Algebraic reconstruction techniques for inverse imaging

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Abstract — **Inverse microwave imaging has the potential to provide low cost solutions for medical imaging applications, such as in breast cancer detection. The mathematical challenges in the solution of inverse problems, namely non-linearity, ill-posedness and non-uniqueness, are well known and cause optimization algorithms to get stuck at local minima(s). In this regard, the initial guess is very crucial in guiding the inversion algorithm to an acceptable solution. We investigate the role of algebraic reconstruction techniques along with box constraints in providing a good initial guess. We propose an inversion algorithm that combines these techniques with a regularization that maximises entropy (using a Gaussian a priori law) and show good reconstructions for contrasts up to 2.**

1 Introduction

In implementing a quantitative method for solving the inverse electromagnetic scattering problem, one considers the electric field integral equation (EFIE)[1]. Here, we encounter non-linear equations with two sets of unknowns inside the scattering object: the permittivity of the object, and the internal electric field. To solve this, we can either search for both sets of unknowns simultaneously or we can search for one set, by taking the other set to be constant and vice-versa. If we use the latter approach, i.e. of solving for the sets of unknowns alternatively, we can approximately linearize our equations and use standard convex optimization techniques.

It is here that we find that an appropriate choice of an initial guess for these unknowns plays a crucial role in how well the rest of the algorithm converges, subject to the strength of the dielectric contrast. Algebraic reconstruction techniques (ART) were proposed for electron and X-ray tomography in the 1970s [2], and we employ them here to provide an initial guess for the permittivity of the object after using the Born approximation to initialize the internal fields. We find this form of initialization to be more effective than some other methods used in the literature.

Beyond the initial guess, the ill-posedness of the problem necessitates suitable regularization techniques in order for the reconstruction algorithms to converge to meaningful solutions. Any available

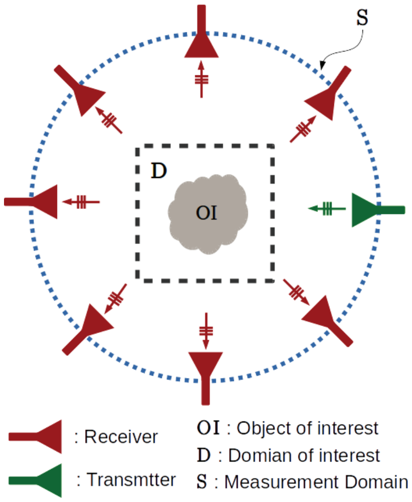

Figure 1: Inverse Problem Setup: The object of interest (OI) is illuminated by a set of receivers and the scattered field for each illumination is measured at each transmitter in order to solve the inverse problem within the domain D.

priori information can easily be added as further constraints to the optimization algorithm. Some of the commonly used regularization terms used are L_1-norm [3], L_2-norm [4], total variation [5], entropy maximization [6], etc.

The paper is organized as follows. Section II formally describes the problem setup. Section III explains the inversion techniques used to solve the inverse problem, where we discuss the ART, its extension for higher contrasts, and the maximum entropy technique of regularization. Results are discussed in Section IV, followed by the conclusion in Section V.

2 Problem Setup

In the inverse scattering problem setup shown in Fig. 1, the EFIE for a two dimentional (2D) scatterer, enclosed in a region D, which is surrounded by a contour S, is given as follows [7]:

$$E_j(r) - E_j^{\mathrm{i}}(r) = k_b^2 \iint_D G(r - r')\chi(r')\, E_j(r')\, d^2r' \quad (1)$$

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978-1-4673-9811-4/16/$31.00 ©2016 IEEE

where $E_j^{\mathrm{i}}(r)$, k_b, $\chi(r') = \epsilon_r(r') - 1$, and G are the incident electric field, propagation constant in the background medium, contrast (with $\epsilon_r(r')$ denoting the relative permittivity at r'), and the 2D Green's function, respectively. The scattered field is denoted as $E_j^{\mathrm{s}}(r) = E_j(r) - E_j^{\mathrm{i}}(r)$, with j denoting the j^{th} transmitter. The number of transmitters is chosen based on the spatially bandlimited nature of the scattered electric field [8].

After discretizing the domain and assuming piece-wise constant approximations for the contrast and field, Equation (1) takes the following form:

$$g_S = G_S\chi, \quad r \in S,\ r' \in D : \text{ Data eqn.} \tag{2a}$$
$$g_D = G_D\chi, \quad r \in D,\ r' \in D : \text{ Object eqn.} \tag{2b}$$

where lower and upper case letters g, G denote the discretization of the left-hand and right-hand side, respectively, of Equation (1). Note that G_S and G_D are functions of the internal electric fields. The inverse problem is to determine χ, given measurements of the scattered fields at all receivers for various transmitters, and knowledge of the incident field.

3 Inversion Techniques

The algorithm we propose to solve the inverse problem is based on the Born iterative method (BIM)[7]. As mentioned earlier, two sets of unknowns, the contrast and the internal fields are involved in solving the inverse problem. We update the contrast and internal electric fields alternatively by keeping one constant during the other's update; the pseudocode for the algorithm is outlined in Figure 2. Different to the BIM, we update the internal fields based on the previous guess of the contrast by solving the forward electromagnetic scattering problem in step 6.

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1:  procedure INVERSION ALGORITHM                ▷ Algorithm to find the object contrast
2:      Initialize u ← u_inc                     ▷ Born approximation
3:      Initialize G_S, g_S, G_D, g_D
4:      Estimate χ                                ▷ Initial guess
5:      while Convergence not achieved do
6:          Find u                                ▷ Solve forward problem
7:          Update G_S, g_S, G_D, g_D
8:          Estimate χ                            ▷ Contrast update
9:      end while
10: end procedure
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Figure 2: Pseudocode for inversion algorithm

The two key steps of this algorithm are steps 4 and 8, corresponding to the initial guess and the contrast-update steps, respectively. In Table 1 we outline the different choices that are utilized depending on the range of the contrast. The details of these methods are discussed in the following subsections.

Contrast	Initial guess	Contrast update
Medium	ART	ART
High	ARTGT	ARTGT or Entropy

Table 1: Inversion Techniques

3.1 Algebraic Reconstruction Technique (ART)

Algebraic reconstruction technique [2] iteratively solves a system of linear equations $G_S\chi = g_S$, equivalent to minimizing a cost function of the form $\|G_s\chi - g_S\|_2^2$ by projecting the past estimate of χ onto the next equation as follows.

$$\chi^{k+1} = \chi^k + \frac{b_i - \langle a_i, \chi^k\rangle}{\|a_i\|^2}\overline{a_i} \tag{3}$$

where $i = k \,\mathtt{mod}\, m + 1$, a_i and b_i are the i^{th} row of G_S and i^{th} element of vector g_S respectively. G_S is a matrix of size $m \times n$, χ is a vector of size n, and g_S is a vector of size n, where m is the number of measurements and n is the number of pixels in the domain D.

The update begins with $\chi = 0$, and after each update, we apply box constraint on χ, i.e. project the value of χ into a range of acceptable values defined as $[\chi_{\mathrm{rmin}}, \chi_{\mathrm{rmax}}]$, and $[\chi_{\mathrm{imin}}, \chi_{\mathrm{imax}}]$ corresponding to projections of the real and imaginary parts of χ, respectively. These 'acceptable' values are derived from a priori knowledge of the problem; for example, from known [9] lower and upper bounds on the permittivity of breast constituents.

3.2 Algebraic Reconstruction Technique with Generalized Tikhonov (ARTGT)

A related technique is to add a 'generalized' Tikhonov type term to the cost function. This is useful if for instance, it is expected that the solution be close to some guess χ_0 (derived from a priori knowledge). The optimization problem becomes of the following form:

$$\arg\min_x \quad \rho^2\|G_s\chi - g_S\|_2^2 + \|\chi - \chi_0\|^2 \tag{4}$$

where ρ is a constant that needs to be determined/specified, and controls the relative importance of measurements versus a priori knowledge. This problem can be cast in the form of a single linear equation by making the substitution $u = \rho(g_S - G_S\chi)$ and $z = \chi - \chi_0$ [10], leading to the following system of equations that can be solved using the ART technique described in the previous sub-section:

$$[I \quad \rho G_S]\,[u \quad z]^T = \rho\,[g_S - G_S\chi_0]\,. \qquad (5)$$

The update begins with $u^{(0)} = 0$ and $z^{(0)} = \chi_0$, and the box constraints are applied as before. We denote this method by the acronym ARTGT.

3.3 Maximum Entropy using Gaussian a priori law

In this sub-section, we explore a different kind of regularization based on entropy considerations. By using Bayes' rule, we maximize the probability distribution of the a posteriori probability of object contrasts given the measurements [6]. The principle of maximum (Shannon) entropy applied to the a priori probability along with some form of a prior knowledge (based on past iterations of the inversion algorithm in our case) gives the following problem:

$$\arg\min_{\chi} \left\{ \frac{1}{\sigma^2}(G_S\chi - g_S)^H(G_S\chi - g_S) + \lambda \sum \chi_i^2 + \mu \sum \chi_i \right\}, \qquad (6)$$

where the paramters λ, μ are defined as $\lambda = \frac{1}{2v}$ and $\mu = -\frac{m}{v}$ and m, v are the mean and variance of χ, respectively, as determined from the χ estimate of the previous iteration. We solve the above equation by using the method of steepest descent and apply box constraints on the contrast at each update.

4 Results

We now present the results of our inversion algorithm based on the following specifications and figure of merit:

Box constraints: Based on the following experimental values of relative permittivity at 3 GHz: [9] malignant tissue : 54-14.5i, fat : 5-0i, fibroconnective tissues : 36-5.64i and matching fluid : 18-0.06i, we get the following box constraints for the contrast: $[-0.73, 2]$ (real part), and $[-0.8, 0]$ (imaginary part).

Simulation details: We consider a domain size of $1.2\lambda \times 1.2\lambda$ and take equi-angularly spaced measurements on a circle of radius 3λ. There are 27 measurements per incidence angle, and as many number of incidence angles. Synthetic data is generated on a uniform grid of side length $\lambda/32$, and to avoid the inverse crime, the inverse problem is solved on a grid of side $\lambda/16$.

Reconstruction Error: The following metric defines the reconstruction error (ζ) as follows [11]:

$$\zeta = \frac{1}{n}\sum_i \frac{|\chi_i^{true} - \chi_i^{est}|}{|1 + \chi_i^{true}|} \times 100\% \qquad (7)$$

where χ^{true}, χ^{est} represent the true and estimated contrast vectors, respectively.

4.1 Choice of initial guess

Some popular choices for initial guesses of contrast that are used in the literature are: the pseudo inverse $(G_S^H G_S)^{-1} G_S^H g_S$ [11], or simply $G_S^H g_S$ [4]. We show comparisons of ART and $G_S^H g_S$ in Figure 3, where a quantitative difference can be seen between the two methods.

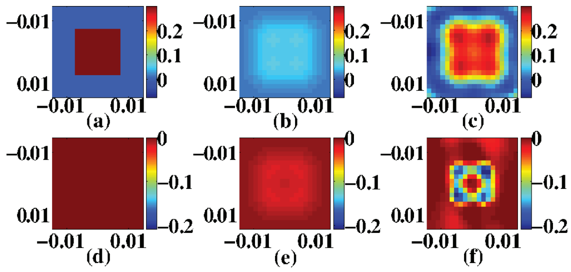

Figure 3: Top row: real parts, bottom row: imaginary parts (of contrast). (a),(d) show the actual object; (b),(e) demonstrates initial guess using $G_S^H g_S$; and (c),(f) demonstrates initial guess using ART.

It is to be noted that the ART-based initial guess of the real part is quite good, both in terms of the magnitude estimated as well as the shape fidelity, where as $G_S^H g_S$ is demonstrably poor. The imaginary part shows some artifacts, but we find that they vanish later in the inversion algorithm.

The above strategy works for 'weak' values of contrast, and as contrasts begin to increase beyond 0.8 ('medium' range), ART starts to become unstable. To increase stability, we find that artificially adding sparsity in the pixel basis by increasing the domain size leads to the improvement in reconstruction. For instance, on increasing the domain side from 1λ to 3λ, the reconstruction error for an object such as Figure 3(a,d) with contrast 1.3 decreased from 35% to 10%.

4.2 Choice of contrast update algorithm

For higher contrasts (> 0.8), adding sparsity does not help, and instead the use of the ARTGT helps to give a stable choice of the initial guess. Presently, we set the initial $\chi_0 = 0$ and $\rho = 2$ in ARTGT. The entropy maximization procedure has the advantage of not having any free parameters (such as ρ), and it the method of choice for the 'contrast update' stage (step 8) of our algorithm.

In Figure (4) we show the results of our algorithm for ARTGT and the entropy based optimization, starting with an initial guess via ARTGT. We also assume a priori knowledge of the object boundaries. We see that the two methods give comparable results.

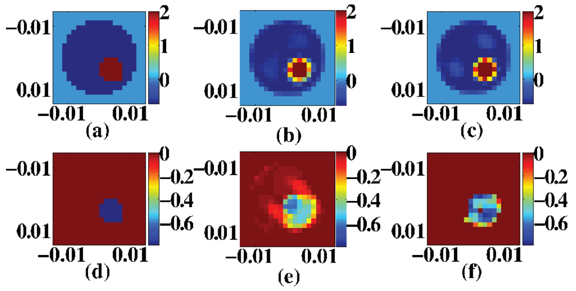

Figure 4: Top row: real parts, bottom row: imaginary parts (of contrast). (a),(d) show the actual object. (b),(e); (c),(f) demonstrates reconstruction using ARTGT and maximum entropy technique, with the priori knowledge of object boundaries giving reconstruction errors of 21% and 22% respectively.

5 Conclusion

ART acts as an good initial guess and reconstruction technique at low contrast. At higher contrasts, ARTGT can be used for both initial guess and for reconstruction. ARTGT works as a very good initial guess for the maximum entropy technique. A priori information can be used to further improve the inversion algorithm.

Acknowledgments

U K Khankhoje would like to acknowledge financial support from the INSPIRE Faculty Award (IFA-13 ENG-60) by the Department of Science and Technology, Government of India.

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