Electric Circuits and Networks	**25-09-2018**

Lecture 22: Tellegen's Theorem, reciprocity and adjoint networks

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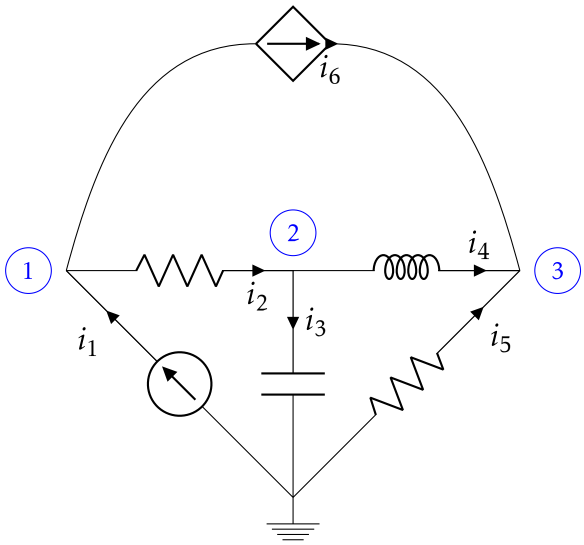
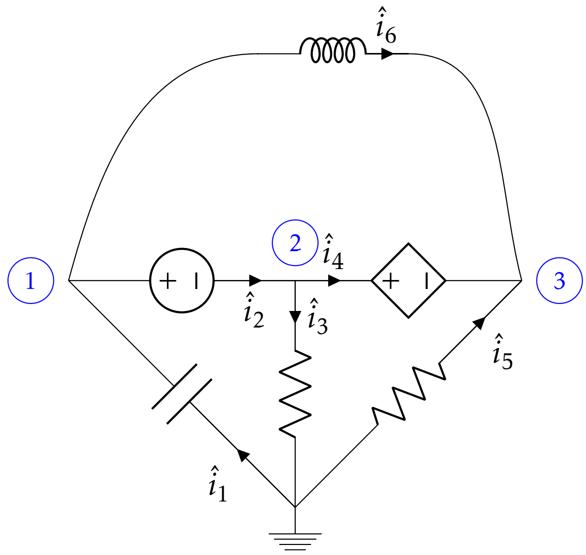
In this class we will take break from "Parameters" and look at an important theorem known as Tellegen's Theorem

Tellegen's Theorem
Consider the following two circuits



Circuit 1



Circuit 2

Although the circuit 1 and 2 are totally different in terms of their components and in their response, both the circuits are topologically same. That is, they have the same number of branches and nodes connected in exactly the same way (graph of the cicuit is the same). For solving a circuit, we need 1) KVL 2) KCL 3) Branch relationship

Apply KCL for circuit 1 (Assume currents leaving the node are positive)

$$\begin{array}{ll} \text{node 1} & -i_1 + i_2 + i_6 = 0 \\ \text{node 2} & -i_2 + i_3 + i_4 = 0 \\ \text{node 3} & -i_4 - i_5 - i_6 = 0 \end{array}$$

$$\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = 0$$

Matrix with entries 1, -1 and 0 is known as **"Reduced incidence matrix"** in which row i represents the coefficient of KCL equation at node i and column j represents the branch j

and have entries 1 (current leaves) , -1 (current enters) or 0 (branch is not connected to that node). Both circuits are different, but have same reduced incidence matrix. If A represents reduced incidence matrix and I_b is the vector containing branch current then KCL gives $AI_b = 0$

Now, apply KVL for circuit 1. According to KVL, the branch voltage equals the difference of node voltages.

$$\begin{aligned}
v_1 &= -v_{n1} \\
v_2 &= v_{n1} - v_{n2} \\
v_3 &= v_{n2} \\
v_4 &= v_{n2} - v_{n3} \\
v_5 &= -v_{n3} \\
v_6 &= v_{n1} - v_{n3}
\end{aligned}$$

where v_{n1}, v_{n2}, v_{n3} are node voltages and $v_1, v_2 ...$ are branch voltage, (Voltage drop along the direction of current)

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ v_{n3} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

i.e.

$$A^T V_N = V_b$$

where A^T is transpose of reduced incidence matrix, V_N is vector containing node voltage and V_b branch voltages

Consider $V_b^T I_b = \sum_{k=1}^{b} v_k i_k$.

$$V_b^T I_b = V_N^T A I_b = 0$$

Makes sense as $\because$ Power deliverd = Power absorbed

Suppose we have two network $N, \hat{N}$ with same directed graph (Reduced incidence matrix). Note that the reference directions for currents and voltages in both networks must be the same. Let $N \to v_i, i_i, \hat{N} \to \hat{v}_i, \hat{i}_b$. So we will have

$$\begin{aligned}
A\hat{I}_b &= 0 \qquad AI_b = 0 \\
A^T \hat{V}_N &= \hat{V}_b \qquad A^T V_N = V_b \\
V_b^T I_b = \ \hat{V}_b^T I_b &= 0 \ = \ \hat{I}_b V_b \ = \ I_b^T V_b
\end{aligned}$$

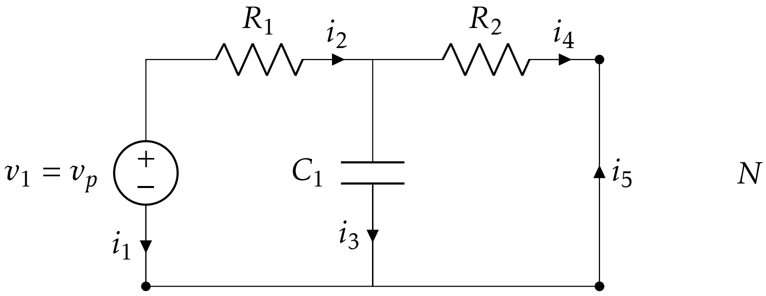
$\hat{V}_b^T I_b, \hat{I}_b V_b$ don't have any physical meaning, but the equation above holds.

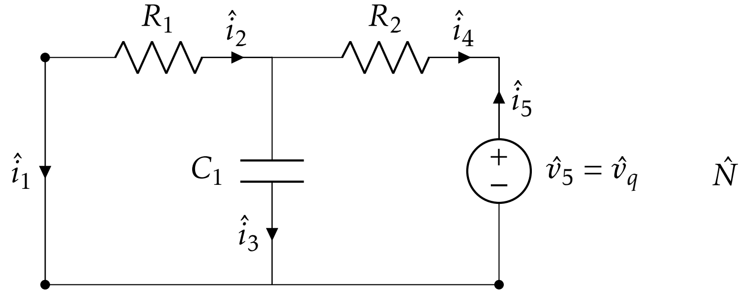
$$\Rightarrow \hat{V}_b^T I_b - V_b^T \hat{I}_b = 0 \tag{1}$$

$$\Rightarrow \sum_{k=1}^{b} \left(\hat{v}_k i_k - v_k \hat{i}_k \right) = 0 \tag{2}$$

Known as "Tellegen's Theorem". It is valid in both 's' and time domain. In the s domain it can give us useful relationships between transfer functions.

Example 1 :





Applying Tellegens theorem in the s domain,

$$\left(\hat{v}_1 i_1 - v_1 \hat{i}_1\right) + \left(\hat{v}_2 i_2 - v_2 \hat{i}_2\right) + \left(\hat{v}_3 i_3 - v_3 \hat{i}_3\right) + \left(\hat{v}_4 i_4 - v_4 \hat{i}_4\right) + \left(\hat{v}_5 i_5 - v_5 \hat{i}_5\right) = 0 \tag{3}$$

All variables are functions of 's'.

Consider N and $\hat{N}$ as two port network then all interior branches containing R, L, C are identical in N and $\hat{N}$
Therefore, $\hat{v}_k = Z\hat{i}_k$ and $v_k = Z_k i_k \implies \hat{v}_k i_k = Z_k i_k \hat{i}_k = v_k \hat{i}_k$. Thie means that the contribution of the internal branches to equation (3) is zero.

Therefore,

$$\left(\hat{v}_1 i_1 - v_1 \hat{i}_1\right) + \left(\hat{v}_5 i_5 - v_5 \hat{i}_5\right) = 0$$

Since $\hat{v}_1 = 0, v_1 = v_p, \hat{v}_5 = \hat{v}_q, v_5 = 0$, we have

$$v_p(s)\hat{i}_1(s) = -\hat{v}_q(s) i_5(s)$$

Or

$$\frac{i_5(s)}{v_p(s)} = \frac{-\hat{i}_1(s)}{\hat{v}_q(s)}$$

The internal network in N and $\hat{N}$ are exactly the same; Therefore the y-matrix for both networks is the same. From the above equation, we conclude $y_{21}(s) = y_{12}(s)$.

In general, if the network contains only R, L and C as internal elements, this will be true. Such networks are called as "Reciprocal Networks"

Exercise : Using same N and $\hat{N}$ Show that $z_{21}(s) = z_{12}(s)$.

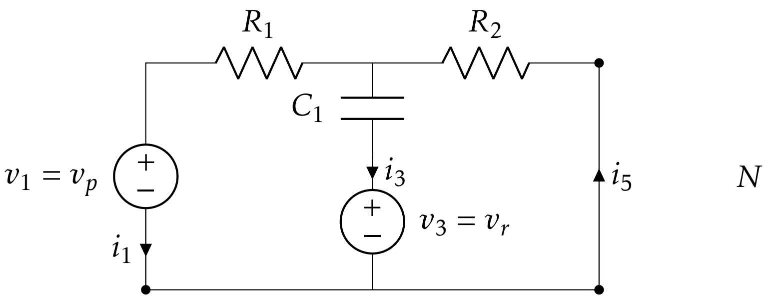
$y_{12}(s), z_{12}(s)$ etc are transfer admittances/impedances. We will see later, when we do hybrid parameters, that similar properties hold for transfer functions as well.

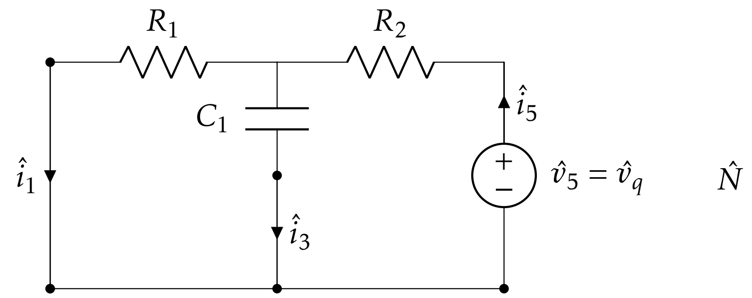
Time Domain
What about in the time domain? Multiplication in the 's' domain is a convolution in the frequency domain. Therefore,

$$\int_0^t v_p(\tau)\hat{i}_1(t-\tau)d\tau = -\int_0^t \hat{v}_q(t-\tau)i_5(\tau)d\tau$$

That is, the network $\hat{N}$ is "integrated backward in time".

Adjoint networks and transfer functions
Supposing we have multipe sources in the network and we need to find the transfer function from each source to the output. To find the transfer function, we need to keep one source active and short (open) all other voltage (current) sources and solve for all the voltages and currents in the network. This has to be repeated for each source, which is painful. It is possible to find all transfer function using one analysis of the adjoint network, $\hat{N}$. Consider Example 1, with one more voltage source added.





Using Tellegen's theorem, we get,

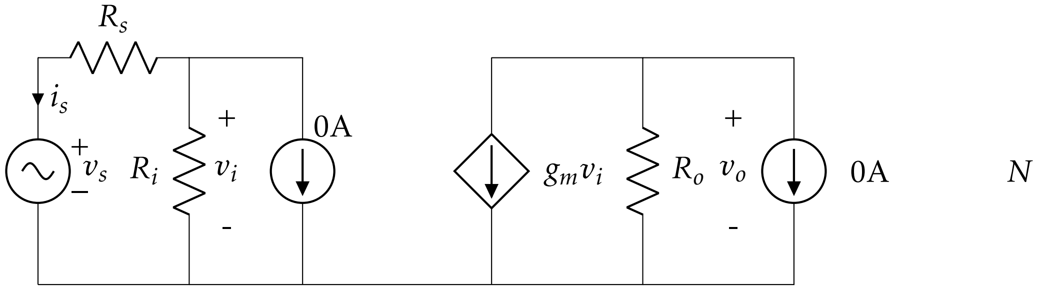
$$\left(\hat{v}_1 i_1 - v_1\hat{i}_1\right) + \left(\hat{v}_3 i_3 - v_3\hat{i}_3\right) + \left(\hat{v}_5 i_5 - v_5\hat{i}_5\right) = 0$$

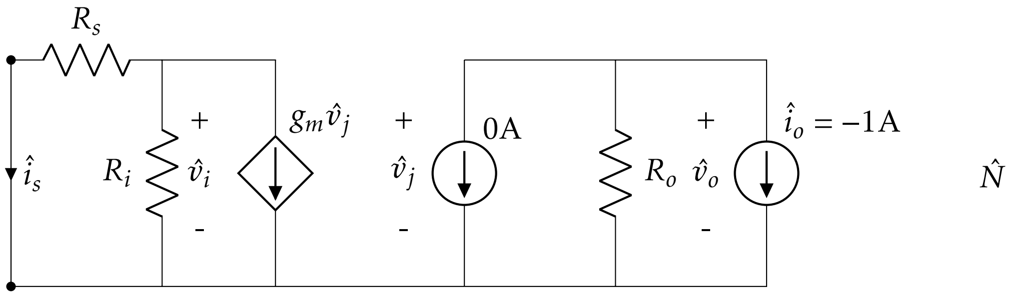
Since $\hat{v}_1 = \hat{v}_3 = 0, v_1 = v_p, v_3 = v_r, \hat{v}_5 = v_q = 1V, v_5 = 0$, we have

$$v_p(s)\hat{i}_1(s) + v_r(s)\hat{i}_3(s) = -i_5(s)$$

If we want to find $H_1(s) = i_5(s)/v_p(s)$ in N, we would set $v_r = 0$ and solve for the network. Setting $v_r = 0$ in the above equation, we get $H_1(s) = -\hat{i}_1(s)$. Similarly, to find $H_2(s) = i_5(s)/v_r(s)$, we set $v_p(s) = 0$ so that $H_2(s) = -\hat{i}_3(s)$. Hence, one analysis of $\hat{N}$ gives both transfer functions. This is particularly useful for sensitivity and noise analysis, where we need to find multiple transfer functions.

Controlled sources
More generally, an adjoint network, $\hat{N}$ can be derived for circuits containing controlled sources and opamps (non-bilateral elements). Assume we have the following circuit





We know that the contributions due to resistors cancel out in the application of Tellegen's theorem. The contribution due to the VCCS is is also zero since

$$v_i g_m \hat{v}_j - \hat{v}_i \; 0 + v_j \; 0 - \hat{v}_j g_m v_i = 0$$

Therefore,

$$v_s \hat{i}_s - v_o = 0$$
$$\implies \frac{v_o}{v_s} = \hat{i}_s$$

Exercises
Find the elements of the adjoint network for other three controlled sources.