Projected Gradient Descent (PGD)

↳ Problem: $\min_{x \in \Omega} f(x)$, e.g.: $\min_{\|x\|_2 \leq 1} \|Ax - b\|^2$

↳ Recap of GD:
① Pick a starting pt $x_0 \in \mathbb{R}^n$
② loop till satisfied
↳ find $-\nabla f$, find α_k
↳ update $x_{k+1} = x_k - \alpha_k \nabla f_k$

↳ PGD is a small modification. $\longrightarrow$ $x_{k+1} = P_\Omega(x_k - \alpha_k \nabla f_k)$

$P_\Omega()$ is a projection operator:

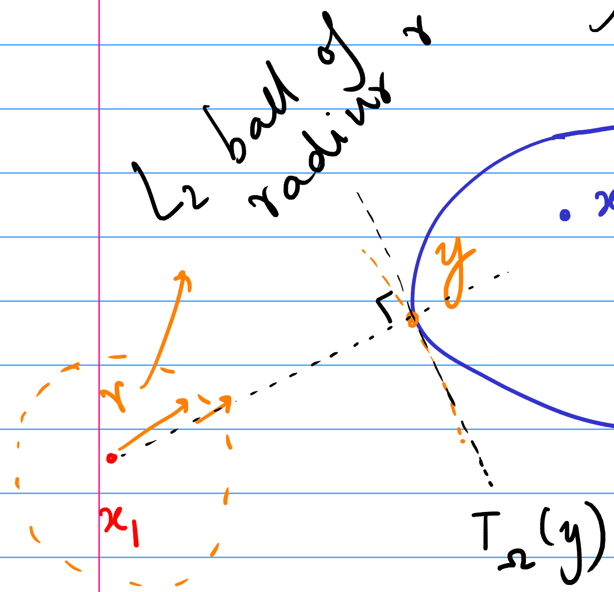
$$P_\Omega(x^*) = \underset{x \in \Omega}{\operatorname{argmin}} \frac{1}{2} \| x - x^* \|_2^2,$$

$$P_\Omega : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

↪ $$P_\Omega(x^*) = \underset{x \in \Omega}{\operatorname{argmin}} \left[\frac{1}{2} \| x - x^* \|_2^2 \right]$$

↪ PGD is useful if P_Ω is easy to compute.

↪ When Ω is a convex set there is a unique soln to the Proj operation.



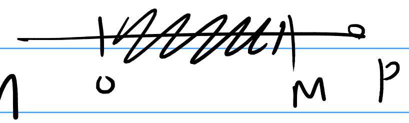
$P_\Omega(x_0) = x_0$,

$P_\Omega(x_1) = ?$

↪ y will be on the boundary of Ω when $x_1 \notin \Omega$.

↪ $(x_1 - y) \perp T_\Omega(y)$ Convex Ω

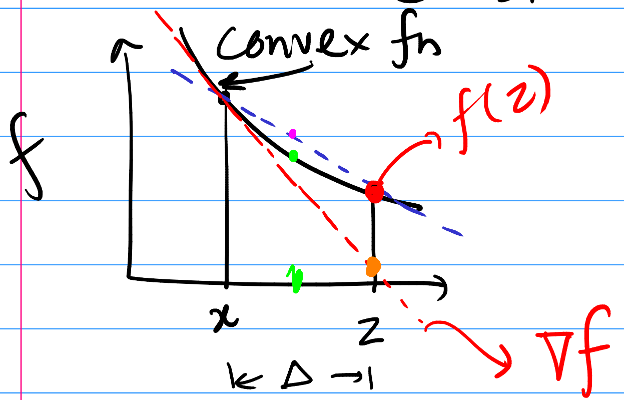
$p \geq 0$, $p \leq M \rightarrow 0 \leq p \leq M$



Assume convexity of f.

↳ Convergence analysis of PGD. (weaker version).

→ Constant step length assumption.

① 

$$f(z) = \underbrace{f(x) + \nabla f(x)^T (z-x)}_{\text{Linear approx of } f \text{ at } x} + O(\Delta^2)$$

$$f(z) \geq f(x) + \nabla f(x)^T (z-x) \leftarrow$$

Choose $z = x^*$, $x = x_k$ → k^{th} iterate.

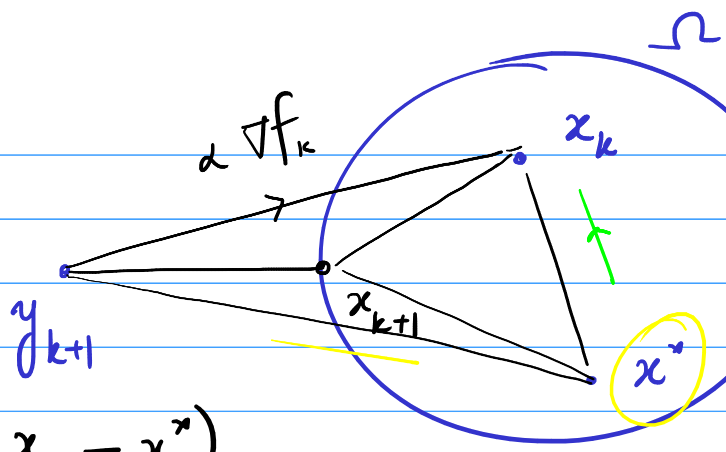
$$f(x_k) - f(x^*) \leq \nabla f(x_k)^T (x_k - x^*) \leftarrow$$

② PGD update: $y_{k+1} = x_k - \alpha \nabla f(x_k)$ usual GD

$$f(x^*) \geq f(x_k) + \nabla f(x_k)^T (x^* - x_k)$$

$$y_{k+1} = x_k - \alpha \nabla f(x_k)$$

$$x_{k+1} = P_\Omega(y_{k+1})$$



$$f_k - f^* \le \frac{1}{\alpha} \underbrace{(\alpha \nabla f_k)^T}_{p} \underbrace{(x_k - x^*)}_{q}$$

$$\|p - q\|^2$$

$$p^T q = \frac{1}{2}\left[\|p\|^2 + \|q\|^2 - \|p - q\|^2\right]$$

$$f_k - f^* \le \frac{1}{2\alpha}\left[\|\alpha \nabla f_k\|^2 + \|x_k - x^*\|^2 - \|\underbrace{\alpha \nabla f_k - (x_k - x^*)}_{x^* - y_{k+1}}\|^2\right]$$

$$\le \frac{1}{2\alpha}\left[\|x_k - x^*\|^2 - \|y_{k+1} - x^*\|^2\right] + \frac{\alpha}{2}\|\nabla f_k\|^2$$

(Since $\|y_{k+1} - x^*\| \ge \|x_{k+1} - x^*\|$, Subst:

$$f_k - f^* \le \frac{1}{2\alpha}\left[\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2\right] + \frac{\alpha}{2}\|\nabla f_k\|^2$$

Telescope the series:

$$f_0 - f^* \leq \frac{1}{2\alpha}\left[\| x_0 - x^*\|^2 - \| x_1 - x^*\|^2\right] + \frac{\alpha}{2}\|\nabla f_0\|^2$$

$\vdots$

$$f_k - f^* \leq \frac{1}{2\alpha}\left[\| x_k - x^*\|^2 - \| x_{k+1} - x^*\|^2\right] + \cdots$$

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$$\sum_{i=0}^{k} f_i - (k+1) f^* \leq \frac{1}{2\alpha}\left[\| x_0 - x^*\|^2 - \| x_{k+1} - x^*\|^2\right] + \frac{\alpha}{2}\sum_{i=0}^{k}\|\nabla f_i\|^2$$

$$k+1\left[\frac{\sum_{i=0}^{k} f_i}{k+1} - f^*\right] \leq \frac{1}{2\alpha}\left[\| x_0 - x^*\|^2\right] + \frac{\alpha}{2}\sum \|\nabla f_i\|^2$$

(2^{nd} term is $-$ve)

↳ Jensen's inequality: (for a convex fn)

$$f\left(\frac{\sum x_k}{k+1}\right) \leq \frac{\sum f(x_k)}{k+1}$$

$$\hookrightarrow \quad f\left(\frac{\sum x_k}{k+1}\right) - f^* \le \frac{\|x_0 - x^*\|^2}{2\alpha(k+1)} + \frac{\alpha}{2(k+1)} \sum_i \|\nabla f_i\|^2$$

Avg of iterates converges.

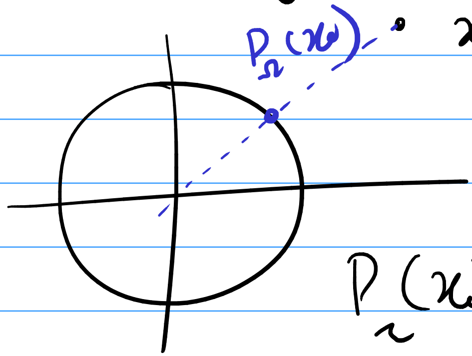
① $\|\nabla f_k\|^2$ must not grow faster than $O(k)$

$\rightarrow$ $\left[f(x_k) - f^* < \frac{M}{k} \right]$

also true, look up.

— x —

① Projection. eg. L_2 ball: $\Omega = \|x\|_2^2 \le 1$. $P_\Omega(x_0) = ?$



$$\left.\begin{array}{ll} \text{When } x_0 \in \Omega, & P_\Omega(x_0) = x_0 \\ \text{When } x_0 \notin \Omega, & P_\Omega(x_0) = \dfrac{x_0}{\|x_0\|_2} \end{array}\right\}$$

$$P_\Omega(x_0) = \frac{x_0}{\max(1, \|x_0\|_2)}.$$