A Perturbative Solution to Plane Wave Scattering from a Rough Dielectric Cylinder

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Abstract— In this paper, we present a perturbative second order closed form solution for the fields scattered from a stochastically rough, infinitely long dielectric cylinder. The electromagnetic boundary conditions at the surface of rough cylinder are perturbed to derive 0th, 1st and 2nd order boundary conditions that can be used to evaluate the scattered fields to the second order in the surface roughness. To validate our perturbative method, we compare it with a method of moments solution of the same problem - which, despite being valid for any arbitrary surface roughness, is computationally more expensive as compared to our analytical method. A region of validity is also estimated numerically, within which the perturbative solution accurately predicts the fields scattered from the rough cylinder. We report this region approximately as $h_0 < \lambda/25$ and $l > 5.84h_0$ for an exponentially correlated rough cylinder, where h_0 is the root mean square roughness, l is the correlation length and λ is the free space wavelength.

1. INTRODUCTION

Scattering from vegetation such as tree trunks, branches and leaves contributes significantly to radar scattering. At low microwave frequencies, branches and tree trunks can accurately be modelled as smooth cylinders for calculating their impact on the incident electromagnetic radiation [1, 2]. At higher frequencies, the surface roughness begins to play an increasingly important role and the cylinder can no longer be treated as smooth.

Numerical methods such as the method of moments or the finite element method allow us to accurately compute the fields scattered from cylinders with arbitrary roughness. However, their use is limited due to the large computational time involved. There have been many analytical studies that attempt to approximate the properties of rough cylinders, for instance treating the surface roughness as a periodic corrugation in a dielectric layer [3, 4] and modelling the scattered fields in the geometric optics approximation [5].

The small perturbation method (SPM) was originally developed to analytically model stochastically rough surfaces [6, 7]. The SPM has also been used to model rough conducting cylinders [8–10] and cylinders with impedance boundary conditions [11]. Although the SPM affords closed form expressions for the scattered fields, it breaks down as the surface roughness increases.

The analytical method developed in this paper is based on the perturbation of the boundary conditions at the surface of the rough cylinder. Several authors have used the SPM to model azimuthally rough conducting cylinders or rough cylinders with impedance surfaces, but there has been no previous report of a perturbative analysis of a rough *dielectric* cylinder with both azimuthally dependant and z dependant roughness. In contrast to the previous attempts to perturbatively analyse the scattering problem, which are based on an integral equation formulation, our approach, based on perturbing the boundary conditions at the cylinder surface, yields not only the scattered fields, but also the fields inside the cylinder. This makes it possible to extend our method to *finite* rough cylinders by using an approach similar to that used by Hulst [12] and Serker [13] in their treatment of finite smooth cylinders.

In Section 2 we derive the perturbed boundary conditions in the cylindrical coordinate system that can be used together with an appropriate eigen-expansion of the fields to obtain a second order solution to the scattering problem. In Section 3, we validate the perturbative method for an azimuthally rough exponentially correlated dielectric cylinder by comparing it with a method of moments based solution of the same problem. We also numerically derive a region of validity in terms of bounds on the root mean square roughness h_0 and the correlation length l of the rough surface within which the perturbative solution accurately captures the scattering properties of the rough cylinder.

2. THE PERTURBATIVE METHOD

The rough dielectric cylinder, shown in Fig. 1, can be mathematically represented by specifying a random process $h(\phi, z)$, which denotes the radial distance of a point on the cylinder surface at azimuthal coordinate ϕ and z coordinate z from the mean smooth cylinder (with radius a). In this section, we derive a perturbed set of boundary conditions that can be used to derive a "Mie" like solution for scattering from this rough dielectric cylinder. The starting point of our analysis are the boundary conditions on the tangential components of the fields:

$$\mathbf{n} \times \Delta\mathbf{V}\big|_{r=a+h} = 0 \text{ for } \mathbf{V} \equiv \mathbf{E} \text{ or } \mathbf{H} \tag{1}$$

where $\Delta\mathbf{V}|_{r=a+h}$ is the discontinuity in $\mathbf{V}$ across the cylinder surface and $\mathbf{n}$ is a normal to the cylinder surface given by:

$$\mathbf{n} = \hat{r} - \left(\frac{1}{a+h}\right)\frac{\partial h}{\partial \phi}\hat{\phi} - \frac{\partial h}{\partial z}\hat{z} \tag{2}$$

Eq. (1) can be re-expressed in terms of the discontinuities in components of $\mathbf{V}$:

$$\Delta V_z\Big|_{r=a+h} = -\frac{\partial h}{\partial z}\Delta V_r\Big|_{r=a+h} \tag{3a}$$

$$(a+h)\Delta V_\phi\Big|_{r=a+h} = -\frac{\partial h}{\partial \phi}\Delta V_r\Big|_{r=a+h} \tag{3b}$$

Use of the Taylor series expansion along with Eq. (3) yields (to the second order in h or it's derivatives):

$$\Delta V_z\Big|_{r=a} + h\frac{\partial \Delta V_z}{\partial r}\Big|_{r=a} + \frac{h^2}{2}\frac{\partial^2 \Delta V_z}{\partial r^2}\Big|_{r=a} = -\frac{\partial h}{\partial z}\left(\Delta V_r\Big|_{r=a} + h\frac{\partial \Delta V_r}{\partial r}\Big|_{r=a}\right) \tag{4a}$$

$$\left(1+\frac{h}{a}\right)\left(\Delta V_\phi\Big|_{r=a} + h\frac{\partial \Delta V_\phi}{\partial r}\Big|_{r=a} + \frac{h^2}{2}\frac{\partial^2 \Delta V_\phi}{\partial r^2}\Big|_{r=a}\right) = -\frac{1}{a}\frac{\partial h}{\partial \phi}\left(\Delta V_r\Big|_{r=a} + h\frac{\partial \Delta V_r}{\partial r}\Big|_{r=a}\right) \tag{4b}$$

To recast Eq. (4) into a more usable form, we expand the functions ΔV_p where $V \equiv E$ or H and $p \equiv r, \phi$ or z into a perturbation series:

$$\Delta V_p = \Delta V_p^{(0)} + \Delta V_p^{(1)} + \Delta V_p^{(2)} \ldots. \tag{5}$$

where $\Delta V_p^{(m)} \sim O(h^m)$. Clearly, $\Delta V_p^{(0)}$ corresponds to the solution of the scattering problem for a smooth cylinder. Eqs. (4) and (5) can then be used to derive the following equations in $\Delta V_p^{(m)}$:

1. *Zeroth Order Boundary Conditions:*

$$\Delta V_z^{(0)}\Big|_{r=a} = 0, \;\; \Delta V_\phi^{(0)}\Big|_{r=a} = 0 \tag{6}$$

2. *First Order Boundary Conditions:*

$$\Delta V_z^{(1)}\Big|_{r=a} = -h\frac{\partial \Delta V_z^{(0)}}{\partial r}\Big|_{r=a} - \frac{\partial h}{\partial z}\Delta V_r^{(0)}\Big|_{r=a} \tag{7a}$$

$$\Delta V_\phi^{(1)}\Big|_{r=a} = -h\frac{\partial \Delta V_\phi^{(0)}}{\partial r}\Big|_{r=a} - \frac{1}{a}\frac{\partial h}{\partial \phi}\Delta V_r^{(0)}\Big|_{r=a} \tag{7b}$$

3. *Second Order Boundary Condtions:*

$$\Delta V_z^{(2)}\Big|_{r=a} = -h\frac{\partial \Delta V_z^{(1)}}{\partial r}\Big|_{r=a} - \frac{h^2}{2}\frac{\partial^2 \Delta V_z^{(0)}}{\partial r^2}\Big|_{r=a} - \frac{\partial h}{\partial z}\Delta V_r^{(1)}\Big|_{r=a} - h\frac{\partial h}{\partial z}\frac{\partial \Delta V_r^{(0)}}{\partial r}\Big|_{r=a} \tag{8a}$$

$$\Delta V_\phi^{(2)}\Big|_{r=a} = -h\frac{\partial \Delta V_\phi^{(1)}}{\partial r}\Big|_{r=a} - \frac{h^2}{2}\frac{\partial^2 \Delta V_\phi^{(0)}}{\partial r^2}\Big|_{r=a} - \frac{1}{a}\frac{\partial h}{\partial \phi}\Delta V_r^{(1)}\Big|_{r=a} + \frac{h}{a^2}\frac{\partial h}{\partial \phi}\Delta V_r^{(0)}\Big|_{r=a} - \frac{h}{a}\frac{\partial h}{\partial \phi}\frac{\partial \Delta V_r^{(0)}}{\partial r}\Big|_{r=a} \tag{8b}$$

In addition to the boundary conditions presented above, it is necessary to use an appropriate eigen-expansion of the scattered fields and the fields inside the cylinder so as to obtain a closed form expression for the scattered fields. Since the boundary conditions have been worked out in the cylindrical coordinates, an appropriate basis for representing the scattered fields would be the cylindrical wave basis. Once the scattered fields are known in terms of $h(\phi, z)$, it is a simple matter to evaluate the statistical properties of the scattered fields as a function of the statistical properties of the rough surface (i.e., the correlation function $R(\phi, z) = \overline{h(\Phi + \phi, Z + z)h(\Phi, Z)}$). For further details and derivations, please refer to the upcoming journal version of this paper [14].

3. RESULTS AND DISCUSSIONS

To validate our analytical approach, we compared the perturbative approach to a Method of Moment (MoM) solution of the same scattering problem. For simplicity, we assume an azimuthally rough cylinder (i.e., h is only a function of ϕ). Additionally, we also assume the cylinder surface to be exponentially correlated:

$$R(\phi) = \overline{h(\Phi + \phi)h(\Phi)} = h_0^2 \exp(-a|\phi|/l) \ \forall \ \phi \in (-\pi, \pi] \tag{9}$$

where $h_0 = (\overline{h^2(\phi)})^{1/2}$ is the root-mean square (RMS) roughness and l is the correlation length of the surface. Additionally, note that $R(\phi)$ is periodic in ϕ with period 2π.

The perturbative solution derived in the previous section is valid if and only if $h(\phi, z)$ and it's derivatives ($\partial h(\phi, z)/\partial\phi$ and $\partial h(\phi, z)/\partial z$) are small. Therefore, the perturbative solution is expected to deviate from the more accurate MoM analysis for large RMS roughness h_0 (resulting in large $h(\phi, z)$ per instance) and small correlation lengths l (resulting in larger derivatives of $h(\phi, z)$ per instance). By an exhaustive comparison between the perturbative solution and the MoM solution, we arrived at the following bounds on h_0 and l for a second order perturbative analysis to be reasonably accurate $-k_0h_0 < 0.25$ and $k_0l > 1.097$. A convenient measure of the derivative of $h(\phi, z)$ is the slope s defined by $s = h_0/l$. In terms of s, the perturbative solution was found to be accurate when $s < 5.84$. Fig. 2 shows the comparison between the ensemble average of the 2D scattering cross section calculated using the method of moments and the perturbative solution for $k_0h_0 = 0.25$, $s = 0.019$ and $k_0h_0 = 0.188$, $s = 5.84$ (these dimensions are just within the validity bounds stated above).

We emphasise that the perturbatively obtained expressions for the scattered fields are very simple to evaluate numerically. Within the region of validity, the perturbative solution is computationally far more efficient than any other numerical method which can be employed to solve the same problem. Also observe that our method directly gives the ensemble average of the scattered fields and derived quantities (such as the scattering cross section) in terms of the auto-correlation

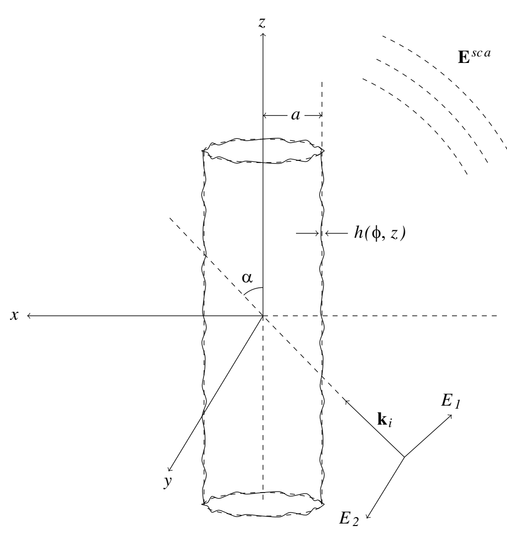
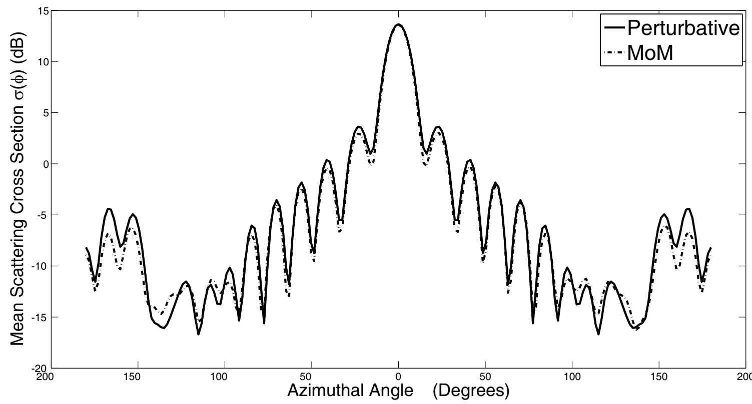
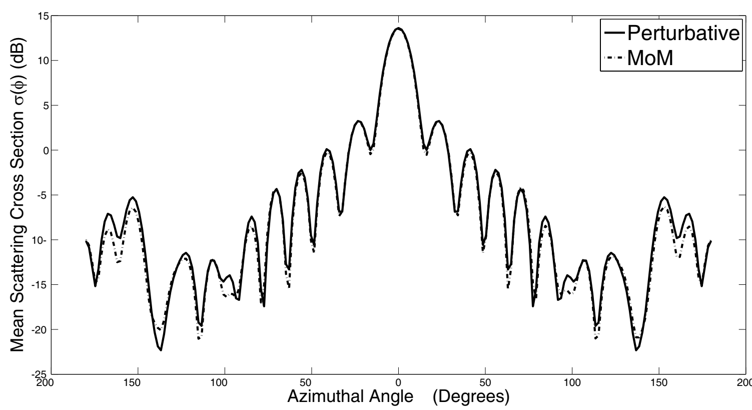


Figure 1: Schematic of the rough cylindrical scatterer with a stochastic roughness $h(\phi, z)$ and illuminated by a plane wave propagating with wavevector $\mathbf{k}_i$ in the xz plane at an angle of α with the x axis. E_1 is the component of the electric field in the xz plane, while E_2 is the component along the y axis.

function of the surface roughness. A numerical analysis of the problem (such as using the method of moments or finite element method) can only yield the scattered fields for a particular *instance* of the rough cylinder, and the simulation would have to be repeated for a large number of instances to obtain the ensemble average of the fields and the derived quantities. For instance, in the MoM results shown in Fig. 2, we needed to calculate the scattering cross section for 100 instances of the rough cylinder to achieve convergence, and for each instance we had to solve a 600×600 dense matrix system to calculate the scattered fields.



(a) $k_0 h_0 = 0.25, s = 0.019$



(b) $k_0 h_0 = 0.188, s = 5.84$

Figure 2: Variation of the mean scattering cross section $(\sigma(\phi))$ with ϕ calculated using the perturbative solution and Method of Moments. The MoM results were generated by averaging the scattering cross section calculated for 100 different instances of $h(\phi)$. In all calculations, the permittivity ϵ of the cylinder was assumed to be $2\epsilon_0$ and the radius a is taken to be $2\lambda_0$.

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