Computational Electromagnetics : Finite Difference Time Domain Methods

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Topics in this module

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History and Central Idea

FDTD IEM, FEM $\rightarrow e^{j\omega t}$

- Simplest, most widely used CEM method.
- Yee 1966 $\rightarrow$ laid the foundation.
- Very useful for time domain formulations: wave prop, pulsed transient phenomena. Switching e.g.s
- Based on differential form of Maxwell's Eqns

$$\frac{\Delta z}{2} f'(z_0) \approx f\left(z_0+\frac{\Delta z}{2}\right) - f(z_0) + O(\Delta z^2)$$

one sided differences.

↳ Linear, isotropic, non dispersive media (change later), time invariant

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H} \qquad \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} - \vec{J} = \varepsilon \frac{\partial \vec{E}}{\partial t}, \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\left.\begin{aligned} f\left(z_0+\frac{\Delta z}{2}\right) &= f(z_0) + \frac{\Delta z}{2} f'(z)\Big|_{z_0} + \cdots \\ f\left(z_0-\frac{\Delta z}{2}\right) &= f(z_0) - \frac{\Delta z}{2} f'(z)\Big|_{z_0} + \cdots \end{aligned}\right\} \Rightarrow \boxed{f'(z_0) = \frac{f\left(z_0+\frac{\Delta z}{2}\right) - f\left(z_0-\frac{\Delta z}{2}\right)}{\Delta z} + O(\Delta z^2)}$$

centred, two point, finite difference.

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2D FDTD formulation: Stencil

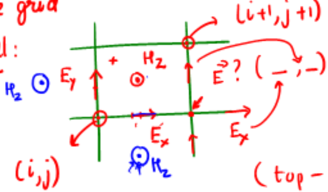
2 pols TE (E_x, E_y, H_z)
TM (H_x, H_y, E_z)

$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$, $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}$ → assume $= 0$

Consider TE

$\varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}$ — ① , $\varepsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$ — ② , $\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$ — ③

Stencil
Space grid
Yee cell:



Convention: $\vec{E}$ discretized along grid lines
$\vec{H}$ at centre.

Notation: $E_x(i+\frac{1}{2}, j) \to E_x(x = x_0 + (i+\frac{1}{2})\Delta x, y = y_0 + j\,\Delta y)$
$E_y(i, j+\frac{1}{2})$, $H_z(i+\frac{1}{2}, j+\frac{1}{2})$ i, j : space index
$E_x^n \to E_x(\ , \ , t = t_0 + n\Delta t)$ n: time index

① ⇒ $\varepsilon_0 \dot{E}_x(i+\frac{1}{2}, j) = \frac{H_z(i+\frac{1}{2}, j+\frac{1}{2}) - H_z(i+\frac{1}{2}, j-\frac{1}{2})}{\Delta y}$

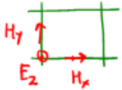
② ⇒ $\varepsilon_0 \dot{E}_y(i, j+\frac{1}{2}) = \frac{H_z(i-\frac{1}{2}, j+\frac{1}{2}) - H_z(i+\frac{1}{2}, j+\frac{1}{2})}{\Delta x}$

left − right

③ $\mu_0 \dot{H}_z(i+\frac{1}{2}, j+\frac{1}{2}) = \left[\frac{E_x(i+\frac{1}{2}, j+1) - E_x(i+\frac{1}{2}, j)}{\Delta y}\right] - \left[\frac{E_y(i+1, j+\frac{1}{2}) - E_y(i, j+\frac{1}{2})}{\Delta x}\right]$

TM pol → (H_x, H_y, E_z)

① — $\frac{\partial \bar{D}}{\partial t} = \nabla\times\bar{H}$, $\frac{\partial \bar{B}}{\partial t} = -\nabla\times\bar{E}$ — ②



2D FDTD formulation: Time Stepping

$\frac{\partial}{\partial t}$ → finite differences.

E^n → time index

in space → staggered by half grid. $(\Delta x, \Delta y)$

in time → " " " grid (Δt)

↳ Say $\vec{E}$ evaluated at integer time grids

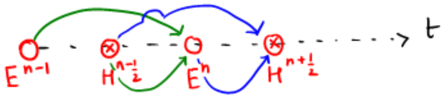
$$\frac{f(z_0+h) - f(z_0-h)}{2h} \to f'(z_0)$$

[avg of z_0+h and z_0-h]

$$① \Rightarrow \varepsilon_0\left[\frac{\bar{E}^n - \bar{E}^{n-1}}{\Delta t}\right] = \nabla\times\bar{H}^{n-1/2} \quad \left(\bar{E}^n = \bar{E}^{n-1} + \frac{\Delta t}{\varepsilon_0}\left[\nabla\times\bar{H}^{n-1/2}\right]\right) — ③$$

$$③ \Rightarrow \mu_0\left[\frac{\bar{H}^{n+1/2} - \bar{H}^{n-1/2}}{\Delta t}\right] = -\nabla\times\vec{E}^n \quad \left(\bar{H}^{n+\frac{1}{2}} = \bar{H}^{n-1/2} - \frac{\Delta t}{\mu_0}\left[\nabla\times\bar{E}^n\right]\right) — ④$$

present | past



leap frog integration

TM case $\vec{D} = D_z(x,y)\hat{z}$ → $\vec{\nabla}\cdot\vec{D} = 0$

$\nabla\cdot\bar{H} = 0$ (TE)

$\frac{\partial\bar{D}}{\partial t} = \nabla\times\bar{H}$

2D FDTD formulation: Divergence Conditions

e.g. $\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t}$ $(\bar{J} = 0)$

$\underbrace{\nabla\cdot\nabla\times\bar{H}}_{\text{always } 0} = \frac{\partial(\bar{\nabla}\cdot\bar{D})}{\partial t} = 0$ TE

⇒ $\nabla\cdot\vec{D} =$ const (time) → Does FDTD respect this?

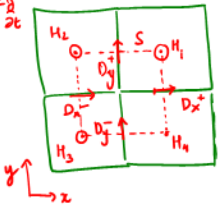
Div thm: (in 2D) → $\iint_S (\bar{\nabla}\cdot\bar{D})\,dx\,dy = \oint_\Gamma \vec{D}\cdot\hat{n}\,dl$

$= D_x^+\Delta y + D_y^+\Delta x - D_x^-\Delta y - D_y^-\Delta x$

↓ $\partial/\partial t$

→ $\dot{D}_x^+\Delta y + \dot{D}_y^+\Delta x - \dot{D}_x^-\Delta y - \dot{D}_y^-\Delta x$

$\iint_S \frac{\partial}{\partial t}(\nabla\cdot\bar{D})\,dx\,dy = \oint \frac{\partial}{\partial t}(\vec{D}\cdot\hat{n})\,dl$



top–bottom: $\dot{D}_x^+ = \frac{H_1 - H_4}{\Delta y}$, $\dot{D}_x^- = \frac{H_2 - H_3}{\Delta y}$

left–right: $\dot{D}_y^+ = \frac{H_2 - H_1}{\Delta x}$, $\dot{D}_y^- = \frac{H_3 - H_4}{\Delta x}$

⊕ → $(H_1 - H_4) + (H_2 - H_1) - (H_2 - H_3) - (H_3 - H_4) = 0$

$\boxed{\Rightarrow \frac{\partial(\bar{\nabla}\cdot\bar{D})}{\partial t} = 0}$ over S

we got it for free!

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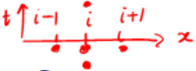
$E^n(i) \to E^{t=n\Delta t}(x = x_0 + i\Delta x)$



$\underline{E^n(i) = \alpha^n e^{-jk\Delta x i}}$.

Stability Criteria – comparing true/computed solutions

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad (\text{wave Eqn}) \to e^{j(kx \pm \omega t)} \qquad (1)$$

$$f(z_0 + \tfrac{\Delta z}{2}) = f(z_0) + \frac{\Delta z}{2} f'(z_0) + \left(\frac{\Delta z}{2}\right)^2 \times \frac{1}{2} f''(z_0) + \dots$$

$$f(z_0 - \tfrac{\Delta z}{2}) = \text{"} - \text{"} + \text{"} - \dots$$

$$f''(z_0) = \frac{f(z_0 + \frac{\Delta z}{2}) - 2f(z_0) + f(z_0 - \frac{\Delta z}{2})}{(\Delta z/2)^2} + \dots \qquad (2)$$

(2) into (1) $\to$

$$\frac{\partial^2 E}{\partial x^2} = \frac{E^n(i+1) - 2E^n(i) + E^n(i-1)}{(\Delta x)^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$= \frac{E^{n+1}(i) - 2E^n(i) + E^{n-1}(i)}{(c\Delta t)^2} \qquad (3)$$

soln should be a wave.

trial soln: $E^n(i) = \alpha^n e^{-jkx}$, $x = i\Delta x$. (4)

correct soln: $\alpha = e^{j\omega\Delta t}$

Q: Put (4) into (3), ask do we get a wave?

$$e^{-jk\Delta x i}\left[\alpha^n e^{-jk\Delta x} - 2\alpha^n + \alpha^n e^{jk\Delta x}\right]$$

$$= e^{-jk\Delta x i}\left[\alpha^{n+1} - 2\alpha^n + \alpha^{n-1}\right] \times \left(\frac{\Delta x}{c\Delta t}\right)^2$$

$$\alpha\left[2\cos(k\Delta x) - 2\right] = \left[\alpha^2 - 2\alpha + 1\right]\left(\frac{\Delta x}{c\Delta t}\right)^2$$

$$\alpha^2 - 2\underbrace{\left[1 - 2\left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right]}_{A}\alpha + 1 = 0$$

Stability Criteria – comparing true/computed solutions

Eqn is: $\alpha^2 - 2A\alpha + 1 = 0$, $\alpha = A \pm \sqrt{A^2 - 1}$ soln.

$E^n(i) = \alpha^n e^{-jk\Delta x i}$

For an oscillatory soln: $|\alpha| = 1$

$\Rightarrow$ $|\alpha| > 1 \rightarrow$ unstable solns

If $|A| < 1$, $\alpha = A \pm j\sqrt{1 - A^2}$, $|\alpha|$?

$|\alpha| = \sqrt{A^2 + (1 - A^2)} = 1 \Rightarrow$ oscillatory soln

$$\alpha = A + j\sqrt{1 - A^2} = e^{j\theta} = e^{j\omega}$$

Needed $|A| \leq 1$

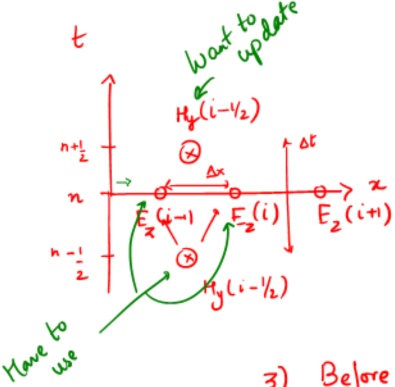
↳ $\boxed{\dfrac{c\Delta t}{\Delta x} \leq 1}$

This condn is the Courant stability criteria.

Say we fix Δx.

then $\Delta t \leq \dfrac{\Delta x}{c}$. → both updates are fixed.

Stability Criteria – Another way (intuitive)



$$\mu_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} \quad \text{(TM case)}$$

1) E_z spaced apart by Δx

2) Time req for signal to go from $(i-1)$ to (i)?

$$\min, \tau = \frac{\Delta x}{c}.$$

3) Before the lapse of τ, I should use my update eqn.

$$H_y^{n+1/2}(i+1/2) = H_y^{n-1/2}(i+1/2) - \frac{\Delta t}{\mu_0}\left[\frac{E_z^n(i) - E_z^n(i-1)}{\Delta x}\right]$$

$\omega = \frac{2\pi}{T}$

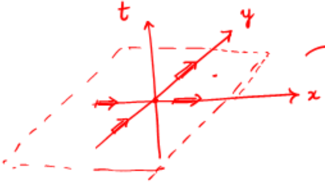
$$\Rightarrow \quad \Delta t \leq \tau \Rightarrow \boxed{\Delta t \leq \frac{\Delta x}{c}}.$$

TE → H_z, E_x, E_y

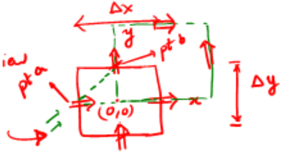
$$\mu_0 \frac{\partial \bar{H}}{\partial t} = -\bar{\nabla} \times \bar{E} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

Stability Criteria in higher dimensions

Take 2D for e.g.



Top view



Eqn: $$\mu_0 \left(\frac{H^{n+1/2} - H^{n-1/2}}{\Delta t} \right) = \left(\frac{E_x^n(i+\frac{1}{2}, j+1) - E_x^n(i+\frac{1}{2}, j)}{\Delta y} \right) - \left(\frac{E_y^n(i+1, j+\frac{1}{2}) - E_y^n(i, j+\frac{1}{2})}{\Delta x} \right)$$

future time

Assume $\Delta x = \Delta y = \Delta s$

$$\text{Min } \tau = \sqrt{\left(\frac{\Delta x}{2}\right)^2 + \left(\frac{\Delta y}{2}\right)^2} \Big/ c = \frac{\Delta s}{\sqrt{2}\, c}$$

For Update eqn to make sense $\Rightarrow$ $\underline{\Delta t \leq \tau}$

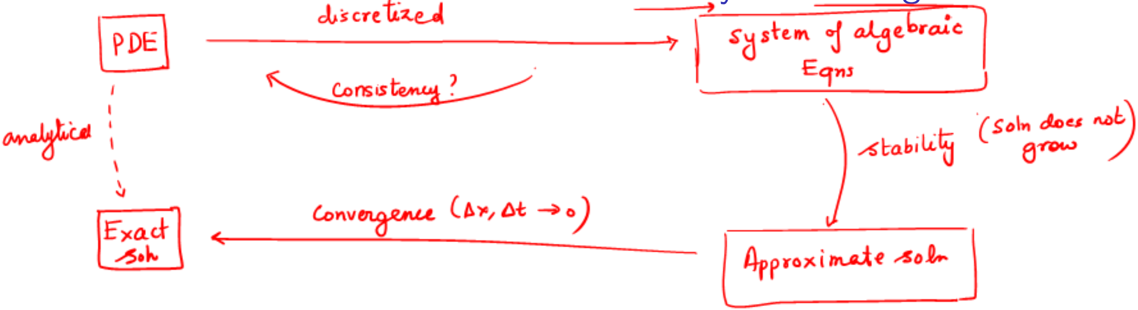
$\Rightarrow \Delta t \leq \frac{\Delta s}{\sqrt{2} c} \Rightarrow$

$$c\,\Delta t \leq \frac{\Delta s}{\sqrt{2}} \quad \leftarrow 2D$$

$$c\,\Delta t \leq \frac{\Delta s}{\sqrt{3}} \quad \leftarrow 3D$$

Higher dimensions ⇒ finer time discretization.

From Stability to Convergence



Theorem by Lax and Richtmyer: "Given a properly posed linear initial value problem and a finite difference approximation to it which satisfies the consistency condition, stability is necessary & sufficient for convergence"

given

imp

Accuracy Considerations – 1D

What is dispersion? → physical.

↳ different freqs → travel with different speeds.

Numerical?

True soln: $\left(\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\right) \to E = e^{j(\underbrace{\omega t - kx}_{\text{phase } \phi})}$ (Right travelling)

$E^n(i) = e^{j(\omega n\Delta t - k i \Delta x)}$

phase velocity → locus of const phase

$\frac{d\phi}{dt} = 0 \Rightarrow \omega - k v_p = 0 \Rightarrow v_p = \frac{\omega}{k} = c$ (in vacuum)

Assume $\alpha = \frac{c\Delta t}{\Delta x} \leq 1 \Rightarrow$ wave-like soln.

$$\frac{E^n(i+1) - 2E^n(i) + E^n(i-1)}{(\Delta x)^2} = \frac{1}{c^2}\,\frac{E^{n+1}(i) - 2E^n(i) + E^{n-1}(i)}{(\Delta t)^2}$$

$$\left[\frac{e^{-jk\Delta x} - 2 + e^{jk\Delta x}}{(\Delta x)^2}\right] = \frac{1}{c^2}\left[\frac{e^{j\omega\Delta t} - 2 + e^{-j\omega\Delta t}}{\Delta t^2}\right]$$

$$\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{(\Delta x)^2} = \frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\omega\Delta t}{2}\right)$$

Dispersion.

Accuracy Considerations – 1D

↳ Check $\Delta x, \Delta t \to 0$

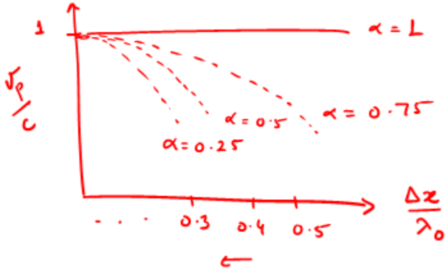
$k^2 = \omega^2/c^2$

phase vel, $v_p = \omega/k = c$.

↳ what when $\Delta x, \Delta t$ finite? $\frac{c\Delta t}{\Delta x} \le 1$

→ when $\alpha = 1 \Rightarrow c\Delta t = \Delta x$, $k = \omega/c$

satisfies dispersion relⁿ



↳ when $\alpha \neq 1$? Solved numerically.

wave travels slower than c.

→ Dispersion (numerical)

- courant parameter $(\alpha = 1)$ $\boxed{?}$
- discretization (fine)

Accuracy Considerations – higher dimensions

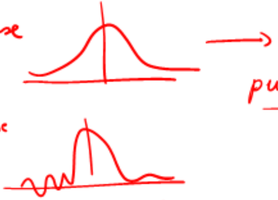
$$\frac{c\,\Delta t}{\Delta s} \le \frac{1}{\sqrt{2}} \quad (2D)$$

$$\frac{c\,\Delta t}{\Delta s} \le \frac{1}{\sqrt{3}} \quad (3D)$$

Here, can't set $\frac{c\,\Delta t}{\Delta s} = 1$

i/p pulse

o/p pulse



pulse distortion.

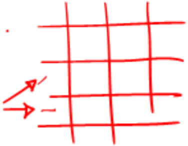
Dispersion reln:

$$\frac{1}{(c\,\Delta t)^2}\sin^2\left(\frac{\omega\,\Delta t}{2}\right) = \frac{1}{(\Delta x)^2}\sin^2\left(\frac{k\,\Delta x}{2}\right) + \frac{1}{(\Delta y)^2}\sin^2\left(\frac{k\,\Delta y}{2}\right)$$

Different directions → diff speeds

Grid Anisotropy.

Only soln → small $\Delta x, \Delta y, \Delta t$.



Topics that were covered in this module

1. Introduction
2. 2D Formulation
3. Numerical Analysis of FDTD

References:

* Ch 12 of Computational Methods for Electromagnetics - Peterson, Ray, Mitra
* Computational Electrodynamics: The Finite-Difference Time-Domain Method – Allen Taflove (the 'Bible' for FDTD)
* Interesting interview by Taflove on Maxwell's equations and FDTD: http://www.eecs.northwestern.edu/images/nphoton.2014.305.pdf