Class Notes - Lectures 9 an 10. Scribe: Jadhav Pradeep

September 1, 2018

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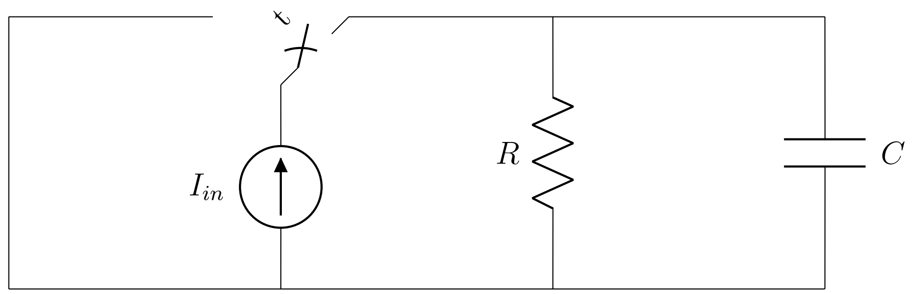
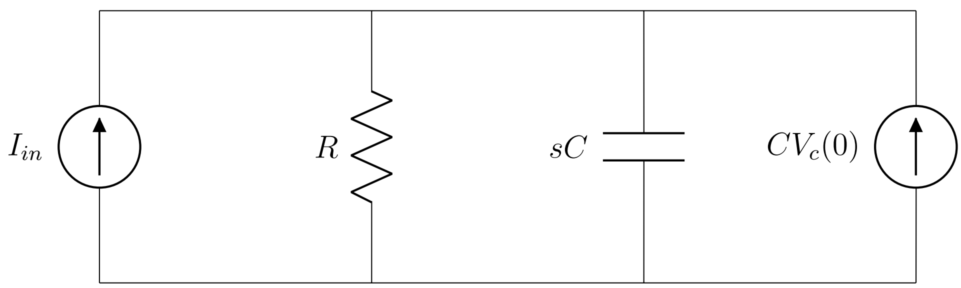


$V_c(0) = 1$V find $V_c(t)$

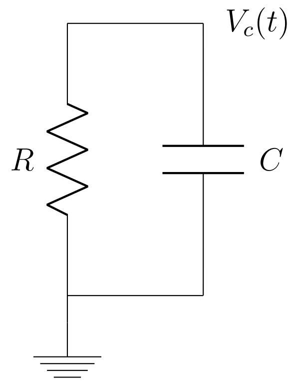


$$I_{in}(s) + CV_c(s) = (\frac{1}{R} + sC)V_c(s)$$
$$V_c(s) = \frac{RI_{in}(s)}{1 + sCR} + \frac{RCV_c(0^-)}{1 + sCR}$$

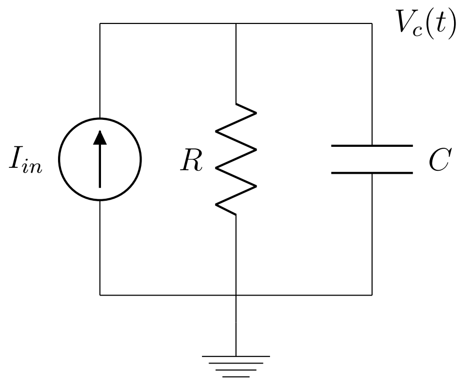
$V_c(s)$=zero state equation + zero input solution

Zero Input :
that means current source open
Applying KCL

$$C\frac{dV_c}{dt} + \frac{V_c}{R} = 0$$
$$V_c = V(0^-)e^{\frac{-t}{RC}}$$



Zero state response: The circuit is linear. First find the impulse response.



$$I_{in}(t) = \delta(t)$$

The claim is the entire current $\delta(t)$ must go through the capacitor. Show this by contradiction. If a fraction $F\delta(t)$ of the current goes through the resistor, we have

$$\begin{aligned} i_R &= F\delta(t) \\ V_R &= RF\delta(t) \\ &= V_c \\ i_C &= CRF\frac{d\delta(t)}{dt} \end{aligned}$$

At t=0, if we apply KCL at the node, we have

$$-\delta(t) + F\delta(t) + CF\dot{\delta}(t) = 0$$

KCL cannot be satisfied as there is no term to cancel $\dot{\delta}(t)$. Only possible solution at t=0 that satisfies KCL is $i_c(0) = \delta(t)$. At t=0

$$\delta(t) = C\frac{dV_c}{dt}$$

Therefore,

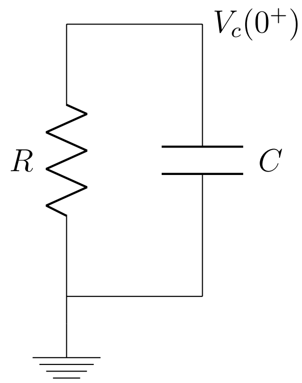
$$V_c(0^+) - V_C(0^-) = \frac{1}{c}\int_{0^-}^{0^+} \delta(t)dt$$

As $V_C(0-) = 0$

$$V_c(0^+) = \frac{1}{C}$$

The current impulse dumps charges onto the capacitor $\implies$ step changes in the voltages across the capacitor

For $t > 0$, once again the input is equal to zero. Therefore, the circuit is as follows

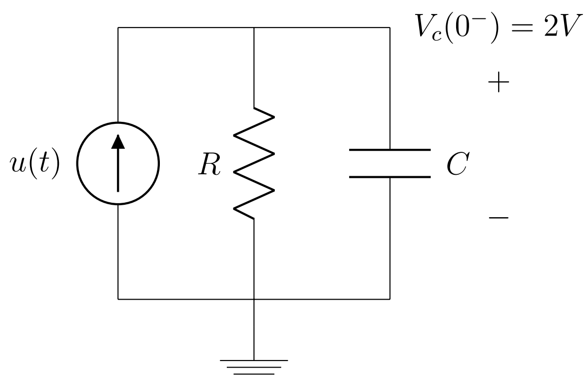


$$\begin{aligned} C\frac{dV_c}{dt} + \frac{V}{R} &= 0 \\ V_c(t) &= V_c(0^+)e^{\frac{-t}{RC}}u(t) \\ &= \frac{1}{C}e^{\frac{-t}{RC}}u(t) \\ &= h(t) \end{aligned}$$

For any arbitrary input current,

$$\begin{aligned} V_c(t) &= \int_0^t h(\tau)I_{in}(t-\tau)d\tau \\ h(t) &= \frac{1}{C}e^{\frac{-t}{RC}}\frac{\Omega}{s} \end{aligned}$$

If $I_{in} = u(t)$ and $V_C(0^-) = 2V$



Zero input solution : $V_C(t) = 2e^{\frac{-t}{RC}}u(t)$

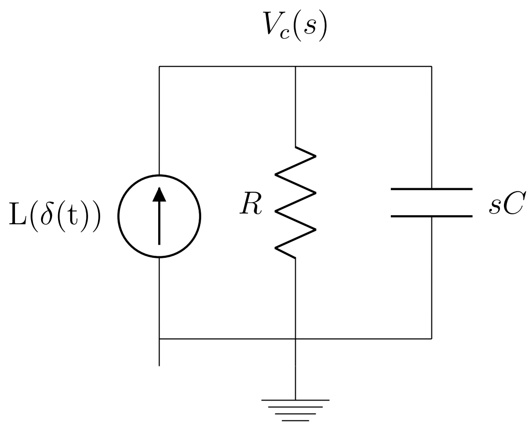
Zero state:

$$\begin{aligned} V_c(t) &= \int_0^{\infty} u(t-\tau)\frac{1}{C}e^{\frac{-\tau}{RC}}\,d\tau \\ &= \int_0^{t} \frac{1}{C}e^{\frac{-\tau}{RC}}\,d\tau \\ &= R(1-e^{\frac{-t}{RC}}) \end{aligned}$$

Total Solution : $V_c(t) = 2e^{\frac{-t}{RC}} + R(1-e^{\frac{-t}{RC}})$, for t>0.

As $t \to \infty$ capacitor is an open ckt and $V_c(t) \to$ R V

We can also get the impulse response as the inverse Laplace transform of the transfer function.



L(δ(t)) = 1

$$\begin{aligned} V(s)(\frac{1}{R}+sC) &= 1 \\ &= \frac{R}{1+sCR} \\ H(s) &= \frac{V_c(s)}{I_{in}(s)} \\ &= \frac{R}{1+sCR} \\ h(t) &= \frac{1}{C}e^{\frac{-t}{RC}}u(t) \end{aligned}$$

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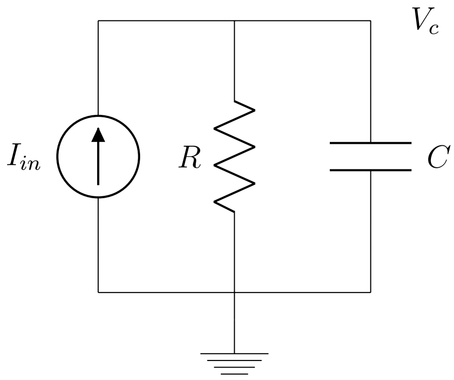
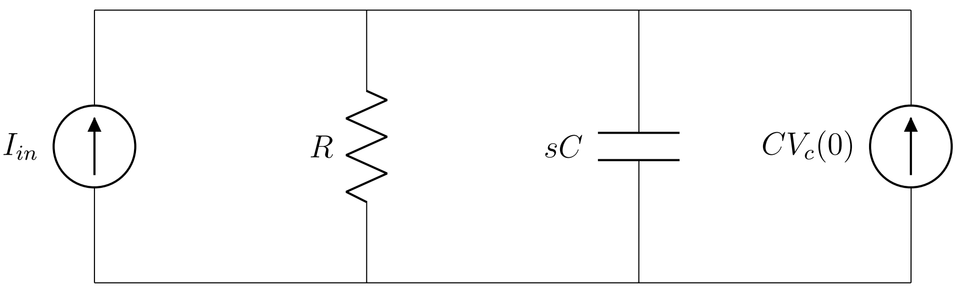
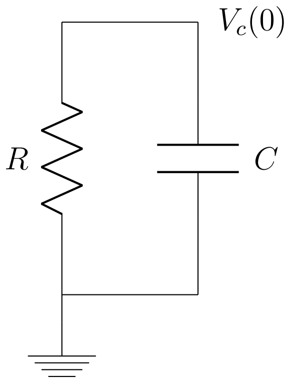
$V_c(0^-) = V_{c0}$





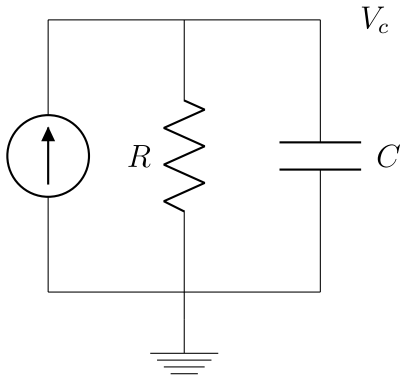
$$
\begin{aligned}
I_{in}(s) + CV_c(s) &= (\frac{1}{R} + sC)V_c(s) \\
V_c(s) &= \frac{RI_{in}(s)}{1+sCR} + \frac{RCV_c(0^-)}{1+sCR} \\
H(s) &= \frac{V_c(s)}{I_{in}(s)} \\
&= \frac{R}{1+sCR} \\
h(t) &= \frac{1}{C}e^{\frac{-t}{RC}}u(t)
\end{aligned}
$$

Zero input Solution :

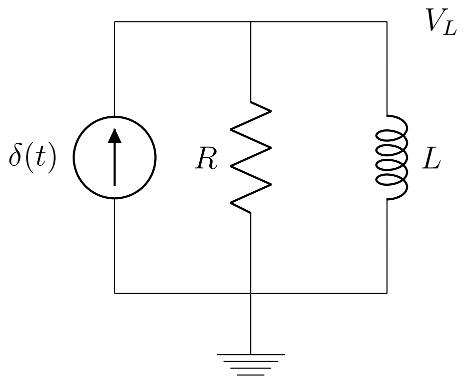


$$
\begin{aligned}
C\frac{dv_c}{dt} + \frac{v}{R} &= 0 \\
V_c(t) &= V_c(0)e^{\frac{-t}{RC}}u(t)
\end{aligned}
$$

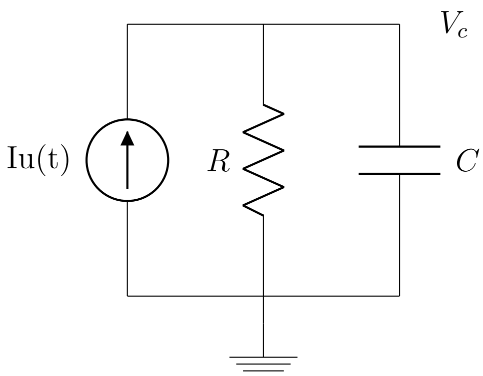
Zero State:
$C\frac{dV_c}{dt} = \delta(t)$



No current/voltage can be $\propto \dot{\delta}(t)$



If $i_L \propto \delta(t)$, $V_L \propto \dot{\delta}(t)$ and KCL not satisfied. The only way you can satisfy KCL at t=0 is if $i_R = \delta(t)$ and $V_L = R\delta(t) = L\frac{di_L}{dt}$



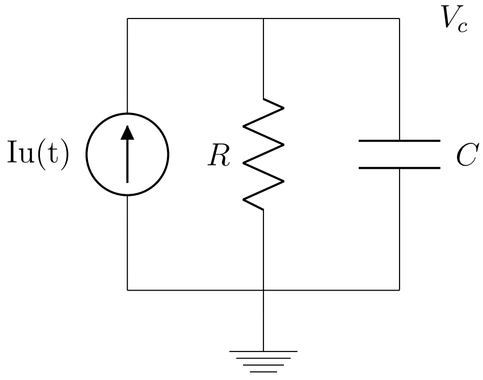
$$\begin{aligned} V_c(t) &= V_{c0}e^{\frac{-t}{RC}} + \mathcal{L}^{-1}(\frac{IR}{s(1+sCR)}) \\ &= V_{c0}e^{\frac{-t}{RC}} + IR(1-e^{\frac{-t}{RC}})t > 0 \end{aligned}$$

As t$\rightarrow \infty$ capacitor is open ckt $V_c(t) \rightarrow IR$.

Natural response (complementary function) + Forced response (particular integral solution)

$$[V_{c0}e^{\frac{-t}{RC}} - IRe^{\frac{-t}{RC}}] + IR$$

Transient Solution (terms containing time constants of the circuit) + Steady state (IR).



KCL in time domain can be written as

$$C\frac{dv_c}{dt} + \frac{v_c}{R} = Iu(t)$$

$$\frac{dv_c}{dt} + \frac{v_c}{CR} = \frac{Iu(t)}{C}$$

Homogeneous solution

$$\frac{dv_c}{dt} + \frac{v_c}{CR} = 0 \quad (1)$$

Guess the solution as $V_{cn}(t) = Ae^{\xi t}$ and substitute in equation (1)

$$A(\xi + \frac{1}{RC})e^{\xi t} = 0$$

$$\xi = \frac{-1}{RC}$$

Therefore the natural response=A$e^{\frac{-t}{RC}}$. The solution to the characteristic equation are the natural frequencies of the system.

Particular Solution or the forced solution is the steady state solution after the input is applied. After a long time, since the input is a constant, the guess for the solution is also a constant. Assume $v_{cp} = K$ and substitute guessed solution in differential equation and solve for K. In this case, $K = IR$. Therefore the total solutionis $V_c(t) = Ae^{\frac{-t}{RC}} + IR$ Solve for A by applying initial conditions at $t = 0^+$ (after the input is applied).

$$V_c(0^+) = A + IR$$

Since $V_c(0^+) = V_c(0^-)$, we have $V_c(t) = (V_{c0} - IR)e^{\frac{-t}{RC}} + IR$.

Question

Assume $I_{in}(t) = e^{-2t}u(t), R = 1\Omega, C = 1F, V_{c0} = 1V$

1 solve in transform domain; draw circuit in s domain, Solve and find inverse transform

2 use h(t) and convolve with input to get zero state solution

3 Natural + Forced response
Answer
1

$$V_c(s) = \frac{1}{s+1} + \frac{1}{(s+1)(s+2)}$$
$$v_c(t) = e^{-t} + (e^{-t} + e^{-2t}), t > 0$$

2 h(t) = e^{-t} ; convolve with e^{-2t} to get the output. **3** Natural response : Ae^{-t} The forced response requires steady state solution of

$$\frac{dv_c}{dt} + V_c = e^{-2t}$$

Guess solution as Be^{-2t} and substitute in the differential equation. This gives $B = -1$. Therefore, $V_c(t) = Ae^{-t} - e^{-2t}$. To apply initial conditions, note that at $t = 0^+, V_c(0^+) = 1$. Therefore, $A = 2$ and $V_c(t) = 2e^{-t} - e^{-2t}$