Descent Directions

$x \in \mathbb{R}^n \rightarrow \min_x f(x)$

✓ Unconst or

Constrained?

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \in \Gamma \quad \leftarrow \quad \min_{x \in \Omega} f(x), \ \cancel{\text{s.t. } x \in \Omega}$$

$\rightarrow \quad p_N = -(\nabla^2 f)^{-1} \nabla f$, $\quad \underline{\underline{\nabla^2 f \text{ P.D.}}}$,

$\Rightarrow \lambda_i\text{'s are} > 0$

$$\underline{\nabla^2 f = Q \Lambda Q^T}$$

$$\begin{bmatrix} \lambda_i\text{'s} \end{bmatrix}$$

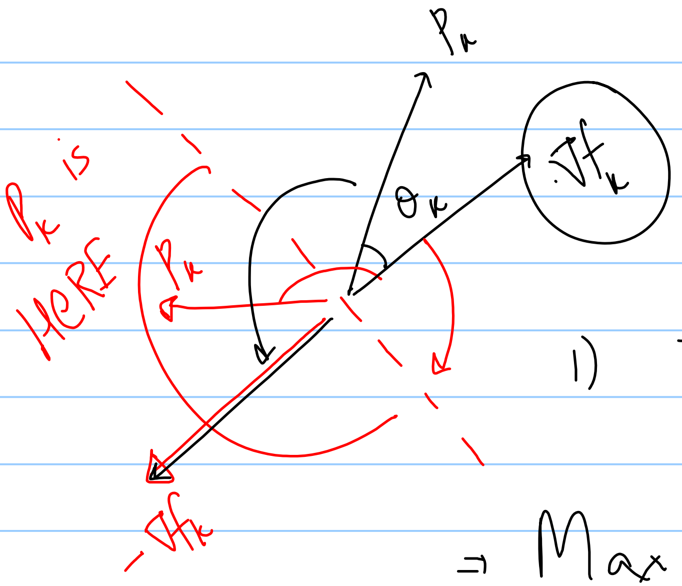
Properties of Descent Directions

① The steepest descent direction is $-\nabla f_k$ $\rightarrow$ $-\nabla f(x_k)$

$$x_k \xrightarrow[p_k]{\varepsilon} x_{k+1}$$

$$f(x_k + \varepsilon p_k) = f(x_k) + \boxed{\varepsilon \nabla f(x_k)^T p_k} + \boxed{O(\varepsilon^2)}$$

$$\nabla f(x_k)^T p_k = \underbrace{\|\nabla f(x_k)\|}_{\geq 0} \underbrace{\|p_k\|}_{\geq 0} \cos(\theta_k)$$



1) $f_{k+1} < f_k$

$$\Rightarrow \nabla f_k^T p_k < 0$$

$\Rightarrow$ Max decrease in f happens when $\theta_k = \pi$

$\Rightarrow \quad p_k = \underbrace{-}_{\text{desc}} \underbrace{\nabla f_k}_{\text{grad}}$ "Steepest" descent

A descent direction makes an angle $\in [-\pi/2, \pi/2]$ with $-\nabla f_k$.

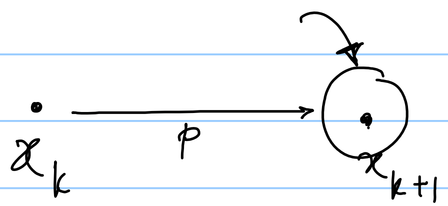
↳ Find the Newton direction

2nd order Taylor: $f(x_k + \varepsilon p_k) = f(x_k) + \varepsilon \nabla f_k^T p_k + \frac{\varepsilon^2}{2} p_k^T \nabla^2 f_k p_k + O(\varepsilon^3)$

$\nabla f_k^T p = \left(\frac{\partial f}{\partial x_1}\right) p_1 + \left(\frac{\partial f}{\partial x_2}\right) p_2 + \cdots$

q_k

If x_{k+1} is the sol'n



$\Rightarrow \nabla_p f(x_k + \varepsilon p_k) = 0$

$\therefore$ Stationary pt

$x^T A x$

$x^T (A^T + A) x$

∇_x

$$\nabla_p \left[f(x_k + \varepsilon p_k) \right] = \nabla_p \left[\quad \right]$$

$$\overset{\shortparallel}{0} = 0 + \varepsilon \nabla f_k + \varepsilon^2 (\nabla^2 f_k) p_k + 0$$

As long as $(\nabla^2 f)$ is P.D $\rightarrow$ $p_k = -\frac{1}{\varepsilon} (\nabla^2 f)^{-1} (\nabla f_k)$ ✓

Left multiply by $(\nabla^2 f)^{-1}$.

$$\boxed{p_N = -(\nabla^2 f)^{-1} \nabla f_k}$$ Newton direction.

Is this a descent direction?

$$p_N^T \nabla f_k < 0 \Rightarrow \nabla f_k^T \Big(\overbrace{-(\nabla^2 f_k)^{-1} \nabla f_k}^{p_N} \Big) < 0$$

$$= \nabla f_k^T p_N \longrightarrow \overbrace{\nabla f_k^T \underbrace{(\nabla^2 f)}^{-1} \nabla f_k > 0}$$

$$\Rightarrow \text{If } \underbrace{\nabla^2 f_k \text{ is P.D}} \text{ is true.}$$

— x —

Trust Region Methods

(1) Construct a model of the fn, m_k

(2) Search for a minima of m_k in the nbd of x_k.

$$\min_{p} \; m_k(x_k + p) \quad , \quad s.t. \; x_k + p \in T$$

$$m_k(x_k + p) = f_k + \nabla f_k^T p + \frac{1}{2} p_k^T B_k p_k$$

Hessian or its approx

—— x ——

$$f(x,y) = e^{-(x-1)^2} \sin(\pi y) \qquad x \in [0,2], \; y \in [0,1]$$