Computational Electromagnetics :

An Introduction to the Finite Element Method

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Topics in this module

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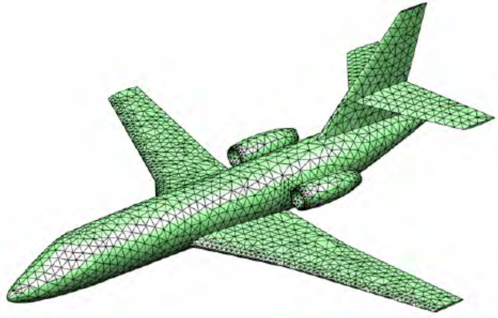
History & Overview of the FEM

Finite Element Method

- Initial work → Courant (1940s)
- FEM → Engineering (1960s)



Key difference v/s Integral Eqns:

- Green's functions: Not there!
- Geometry: ✓
- Computation: IE: $O(n^3)$ $\to$ $O(n^{2.4})$: FEM $O(n)$
- System of eqns: sparse
- Storage: IE: $O(n^2)$: FEM $O(n)$

A recap of the Method of Moments . . .

operator, unknown, known

$$\mathbb{L}\,\phi(r) = f(r) \quad + \text{B.C.}$$

$$\mathbb{L} = \frac{d^2}{dr^2} \to \frac{d^2\phi(r)}{dr^2} = f(r)$$

1) $\phi(r) = \sum_{n=1}^{N} \phi_n b_n(r)$ (Basis)

$$\sum \phi_n \mathbb{L}\, b_n(r) = f(r)$$

2) Testing fns $t_m(r)$: Take inner product

$$\sum_{n=1}^{N} \phi_n \left(t_m(r), \mathbb{L}\, b_n(r) \right) = \left(t_m(r), f(r) \right)$$

3) Simplest testing fn $t_m(r) = \delta(r - r_m)$

$$\sum_{n=1}^{N} \phi_n \,\mathbb{L}\, b_n(r)\Big|_{r=r_m} = f(r_m)$$

Point matching (FDM)

Finite difference methods

$$\int \delta(r-r_m)\,\frac{d^2}{dr^2}\phi(r)\,dr = \frac{d^2\phi(r)}{dr^2}\Big|_{r=r_m}$$

$$\frac{\phi(r_{m+1}) - 2\phi(r_m) + \phi(r_{m-1})}{2\,(r_{m+1} - r_{m-1})^2} = f(r_m)$$

$$\mathbb{L}\,\phi(r)\Big|_{r=r_m} = f(r_m)$$

sub domain basis/testing fns.

FEM

Casting MoM as a 'weighted residual' method

Sets $t_m(r) = b_m(r)$ → Galerkin's method.

$$\phi(r) = \sum_{n=1}^{N} \phi_n b_n(r)$$

$$\mathbb{L}\phi(r) = f(r) \rightarrow \mathbb{L}\phi(r) - f(r) = 0,$$

support of $b_m(r)$

$$\rightarrow \int_{\Omega_m} \underbrace{b_m(r)}_{\text{weight}} \{ \underbrace{\mathbb{L}\phi(r) - f(r)}_{\text{residual}} \} dr = 0$$

$\Omega_m \updownarrow \Omega$

repeat for $m = 1, \ldots, N$

→ System of eqns $Ax = c$

$$\frac{1}{\mu(r)} \nabla \times E(r) = -j\omega \bar{H}(r)$$

$$\nabla \times \left[\frac{1}{\mu(r)} \nabla \times E(r) \right] = -j\omega \cdot j\omega \varepsilon(r) \bar{E}(r)$$

$(J=0)$

$$\nabla \times \frac{1}{\mu_r(r)} \nabla \times \bar{E}(r) - k_0^2 \varepsilon_r(r) \bar{E}(r) = 0$$

$$\underbrace{\left[\nabla \times \frac{1}{\mu_r(r)} \nabla \times - k_0^2 \varepsilon_r(r) \right]}_{\mathbb{L}} \underbrace{\bar{E}(r)}_{\phi} = 0$$

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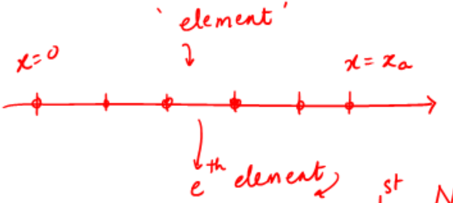


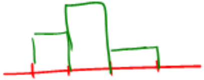
Elements in one dimension

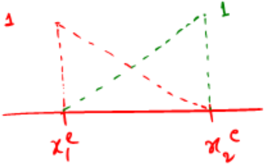
Lagrange polynomials.

1st $N_1^e(x) = \dfrac{x_2^e - x}{x_2^e - x_1^e}$

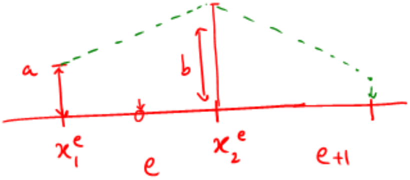
2nd $N_2^e(x) = \dfrac{x - x_1^e}{x_2^e - x_1^e}$

only over element e
zero outside
shape / basis fns





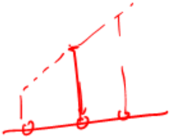
→ $a\,N_1^e(x) + b\,N_2^e(x)$



Elements in two dimensions

node values are unknowns.

Node based elements.



elements:

shape fns

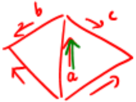
3 shape fns.

$E_z(x,y)$, $\vec{H}$

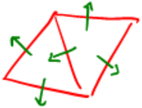
2 shape fns

Edge based elements

Nedelec.

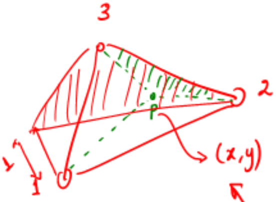


Fluid mechanics



Elements in two dimensions

1st order node based element



$$N_1^e = \frac{\Delta_1}{\Delta} = \frac{\text{Area } \Delta P23}{\text{Area } \Delta 123}$$

$N_1^e(x,y) \longrightarrow$ value at Node 1 : 1

Node 2, 3 : 0

along 2 - 3 : 0

$$= \frac{(x_2y_3 - x_3y_2) - x(y_3 - y_2) + y(x_3 - x_2)}{(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)}$$

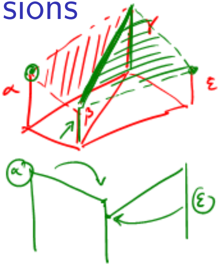
$$= \frac{a_1 + b_1x + c_1y}{2\Delta}$$

Similarly:

$$N_2^e = \frac{\Delta_2}{\Delta} = \frac{\text{Area } P13}{\text{Area } 123}$$

$$N_3^e = \frac{\Delta_3}{\Delta} = \frac{\text{Area } P12}{\text{Area } 123}$$

any fn $u(x,y) = \alpha N_1^e + \beta N_2^e + \gamma N_3^e$



Topics that were covered in this module

1. Overview of the FEM

2. Basis functions

Reference: