Convergence Analysis → Rate?

For simplicity, assume c.f. is convex, quadratic

$\Rightarrow$ exact line search possible

$$\left.\frac{d\phi(\alpha)}{d\alpha}\right|_{\alpha=\alpha_k} = 0 \quad , \text{ given } p_k.$$

$$f(x) = \frac{1}{2} x^T Q x - b^T x \quad — (1)$$

Q is sym, pos. def.

$$\nabla f(x) = Qx - b \quad — (2)$$

Assume $p_k = -\nabla f_k$

$$\phi(\alpha) = f(x_{k+1}) = f(x_k - \alpha_k \nabla f_k)$$

Exact line search $\rightarrow$ $\phi'(\alpha) = 0$

$$= \nabla f(x_{k+1})^T p_k = - \underbrace{\nabla f_{k+1}^T \nabla f_k}$$

$$x_{k+1} = x_k - \underbrace{\left(\frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k} \right)}_{\alpha_k} \nabla f_k$$

$\| x_k - x^* \|_2$ stationary pt

Define $\|a\|_Q^2 = a^T Q a$ P.D.

[At the soln $\nabla f(x^*) = 0$

$\Rightarrow$ $\underbrace{Qx^* - b = 0}$]

$$b^T = x^{*T} Q$$

$$\frac{1}{2}\|x_k - x^*\|_Q^2 = \frac{1}{2}\left[(x_k - x^*)^T Q (x_k - x^*)\right]$$

$$= \frac{1}{2}\left[x_k^T Q x_k + x^{*T} Q x^* - \underbrace{x^{*T} Q}_{b^T} x_k - x_k^T \underbrace{Q x^*}_{b}\right]$$

$$f(x) = \frac{1}{2} x^T Q x - b^T x$$

$$= \left[\frac{1}{2} x_k^T Q x_k - b^T x_k\right] + \frac{1}{2} x^{*T} Q x^*$$

$$= f(x_k) - f(x^*)$$

$$b = Q x^*$$

$$Q x^* = b$$

$$x^* = Q^{-1} b$$

$$f(x^*) = \frac{1}{2} b^T Q^{-1T} Q Q^{-1} b - b^T Q^{-1} b$$

$$= \frac{1}{2} b^T (Q^{-1}) b - b^T Q^{-1} b = -\frac{1}{2} b^T Q^{-1} b$$

$$\frac{1}{2}\|x_k - x^*\|_Q^2 = f(x_k) - f(x^*)$$

Luenberger

$$\|x_{k+1} - x^*\|_Q^2 \leq \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)\|x_k - x^*\|_Q^2$$

λ_i's are eigen values of Q

$$\frac{\|\Delta x_{k+1}\|}{\|\Delta x_k\|} \leq r, \quad r \in (0,1) \qquad r = \sqrt{\left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)}$$

$\Rightarrow$ Convergence is linear.

$$K(Q) = \frac{\sigma_{max}}{\sigma_{min}} = \frac{\lambda_n}{\lambda_1} \quad \text{Cond no.}$$