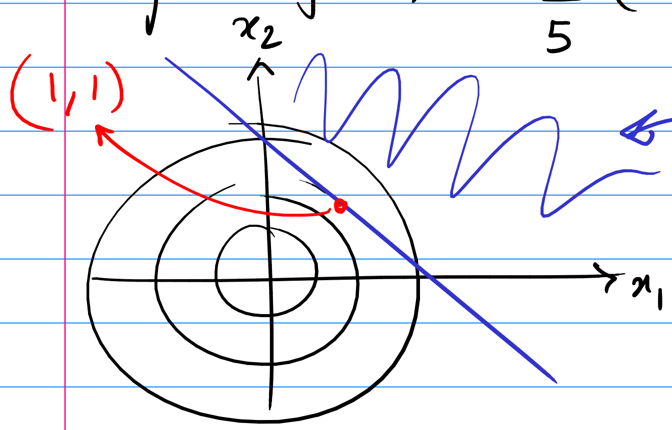
KKT & Duality

↳ e.g. $f(x) = \frac{2}{5}(x_1^2 + x_2^2)$, **P1**: $\min f(x)$ s.t. $c(x) = x_1 + x_2 - 2 \geq 0$



feasible: $\mathcal{L}(x,\lambda) = f(x) - \lambda c(x)$

As per KKT thm: $\nabla_x f - \lambda \nabla c(x) = 0$, $\underline{\lambda \geq 0}$

Comp → $\lambda c(x) = 0$

$$\frac{4}{5}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0, \quad \lambda \geq 0$$

e.g. $\lambda = 1 \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 5/4 \end{pmatrix} \rightarrow c(x) \neq 0$ ✗

$\lambda = 4/5 \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow c(x) = 0$ ✓

primal approach

$$\mathcal{L}(x,\lambda) = f(x) - \sum \lambda_i c_i(x)$$

Solve it differently: (2 steps)

Step 1) Lagrangian dual fn, $q(\lambda) = \left(\min_x\right) \mathcal{L}(x,\lambda)$

$\hookrightarrow \nabla_x \mathcal{L}(x,\lambda) = 0 \qquad \frac{4}{5}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underbrace{x_1^* = 5\lambda/4, \; x_2^* = 5\lambda/4.}$

$$q(\lambda) = \frac{2}{5}\left(x_1^{*2} + x_2^{*2}\right) - \lambda\left(x_1^* + x_2^* - 2\right)$$

$$= \frac{4}{5}\left(\frac{5\lambda}{4}\right)^2 - \lambda\left(\frac{10\lambda}{4} - 2\right) = -\frac{5}{4}\lambda^2 + 2\lambda$$

Concave.

Step 2) find $\lambda \geq 0$, which maximizes $q(\lambda)$.

$\max_\lambda q(\lambda)$, s.t. $\lambda \geq 0 \Rightarrow -\frac{10}{4}\lambda + 2 = 0 \Rightarrow \lambda^* = \frac{4}{5}.$

$\Rightarrow x_1^* = x_2^* = 1.$

"dual approach"

Primal problem: $\left(\min_x f(x) \quad \text{s.t. } c(x) \geq 0 \right)$

Dual problem: $\left(\begin{array}{l} \max_\lambda q(\lambda) \quad \text{s.t. } \lambda_i \geq 0 \quad i \in \mathcal{I} \\ \text{where } q(\lambda) = \min_x \mathcal{L}(x, \lambda) \end{array} \right)$

↳ for convex problems, both give same soln.
dual is generally easier to solve.

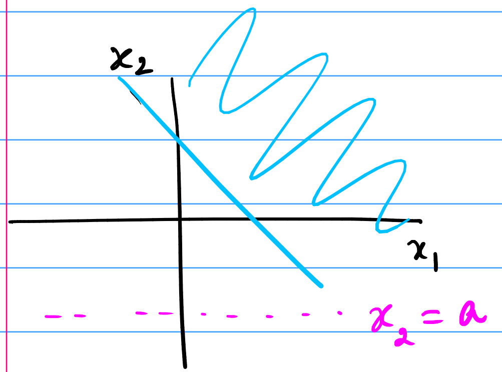
$\min_x f \quad s.t. \; c(x) \geq 0$

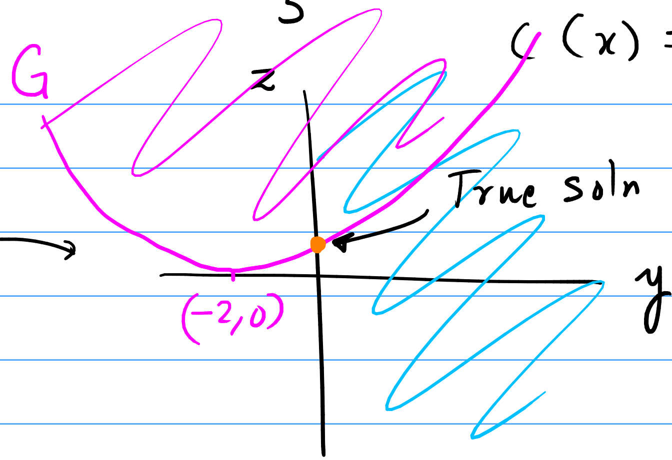
$f(x) = \frac{2}{5}(x_1^2 + x_2^2)$

↳ Geometric intuition

$c(x) = x_1 + x_2 - 2 \geq 0$

$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} c(x) \\ f(x) \end{pmatrix}$





$$y = x_1 + x_2 - 2 = x_1 + a - 2$$
$$z = \frac{2}{5}(x_1^2 + x_2^2) = \frac{2}{5}(x_1^2 + a^2)$$

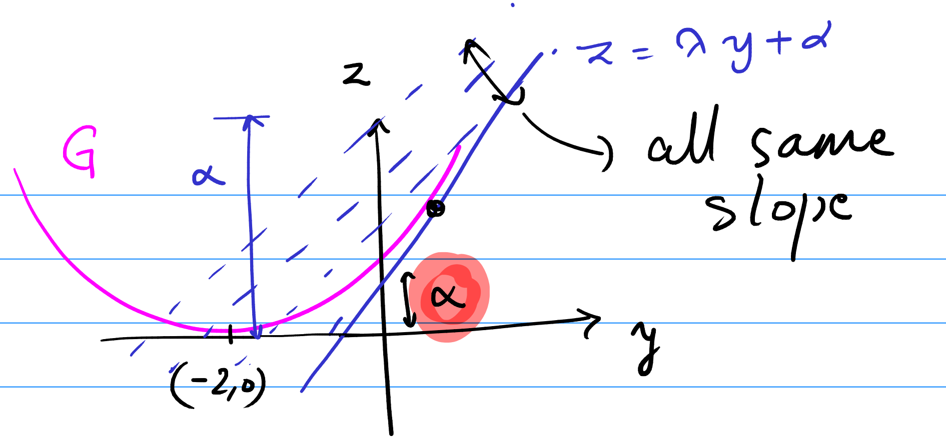
$$z = \frac{2}{5}\left[(y - a + 2)^2 + a^2\right]$$
$$\left(z - \frac{2}{5}a^2\right) = \frac{2}{5}\left(y - (a-2)\right)^2$$

$a = 0 \Rightarrow z = \frac{2}{5}(y+2)^2$

↳ What is the primal problem? $\min z, \; s.t. \; y \geq 0$

Can be easily seen graphically

What is the dual problem?



$\mathcal{L}(x,\lambda) = f(x) - \lambda\, C(x)$

$\alpha \quad = \quad z - \lambda y$

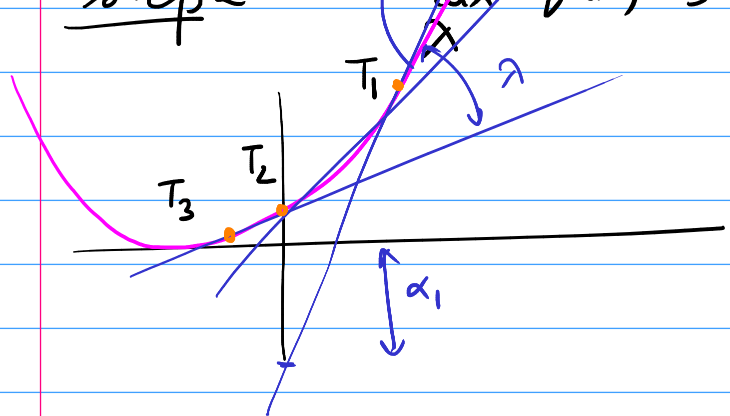
$$\boxed{z = \lambda y + \alpha}$$

Step 1: $q(\lambda) = \underbrace{\min_{x \in \mathcal{F}}}\ \mathcal{L}(x,\lambda) = \min_{\substack{y, z \in G \\ y \geq 0}} [\underbrace{z - \lambda y}_{\alpha}]$

α: intercept

Step 2: $\max q(\lambda)$ s.t. $\lambda \geq 0 \longrightarrow \max_{\lambda} [\underbrace{z^+ - \lambda y^+}_{\alpha^+}]$

(+ : from prev step)



Happens at T_2.

T_2 is the soln to the dual problem.

Primal soln = Dual soln.

strong duality

Update on defn of q. for some λ, $q(\lambda) \to -\infty$

Domain of q: $\mathcal{D} = \{\lambda \mid q(\lambda) > -\infty\}$

2 results about the dual problem:

① q is concave

② $\mathcal{D}$ is convex

$\Big\}$ for <u>any</u> f, c