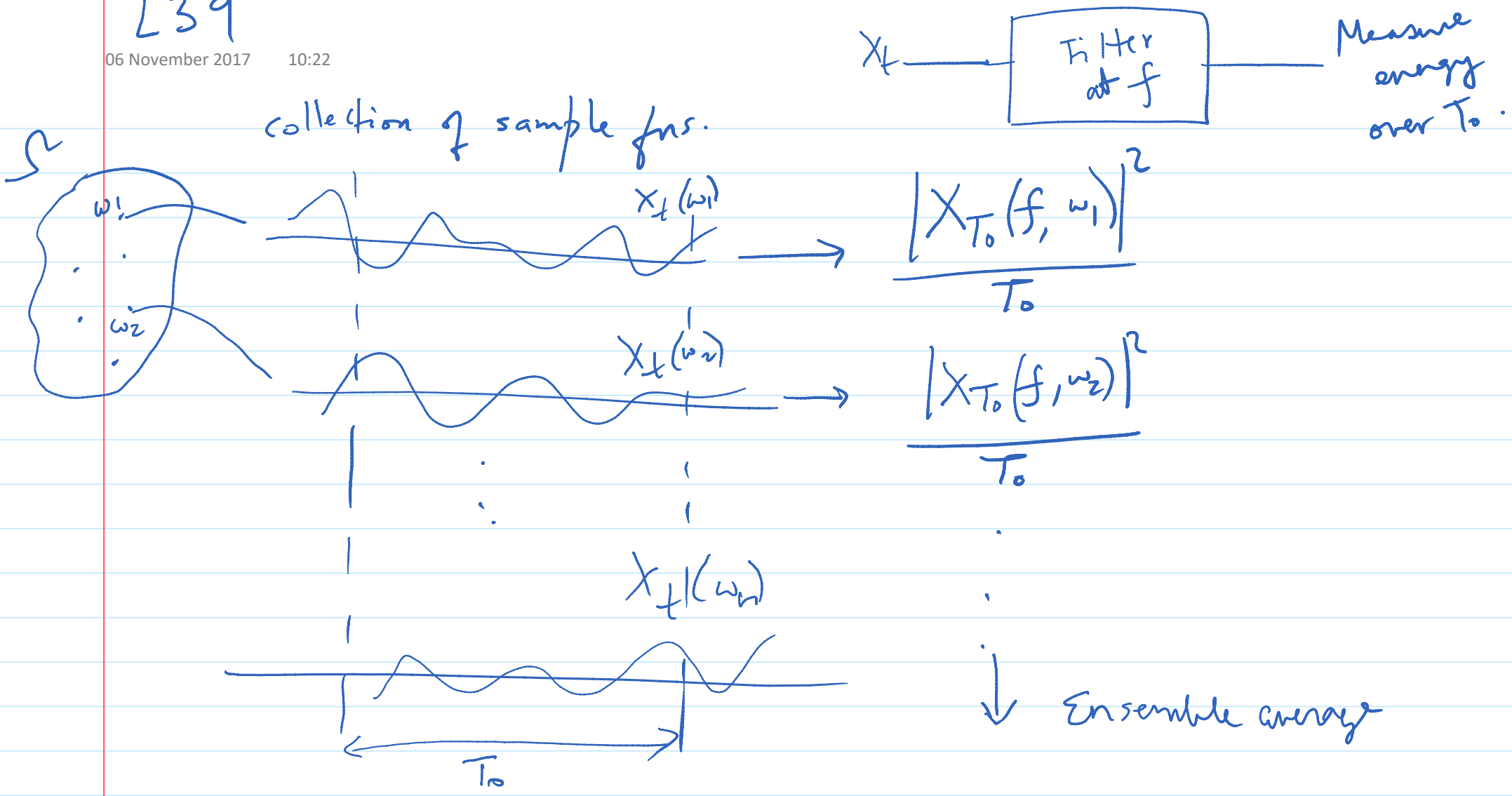

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It can be shown that

$$\lim_{T_0 \to \infty} E\left[\frac{|X_{T_0}(f)|^2}{T_0}\right] = \mathcal{F}\left[R_X(\tau)\right] \triangleq S_X(f)$$

(Wiener–Khintchine Thm)

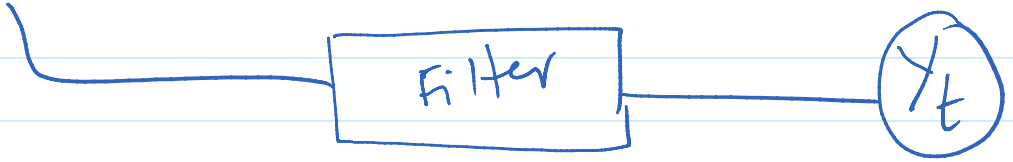
$S_X(f)$: Power spectral density of X_t.

X_t is W.S.S. $R_X(t_1 - t_2)$, τ.

Example: $X_t = X_1 \cos 2\pi f_c t - X_2 \sin 2\pi f_c t$

$$R_X(\tau) = \cos 2\pi f_c \tau \iff S_X(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

X_t is W.S.S., $R_X(\tau)$ bandlimited



$X_t - Y_t \triangleq Z_t$ → Power = 0.

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- To determine Bandwidth of a W.S.S. random process, find the bandwidth of the autocorrelation fn.

Power spectral density $= \mathcal{F}[R_X(\tau)]$

Properties

(1) $S_X(f) \geq 0 \quad \forall f$

(2) [Real X_t] $S_X(f) = S_X(-f)$

$R_X(\tau) = R_X(-\tau)$

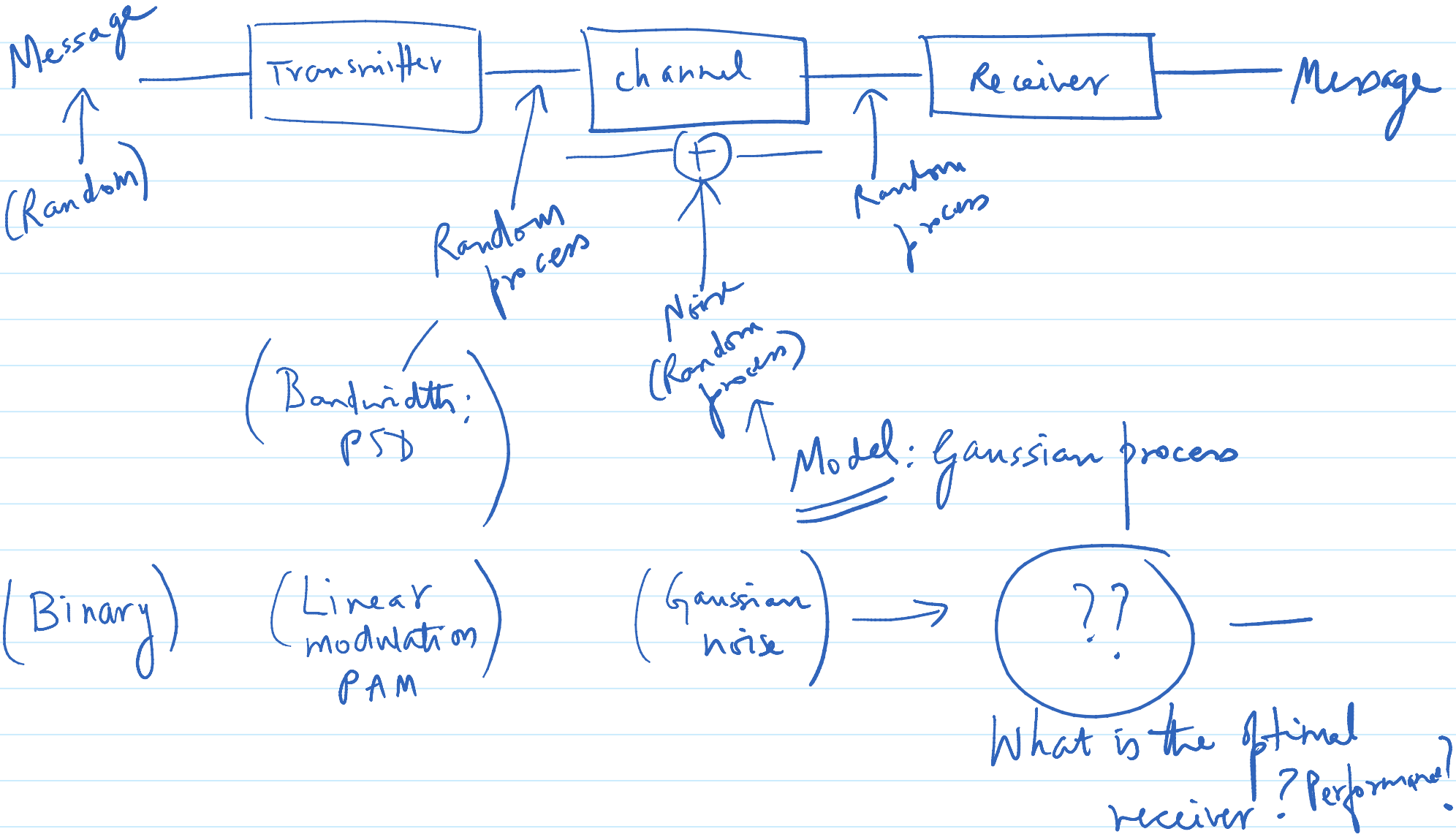
$E[X_{t_1} X_{t_2}]$

$t_1 - t_2 = \tau$

- Relation between passband and baseband random processes

L39

06 November 2017 10:44



Lecture 40

07 November 2017 09:00

Gaussian random processes

$\{X_t, t \in T\}$ is a Gaussian random process if

for any $n, t_1, t_2, \ldots, t_n, a_1, \ldots, a_n$

$$a_1 X_{t_1} + a_2 X_{t_2} + \cdots + a_n X_{t_n} \sim N(\cdot, \cdot)$$

$$\Rightarrow \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_n} \end{pmatrix} \sim \text{Jointly Gaussian for any } n, t_1, t_2, \ldots, t_n.$$

Example: $X_t = X_1 \cos 2\pi f_c t - X_2 \sin 2\pi f_c t$ $\quad X_1, X_2$ i.i.d $N(0,1)$

↳ Gaussian r.p.

L40

07 November 2017 09:08

To characterize a Gaussian random process, we need to specify only the mean fn. and the autocorrelation

$$\begin{cases} m_X(t) = E[X_t] \\ R_X(t_1, t_2) = E[X_{t_1} X_{t_2}] \end{cases}$$

Using these 2 fns, we can write the joint pdf (for any $n, t_1, \dots, t_n$) of $\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} \sim N(\cdot, \cdot)$

$$\begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_n) \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$(i,j)^{th}$ entry $R_X(t_i, t_j) - m_X(t_i) m_X(t_j)$

L40

07 November 2017 09:12

If a Gaussian r.p. is W.S.S. ($m_X(t) = m_X$, $R_X(t_1, t_2) = R_X(\tau)$)

$$\text{Joint PDF of} \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_n} \end{pmatrix} \sim N\left(\begin{bmatrix} m_X \\ m_X \\ \vdots \\ m_X \end{bmatrix}, \begin{bmatrix} & \\ & \end{bmatrix} \right)$$

$(i,j)^{th}$ entry $R_X(t_i - t_j) - m_X^2$

fn. of time differences

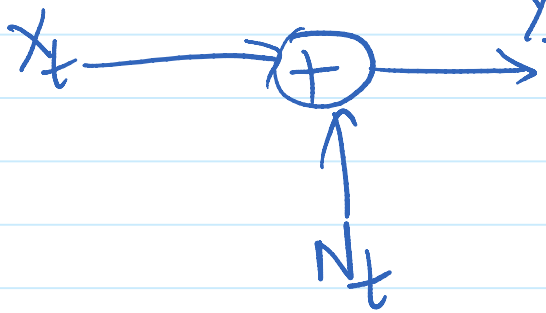
$$= \text{Joint pdf of} \begin{pmatrix} X_{t_1+d} \\ X_{t_2+d} \\ \vdots \\ X_{t_n+d} \end{pmatrix}$$

$\Rightarrow$ A W.S.S. Gaussian r.p. is <u>also</u> stationary.

L40

07 November 2017 09:16

Noise model



(Noise Mechanisms: thermal noise, Shot noise, ...)

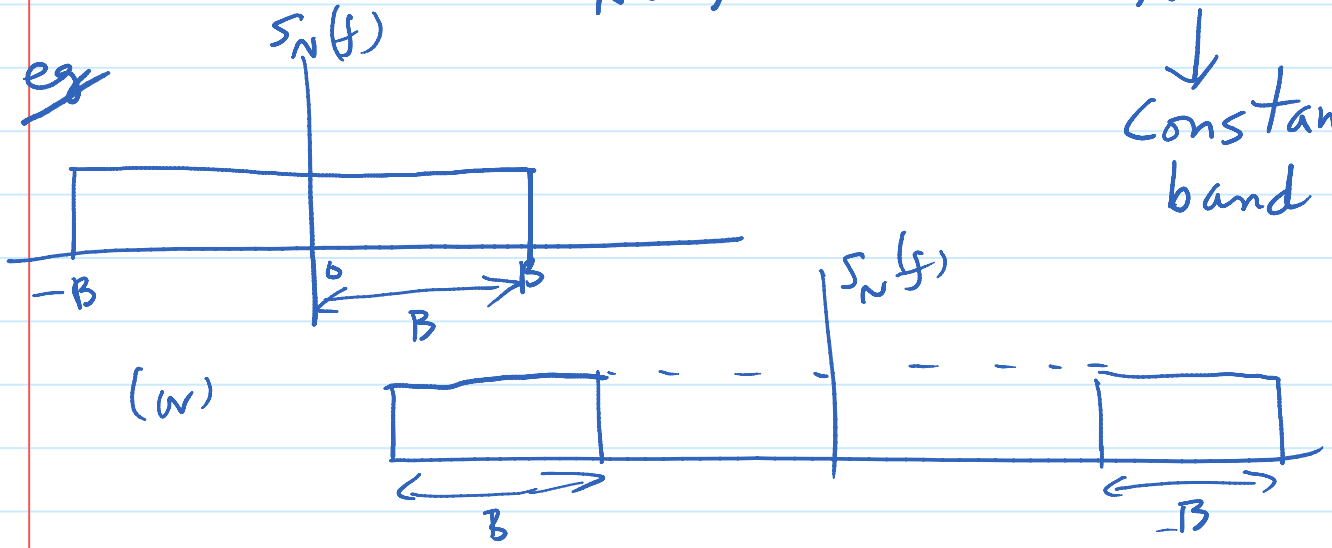
N_t : Gaussian random process

- $m_N(t) = 0$

$R_N(\tau) \overset{\mathcal{F}(\ )}{\Longleftrightarrow} S_N(f)$

↓

Constant in the band of interest



$N_{t_1} \sim N(0, \sigma_{t_1}^2)$ → σ^2

$\sigma^2 = R_N(0)$

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$$R_N(\tau) = \int_{-\infty}^{\infty} S_N(f)\, df = (\text{constant})(2B)$$

$$\frac{N_0}{2} \text{ watts/Hz}$$

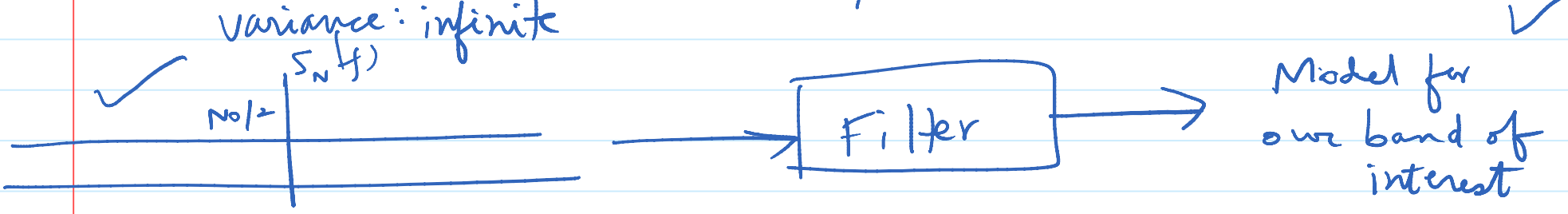
$$= N_0 B$$

B: BW of the system

N_0: Thermal noise model

Boltzmann constant $\times$ room temperature

1.38×10^{-23} J/K $\times$ 290 K



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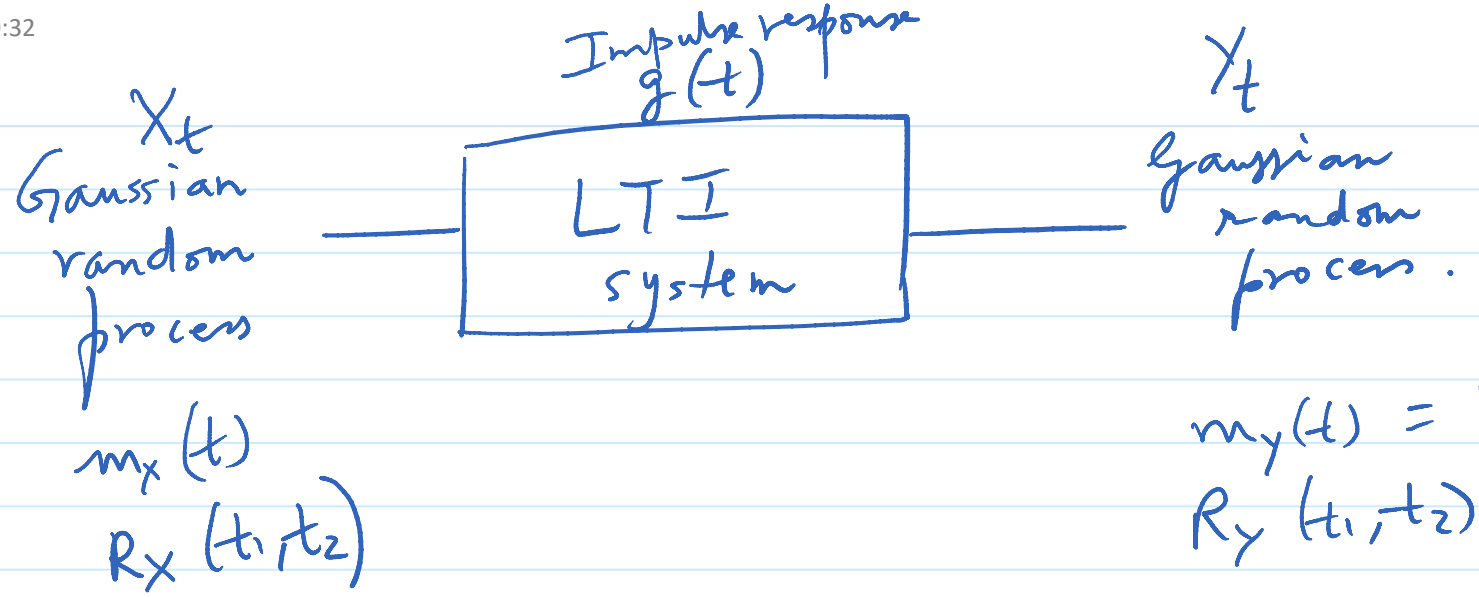

$$m_y(t) = E[Y_t] = E\left[\underbrace{\int X_{t-\tau}\, g(\tau)\, d\tau}_{\text{random variable}}\right]$$

m_y

$$= \int \underbrace{E[X_{t-\tau}]}\, g(\tau)\, d\tau = m_x(t) * g(t)$$

m_x

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$$R_y(t_1, t_2) = E[Y_{t_1} Y_{t_2}]$$

$$\downarrow$$

$$R_y(\tau) = E\left[\left(\int X_{t_1-\tau} g(\tau)\, d\tau\right)\left(\int X_{t_2-\eta} g(\eta)\, d\eta\right)\right]$$

$$= \iint \underbrace{E[X_{t_1-\tau} X_{t_2-\eta}]}\, g(\tau)\, g(\eta)\, d\tau\, d\eta$$

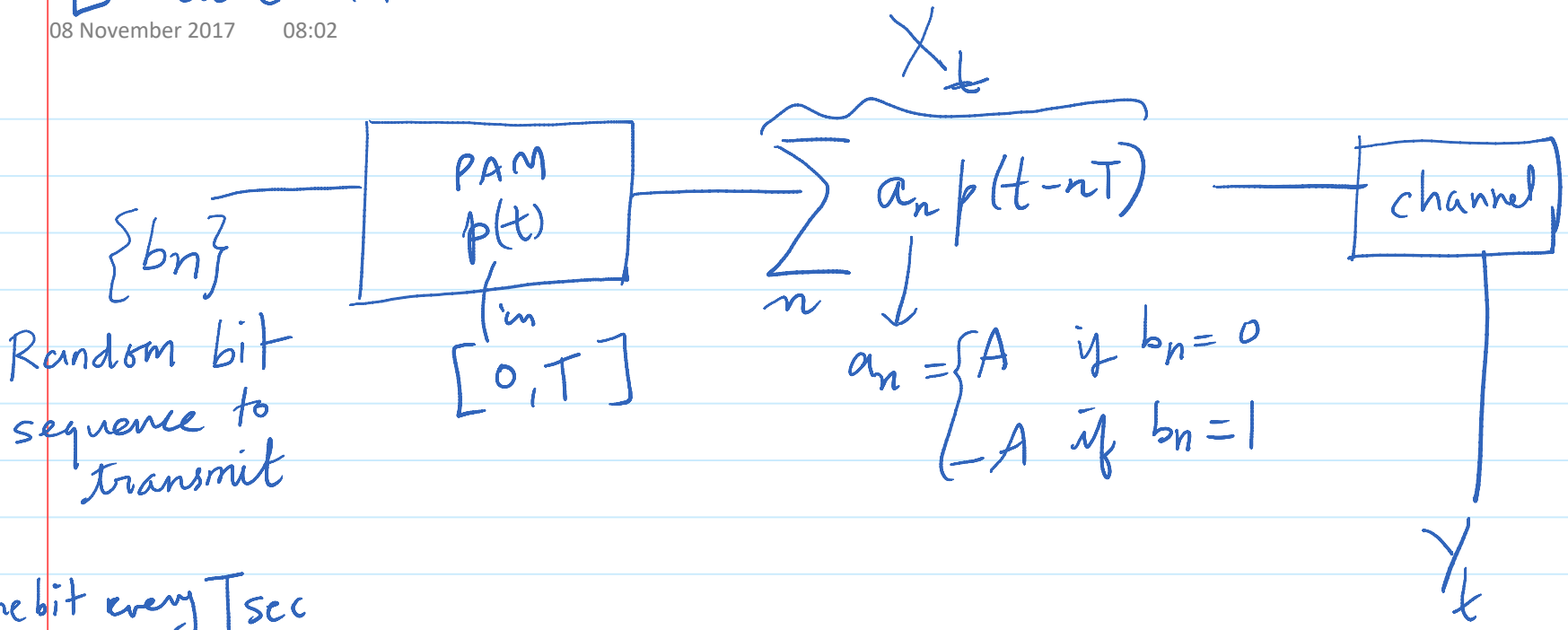
$$\downarrow X_t \text{ is W.S.S.}$$

$$R_x(t_1 - t_2 - \tau + \eta)$$

$$= R_x(\tau) * g(\tau) * g(-\tau)$$

Lecture 41

08 November 2017 08:02

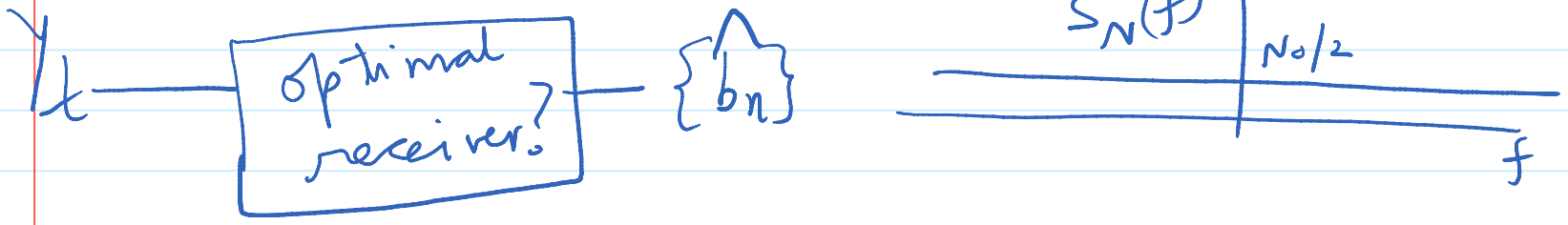


- one bit every T sec

$$Y_t = X_t + N_t$$

N_t: White Gaussian random process

$m_N(t) = 0$



L41

08 November 2017 08:08

Step 1 : Projection on signal space

Here $p(t)$ $(0^{th}$ bit$)$

$p(t-nT)$ $(n^{th}$ bit$)$

$$y_n = \langle y_t, p(t-nT) \rangle = \int y_t \, p(t-nT)\, dt$$

$$= \int (x_t + N_t)\, p(t-nT)\, dt$$

$$= \int \left(\sum_m a_m p(t-mT) + N_t \right) p(t-nT)\, dt$$

L4

08 November 2017 08:11

$$= \sum_m a_m \underbrace{\int p(t-mT)\, p(t-nT)\, dt}_{= \begin{cases} 0 & m \neq n \\ E_p & m = n \end{cases}} + \underbrace{\int N_t\, p(t-nT)\, dt}$$

$$= \underbrace{a_n E_p} + \underbrace{\int N_t\, p(t-nT)\, dt}$$

Gaussian Random variable

Mean $= \int \underline{n_N(t)}\, p(t-nT)\, dt = 0$

Variance = ?

L41

08 November 2017 08:15

Noise Variance $= E\left[\left(\int N_t\, p(t-nT)\, dt\right)\left(\int N_s\, p(s-nT)\, ds\right)\right]$

$$= \iint \underbrace{E[N_t N_s]}\, p(t-nT)\, p(s-nT)\, dt\, ds$$

$\downarrow$

$= 0 \quad t \neq s.$

$$S_N(f) = \frac{N_0}{2}$$

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$= \iint \frac{N_0}{2}\delta(t-s)\, p(t-nT)\, p(s-nT)\, dt\, ds$$

$$= \int \frac{N_0}{2}\, p(t-nT)\, p(t-nT)\, dt = \frac{N_0}{2} E_p .$$

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$$Y_n = E_p a_n + N_n$$

$N_n \sim N\left(0, \frac{N_0}{2} E_p\right)$

Let $E_p = 1$

Claim: Y_n is sufficient for optimal detection of b_n.

$\left[Y_t - Y_n p(t-nT)\right]$: part of the received signal not in the signal space

$P\left[b_n = 1 \,/\, \{Y_t\}\right]$, $P\left[b_n = 0 \,/\, \{Y_t\}\right]$

$\downarrow =$ $\downarrow =$

$P\left[b_n = 1 \,/\, Y_n\right]$ $P\left[b_n = 0 \,/\, Y_n\right]$

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08 November 2017 08:27

To show $Y_t - Y_n p(t-nT)$ is irrelevant, show that it is independent of b_n given Y_n.

$$Y_t = \sum_m a_m p(t-mT) + N_t$$

$$Y_t - Y_n p(t-nT) = \sum_m a_m p(t-mT) - Y_n p(t-mT) + N_t$$

$$= \underbrace{\sum_{\substack{m \\ m \neq n}} a_m p(t-mT) - N_n p(t-nT) + N_t}$$

For each t, show that ( ) & N_n are uncorrelated

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$$E\left[\left(\sum_{\substack{m \\ m\neq n}} a_m p(t-mT) - N_n p(t-nT) + N_t\right) N_n\right] = 0$$

$\longrightarrow$ $Y_t - Y_n p(t-nT)$ has no information about b_n.

* No signal component, only noise
* Noise is uncorrelated with N_n.

Step 2

We have $\dot{Y}_n = \underline{\underline{\dot{a}_n}} + \dot{N}_n$ (for $E_p = 1$)

(the vector channel) $\uparrow$

$+A$ or $-A$.

$N\left(0, \frac{N_0}{2}\right)$

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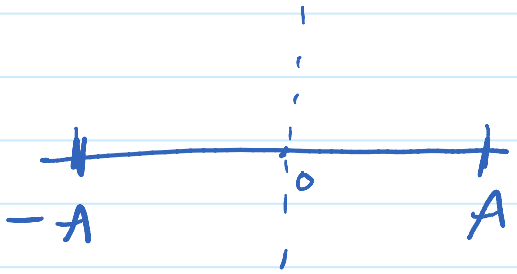
Compute $P[b_n = 0 / Y_n]$ and $P[b_n = 1 / Y_n]$

If $P[b_n = 0 / Y_n] > P[b_n = 1 / Y_n]$, then $\hat{b}_n = 0$

else $\hat{b}_n = 1$.



[Compute using Bayes' rule]

$Y_n >$ threshold $\hat{b}_n = 0$

else $\hat{b}_n = 1$

21 August 2017 09:12